Wild solenoids and tilings

Steve Hurder

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University of Illinois at Chicago
www.math.uic.edu/∼hurder

Set-up:

- G a group (or $\mathrm{pseudo}\star\mathrm{group} \equiv$ a pseudogroup without the disjoint union property for maps),
- $\mathfrak{X}$ a Cantor space,
- $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ a minimal action.

Program:

Define and calculate invariants which distinguish such actions up to an appropriate notion of equivalence.

Applications:

- Classification of weak solenoids up to homeomorphism
- Classification of tiling spaces
- Invariants of iterated polynomial equations

Definition: Let $\varphi_i \colon G_i \times \mathfrak{X}_i$ be minimal Cantor actions, for $i = 1, 2$. Say that φ_1 is return equivalent to φ_2 if there exist

- clopen subsets $U_i \subset \mathfrak{X}_i$ for $i = 1, 2$
- homeomorphism $h \colon U_1 \to U_2$

such that h induces an isomorphism $\alpha_h \colon G_1|U_1 \to G_2|U_2$ of the restricted pseudo⋆groups

$$G_i|U_i = \{g_U \colon U \to V \mid g \in G_i \ , \ U \subset Dom(g) \ , \ V = g(U) \subset U_i\}$$

Remark: When G_i is a group and $U_i = \mathfrak{X}_i$ for $i = 1, 2$, this reduces to the notion of topological conjugacy of the actions, allowing an isomorphism $\alpha_h \colon G_1 \to G_2$ intertwining the actions.

"Analytic" Cantor actions:

Let $U, V \subset \mathfrak{X}$ be clopen subsets of a Cantor space $\mathfrak{X}$.

• A homeomorphism $h\colon U \to V$ is quasi-analytic (QA) if either $U = V$ and h is the identity map, or for every *clopen* subset $W \subset U$ the fixed-point set of the restriction $h|W\colon W \to h(W) \subset V$ has no interior.

• A homeomorphism $h\colon U \to V$ is locally quasi-analytic (LQA) if for each $x \in U$ there exists a clopen neighborhood $x \in U' \subset U$ such that the restriction $h_{U'}\colon U' \to V' = H(U')$ is QA.

• A pseudo⋆group action $\varphi\colon G\colon \mathfrak{X} \to \mathfrak{X}$ is LQA if for each $x \in \mathfrak{X}$, there exists a clopen neighborhood $x \in U$, such that each map in the restricted pseudo⋆group $G|U$ is QA.

Remarks:

• A free action $G \times \mathfrak{X} \to \mathfrak{X}$ is QA.

• A minimal action of $G = \mathbb{Z}^n$ is always QA.

• The pointed automorphism group of an equi-valent tree (acting on the Cantor set of ends) is not LQA. Such actions are also not finitely generated.

Proposition: The LQA property is an invariant of return equivalence for minimal $\mathrm{pseudo}{\star}\mathrm{group}$ actions.

Let $\varphi\colon G\colon \mathfrak{X} \to \mathfrak{X}$ be a $\text{pseudo}\star\text{group}$ action. A point $x \in \mathfrak{X}$ is said to be a *holonomy point* for the action if for all $g \in G$ with $x \in Dom(g)$, $g(x) = x$, and for all open $x \in U \subset Dom(g)$, the restriction $g|U\colon U \to g(U)$ is not the identity.

The following result was first shown for foliation $\text{pseudo}\star\text{group}$'s by Epstein, Millett and Tischler in 1977:

Proposition: For $\varphi\colon G\colon \mathfrak{X} \to \mathfrak{X}$ a $\text{pseudo}\star\text{group}$ action, then

$$Holo(\varphi) = \{x \in \mathfrak{X} \mid x \text{ is a holonomy point}\}$$

is meager; in particular, it has empty interior in $\mathfrak{X}$.

Definition: Let $\varphi_i \colon G_i \times \mathfrak{X}_i$ be minimal Cantor actions, for $i = 1, 2$. Say that φ_1 is topologically orbit equivalent (TOE) to φ_2 if there exist a homeomorphism $h \colon \mathfrak{X}_1 \to \mathfrak{X}_2$ such that for all $x \in \mathfrak{X}_1$

$$h\left(\{\varphi_1(g)(x) \mid g \in G_1\}\right) = \{\varphi_2(g')(h(x)) \mid g' \in G_2\}$$

Definition: A continuous cocycle $\phi \colon G_1 \times \mathfrak{X}_1 \to G_2$ over the action φ_1 is said to be a rearrangement cocycle for a TOE $h \colon \mathfrak{X}_1 \to \mathfrak{X}_2$ between actions φ_1 and φ_2 if we have

$$h(\varphi_1(g)(x)) = \varphi_2(\phi(g, x))(h(x)) \quad \text{for all } g \in G_1 \ , \ x \in \mathfrak{X}_1 \ . \quad (1)$$

Lemma: For $i = 1, 2$, let $\varphi_i \colon G_i \times \mathfrak{X}_i$ be a free minimal Cantor action. Let $h \colon \mathfrak{X}_1 \to \mathfrak{X}_2$ be a TOE between φ_1 and φ_2. Then there exists a rearrangement cocycle $\phi \colon G_1 \times \mathfrak{X}_1 \to G_2$ for h.

Proof. The formula (1) defines ϕ as the actions are free.

Proposition: For $i = 1, 2$, let $\varphi_i \colon G_i \times \mathfrak{X}_i$ be an LQA minimal Cantor action. Let $h \colon \mathfrak{X}_1 \to \mathfrak{X}_2$ be a TOE between φ_1 and φ_2. Then there exists a rearrangement cocycle $\phi \colon G_1 \times \mathfrak{X}_1 \to G_2$ for h.

Proof. The LQA assumption implies that there exists a dense φ_2-invariant subset $\mathfrak{X}_2' \subset \mathfrak{X}_2$ on which the action is free. Then $\mathfrak{X}_1' = h^{-1}(\mathfrak{X}_2') \subset \mathfrak{X}_1$ is dense and φ_1-invariant. Use the formula (1) to define ϕ on $\mathfrak{X}_1'$, then extend to all of $\mathfrak{X}_1$ by continuity.

We apply these concepts for a special class of actions.

Definition: A Cantor action $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ is equicontinuous if for some metric $d_{\mathfrak{X}}$ on $\mathfrak{X}$, for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$d_{\mathfrak{X}}(x, y) < \delta \implies d_{\mathfrak{X}}(\varphi(g)(x), \varphi(g)(y)) < \epsilon \quad \text{for all } g \in G.$$

Theorem: (folkore) Assume that G is finitely generated. Given $x \in \mathfrak{X}$, then the equicontinuous minimal Cantor action φ is conjugate to a generalized odometer (X_∞, G, Φ) associated to a properly descending chain of subgroups of finite index in G,

$$G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_\ell \supset \cdots$$

where $X_\infty \equiv \varprojlim \, \{p_\ell\colon G/G_{\ell+1} \to G/G_\ell \mid \ell > 0\}$ has the natural left G-action, but need not be a profinite group. Moreover, the conjugating homeomorphism $\tau\colon X_\infty \to \mathfrak{X}$ satisfies $\tau(e_\infty) = x$, where $e_\infty \in X_\infty$ is the coset of the identity $e \in G$. The map τ represents "coordinates" for $\mathfrak{X}$ around x.

The following result was proved by Cortez and Medynets:

Theorem: Let $\varphi_i \colon G_i \times \mathfrak{X}_i \to \mathfrak{X}_i$ be free equicontinuous minimal Cantor actions, for $i = 1, 2$. Let $h \colon \mathfrak{X}_1 \to \mathfrak{X}_2$ be a TOE between φ_1 and φ_2. Then h induces a return equivalence between the actions φ_1 and φ_2.

In fact, a stronger statement is true:

Theorem: Let $\varphi_i \colon G_i \times \mathfrak{X}_i \to \mathfrak{X}_i$ be an LQA equicontinuous minimal Cantor action, for $i = 1, 2$. Let $h \colon \mathfrak{X}_1 \to \mathfrak{X}_2$ be a TOE between φ_1 and φ_2. Then h induces a return equivalence between the actions φ_1 and φ_2.

Proof. The existence of a continuous rearrangement cocycle is a consequence of the LQA hypothesis. The proof then follows as in Cortez and Medynets [2016] and Clark and Hurder [2013].

A Cantor action $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ of a group G defines a homomorphism $\widehat{\varphi}\colon G \to H_\varphi \subset \mathrm{Homeo}(\mathfrak{X})$.

Definition: The Ellis (enveloping) semigroup $E(\varphi) \subset \mathrm{Homeo}(\mathfrak{X})$ is the closure of the group H_φ in the topology of pointwise convergence on maps.

The Ellis semi-group of an action φ seems very difficult to calculate in general, except in the following case:

Proposition: Let φ be an equicontinuous Cantor action. Then $E(\varphi)$ is the closure $\overline{H_\varphi}$ of H_φ in the *uniform topology on maps*.

Lemma: The left action of $\overline{H_\varphi}$ on $\mathfrak{X}$ is transitive if φ is minimal.

Proof. Let $x \in \mathfrak{X}$, then the orbit $\overline{H^\varphi} \cdot x$ is a closed subset of $\mathfrak{X}$ that contains a dense orbit under the action of $\widehat{\varphi}(G)$.

Let $\overline{H^\varphi}_x = \{h \in \overline{H^\varphi} \mid h(x) = x\}$ be the isotropy group of $x \in \mathfrak{X}$.

Corollary: $\mathfrak{X} \cong \overline{H^\varphi}/\overline{H^\varphi}_x$.

Note that group $\overline{H^\varphi}_x$ is independent of the choice of basepoint $x \in \mathfrak{X}$, up to topological isomorphism of groups.

The normal core N of a subgroup $H \subset G$ is the largest subgroup $N \subset H$ which is normal in G.

Proposition: The normal core of $\overline{H^\varphi}_x$ in $\overline{H^\varphi}$ is trivial.

Proof. The action of $\overline{H^\varphi}$ on $\mathfrak{X}$ is effective, the normal core of $\overline{H^\varphi}_x$ acts trivially on the quotient space $\mathfrak{X} \cong \overline{H^\varphi}/\overline{H^\varphi}_x$, hence it is trivial.

Let G be finitely generated, $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ an equicontinuous minimal Cantor action, and $x \in \mathfrak{X}$ a choice of basepoint.

Then the action is conjugate to a generalized odometer (X_∞, G, Φ) associated to a properly descending chain of subgroups $\{G_\ell \mid \ell \geq 0\}$ of finite index in G, where the conjugating homeomorphism $\tau\colon X_\infty \to \mathfrak{X}$ satisfies $\tau(e_\infty) = x$.

Let $C_\ell \subset G_\ell$ be the normal core of G_ℓ in G, then C_ℓ has finite index in G. Define the profinite group

$$G_\infty \equiv \varprojlim \; \{q_\ell\colon G/C_{\ell+1} \to G/C_\ell \mid \ell > 0\} \; .$$

The left action of G on G_∞ yields a minimal Cantor action $(G_\infty, G, \Phi_\infty)$ and homomorphism $\widehat{\Phi}_\infty\colon G_\infty \to \mathrm{Homeo}(\mathfrak{X})$.

Theorem: Let $\varphi \colon G \times \mathfrak{X} \to \mathfrak{X}$ be an equicontinuous minimal Cantor action. Then $\overline{H^{\varphi}} = \widehat{\Phi}_{\infty}(G_{\infty})$.

Theorem: There is a natural isomorphism

$$\overline{H^{\varphi}}_x \cong \mathcal{D}_x \equiv \varprojlim \; \{\pi_{\ell} \colon G_{\ell+1}/C_{\ell+1} \to G_{\ell}/C_{\ell} \mid \ell \geq 0\} \; . \qquad (2)$$

The profinite group $\mathcal{D}_x$ defined by the right-hand-side of (2) is called the <u>discriminant</u> of the group chain $\{G_{\ell} \mid \ell \geq 0\}$ at x.

Corollary: The isomorphism class of the discriminant group $\mathcal{D}_x$ is independent of the choice of basepoint $x \in \mathfrak{X}$ and the group chain $\{G_{\ell} \mid \ell \geq 0\}$ used to define the odometer model of the action φ.

Problem: Show that the isomorphism class of the discriminant group an invariant of return equivalence.

A minimal Cantor action is wild if there exist "hidden symmetries" in the large scale structure of the associated colored tiling.

Let $(\mathfrak{X}, G, \varphi)$ be conjugate to the odometer (X_∞, G, Φ) associated to a properly descending chain of subgroups $\{G_\ell \mid \ell \geq 0\}$ of finite index in G, with basepoint x. Corresponding to each subgroup G_ℓ for $\ell \geq 0$, there is a clopen neighborhood U_ℓ of x with $x \in U_{\ell+1} \subset U_\ell$ and $G_\ell = G|U_\ell$.

Introduce the nested clopen neighborhood subgroups of $\widehat{e} \in G_\infty$

$$\widehat{U}_\ell \equiv \varprojlim \; \{\pi_\ell \colon G_\ell/C_{\ell+k+1} \to G_\ell/C_{\ell+k} \mid k > 0\} \subset G_\infty \, .$$

Note that $\mathcal{D}_x \subset \widehat{U}_\ell$ and $U_\ell \cong \widehat{U}_\ell/\mathcal{D}_x$ for all $\ell \geq 0$.

The isotropy action of $\mathcal{D}_x$ on U_ℓ is induced by the adjoint action: for $\widehat{h} \in \mathcal{D}_x$ and $y \in U_\ell$, let $y = \widehat{g}\,\mathcal{D}_x$ for $\widehat{g} \in \widehat{U}_\ell$, then

$$\widehat{h} \cdot y = Ad(\widehat{h})(\widehat{g})\,\mathcal{D}_x = \widehat{h}\;\widehat{g}\;\widehat{h}^{-1}\,\mathcal{D}_x\,.$$

The "trivial core" property of $\mathcal{D}_x$ implies that the action of $\widehat{h} \in \mathcal{D}_x$ on $X_\infty \cong G_\infty/\mathcal{D}_x$ is non-trivial for all non-trivial $\widehat{h}$.

Definition: An equicontinuous minimal Cantor action φ is wild if for all $\ell > 0$ there exists non-trivial $\widehat{h} \in \mathcal{D}_x$ and $\ell' > \ell > 0$ such that the left isotropy action of $\widehat{h}$ on $U_{\ell'}$ is trivial.

Definition: A minimal Cantor action φ is wild if its maximal equicontinuous factor is wild.

An action which is not wild is said to be stable.

Theorem: Let $\varphi\colon G_i \times \mathfrak{X} \to \mathfrak{X}$ be a stable equicontinuous minimal Cantor action. Then for all suitably small clopen subsets $U \subset \mathfrak{X}$ the discriminant group of the restricted action $G|U$ is independent (up to isomorphism) of the choice of U.

The limiting isomorphism class of the discriminant for a stable action is called the stable discriminant of the action.

Corollary: Let $\varphi\colon G_i \times \mathfrak{X} \to \mathfrak{X}$ be a stable equicontinuous minimal Cantor action. Then the isomorphism class of the discriminant group an invariant of return equivalence.

That is, any minimal action which is return equivalent to φ is equicontinuous and stable, and their stable discriminant groups are isomorphic.

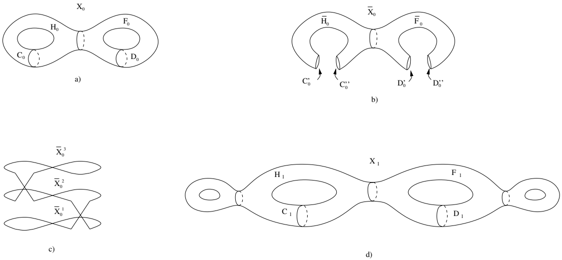
Theorem: (Hurder & Lukina [2017]) Let $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ be an equicontinuous minimal Cantor action, where G is finitely generated. Then the action φ is stable if and only if the action of G_∞ on X_∞ satisfies the LQA property.

Proof. Suppose the action of G_∞ on X_∞ satisfies the LQA property. Choose a metric $d_{\mathfrak{X}}$ on $\mathfrak{X}$ and choose group chain model for the action. Then there exists $\epsilon > 0$ such that if $U \subset \mathfrak{X}$ is a clopen subset with $\mathrm{diam}(U) < \epsilon$ then the restricted $\mathrm{pseudo}{\star}\mathrm{group}$ $G|U$ is QA. Let $\ell > 0$ be sufficiently large so that $\mathrm{diam}(U_\ell) < \epsilon$. Then for all non-trivial $\widehat{h} \in \mathcal{D}_x$ and $\ell' > \ell > 0$ the isotropy action of $\widehat{h}$ is non-trivial on $U_{\ell'}$.

The converse follows in a similar fashion.

Constructions of wild Cantor actions of finitely generated groups.

Example 1: Schori [1966] gave first example of a non-homogeneous weak solenoid. It is obtained by taking repeated 3-fold coverings starting with a closed surface Σ_2 of genus 2.



Proposition: (Hurder & Lukina [2017]) The monodromy action of $G = \pi_1(\Sigma_2, b_0)$ on the Cantor fiber of the solenoid over Σ_2 is wild.

Example 2: Arboreal actions of Galois groups

The analogy between theory of finite coverings and Galois theory of finite field extensions suggests looking for examples of minimal Cantor actions arising from purely arithmetic constructions.

• R.W.K. Odoni [1985] began the study of arboreal representations of absolute Galois groups on the rooted trees formed by the solutions of iterated polynomial equations.

• 2013 survey paper by Rafe Jones gives introduction to this program from the point of view of arithmetic dynamical systems and number theory.

• Lukina [2017] gives concrete examples of polynomials with non-trivial discriminant invariants for their arboreal actions, and also describes some wild arboreal actions.

Example 3: Wild actions of arithmetic lattices. Lubotzky [1993] showed that the profinite completions of higher rank arithmetic lattices contain arbitrary products of finite torsion groups.

$\mathbf{SL}_N(\mathbb{Z}) = N \times N$ matrices with integer entries and determinant 1

$\widehat{\mathbf{SL}_N(\mathbb{Z})}$ profinite completion of $\mathbf{SL}_N(\mathbb{Z})$

$\mathcal{P}$ = set of primes

$$\widehat{\mathbf{SL}_N(\mathbb{Z})} \equiv \varprojlim \mathbf{SL}_N(\mathbb{Z}/M\mathbb{Z}) \cong \mathbf{SL}_N(\widehat{\mathbb{Z}}) \cong \prod_{p \in \mathcal{P}} \mathbf{SL}_N(\widehat{\mathbb{Z}}_p) \ , \quad (3)$$

Let $G \subset \mathbf{SL}_N(\mathbb{Z})$ be a finite-index, torsion free subgroup.

Then G is finitely generated, and its profinite completion $\widehat{G}$ is a clopen subgroup of $\widehat{\mathbf{SL}_N(\mathbb{Z})}$, hence there is a cofinite $\mathcal{P}' \subset \mathcal{P}$, with

$$\prod_{p \in \mathcal{P}'} \mathbf{SL}_N(\widehat{\mathbb{Z}}_p) \subset \widehat{G} \subset \prod_{p \in \mathcal{P}} \mathbf{SL}_N(\widehat{\mathbb{Z}}_p)$$

Set $\widehat{H} = \prod_{p \in \mathcal{P}'} \mathbf{SL}_N(\mathbb{Z}/p\mathbb{Z})$. Then there is a homomorphism with dense image $\alpha \colon G \to \widehat{H}$. For each $p \in \mathcal{P}'$, choose $D_p \subset \mathbf{SL}_N(\mathbb{Z}/p\mathbb{Z})$ with trivial normal core. Set $\mathcal{D} = \prod_{p \in \mathcal{P}'} D_p$.

Theorem: (Hurder & Lukina [2017]) For a closed subgroup $\mathcal{D} \subset \widehat{H}$ as above, the induced action $\varphi_{\alpha,\mathcal{D}}$ of G on $\widehat{H}/\mathcal{D}$ by α satisfies:

- The action $\varphi_{\alpha,\mathcal{D}}$ is minimal and equicontinuous;
- The action $\varphi_{\alpha,\mathcal{D}}$ is wild for suitable choices of $\mathcal{D}$;
- The actions $\varphi_{\alpha,\mathcal{D}}$ for uncountably many such choices of $\mathcal{D}$ are not return equivalent.

Conclusion: The methods for constructing wild actions are not systematically understood, yet they seem to be an interesting class of minimal Cantor actions worthy of further study.

Finally, we give a "geometric" interpretation of the notion of a wild action, as the existence of "hidden symmetries" in the periodic tilings associated to periodic semi-cocycles over the action.

Let $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ be an equicontinuous minimal Cantor action, where G is finitely generated. Let $x \in \mathfrak{X}$, and $(\mathfrak{X}, G, \varphi)$ be conjugate to the odometer (X_∞, G, Φ) associated to a properly descending chain of subgroups $\{G_\ell \mid \ell \geq 0\}$ of finite index in G, with basepoint x.

For $\ell \geq 0$, recall that $U_\ell \subset \mathfrak{X}$ is the clopen subset satisfying

$$G_\ell = \{g \in G \mid \varphi(g)(U_\ell) \cap U_\ell \neq \emptyset\} \ .$$

In particular, $G_0 = G$ and $U_0 = \mathfrak{X}$.

We next construct a periodic semi-cocycle for a group acting cocompactly on the Poincaré disc $\mathbb{D}^2$ of constant curvature -1.

Let $\{g_1, \ldots, g_k\} \subset G$ be a generating set. Let Σ_k be a closed surface of genus k with a Riemannian metric of constant negative curvature -1. Let $\pi \colon \widetilde{\Sigma}_k \to \Sigma_k$ be the universal covering of Σ_k then $\widetilde{\Sigma}_k$ is isometric to the Poincaré disk $\mathbb{D}^2$, and the fundamental group $\Gamma_k \equiv \pi_1(\Sigma_k, b_0)$ acts on $\mathbb{D}^2$ on the right by deck transformations, which are isometries.

Let $\tau_k \colon \Gamma_k \to \star_k \mathbb{Z}$ be a surjective map onto the free group on k generators. Let $\xi_k \colon \star_k \mathbb{Z} \to G$ be the surjective map defined by the choice of generating set for G, then the composition $\sigma_k \equiv \xi_k \circ \tau_k \colon \Gamma_k \to G$ is a surjection.

For $\ell \geq 0$ set $\Gamma_{k,\ell} = \sigma_k^{-1}(G_\ell)$ which has finite index in Γ_k.

Let $M_{k,\ell} \to \Sigma_k$ be the finite covering associated to $\Gamma_{k,\ell}$ so $\pi_1(M, b) \cong \Gamma_{k,\ell}$, which is again covered by $\mathbb{D}^2 = \widetilde{\Sigma}_k$. Choose a convex fundamental domain $T_{k,\ell} \subset \mathbb{D}^2$ for this covering. Then $\mathbb{D}^2$ has a periodic tiling by $T_{k,\ell}$ invariant under the deck transformations by elements of $\Gamma_{k,\ell}$.

The covering $M_{k,\ell} \to \Sigma_k$ is the tiling space for the periodic tiling $\mathcal{T}_{k,\ell}$ of $\mathbb{D}^2$ associated to the tile $T_{k,\ell}$ and the semi-cocycle

$$f_{k,\ell} \colon \Gamma_k \to X_\ell = G/G_\ell$$

defined by composing σ_k with the quotient map.

As $\ell \to \infty$, the sequence of coverings $M_{k,\ell} \to \Sigma_k$ converge to the weak solenoid $\mathcal{S}_\varphi \to \Sigma_k$ defined by the suspension of the induced action $\varphi \circ \sigma_k \colon \Gamma_k \times \mathfrak{X} \to \mathfrak{X}$:

$$\mathcal{S}_\varphi \equiv (\mathbb{D}^2 \times \mathfrak{X})/(w \cdot \gamma^{-1}, x) \sim (w, \varphi \circ \sigma_k(\gamma)(x)) \ .$$

The automorphism group of $\mathcal{S}_\varphi$ is typically trivial without additional assumptions on G, such as $G = \mathbb{Z}^n$.

Assume that the action φ is wild. Then there exists $\ell' > \ell$, and $\widehat{h} \in \mathcal{D}_x \subset G_\infty$ such that the left isotropy action of $\widehat{h}$ on $U_\ell = \widehat{U}_\ell/\mathcal{D}_x$ is non-trivial, but the isotropy action on $U_{\ell'} = \widehat{U}_{\ell'}/\mathcal{D}_x$ is trivial.

Let $\pi\colon M_{k,\ell'} \to M_{k,\ell}$ be the covering associated to $\Gamma_{k,\ell'} \subset \Gamma_{k,\ell}$ whose fiber is the coset space $X_{\ell,\ell'} = G_\ell/G_{\ell'}$.

Choose a convex fundamental domain $T_{k,\ell'} \subset \mathbb{D}^2$ for this second covering, which is a union of contiguous translates of the tile $T_{k,\ell}$.

Claim: There exists $\gamma \in \Gamma_k$ whose covering translation action on $\mathbb{D}^2$ permutes the tiling by $T_{k,\ell}$ but preserves the tiling by $T_{k,\ell'}$.

Remark: The action of γ on the tiling $\mathcal{T}_{k,\ell}$ preserves the super-tile structure given by the tiles in $\mathcal{T}_{k,\ell'}$. This is a "hidden symmetry" of the tiling $\mathcal{T}_{k,\ell}$.

Now recall that $\widehat{h} \in \mathcal{D}_x \subset G_\infty$ is given. Definition (2) of $\mathcal{D}_x$ implies there exists $h_{\ell'} \in G_{\ell'}$ so that

$$\widehat{h} \; mod \; C_{\ell'} = h_{\ell'} \; mod \; C_{\ell'}$$

Let $\overline{h_{\ell'}} \in G_{\ell'}/C_{\ell'}$ be its image in the finite group $G_{\ell'}/C_{\ell'}$. Then by the choice of $\widehat{h}$ the left action of $\overline{h_{\ell'}}$ commutes with the left action of $G_{\ell'}$ on the coset space $X_{\ell,\ell'} = G_\ell/G_{\ell'}$.

Let $\gamma \in \Gamma_{k,\ell'}$ be chosen such that $\sigma_k(\gamma) = h_{\ell'}$.

Remark: A similar analysis can be applied to the non-periodic Toeplitz semi-cocycles over Γ_k as constructed by Krieger [2010] and Cortez and Medynets [2016].

References

A. Clark and S. Hurder, *Homogeneous matchbox manifolds*, **Transactions A.M.S.**, 365:3151–3191, 2013.

M.-I. Cortez and S. Petite, *G-odometers and their almost one-to-one extensions*, **J. London Math. Soc.**, 78:1–20, 2008.

M.-I. Cortez and K. Medynets, *Orbit equivalence rigidity of equicontinuous systems*, **J. London Math. Soc.**, 94:545–556, 2016.

J. Dyer, S. Hurder and O. Lukina, *Molino theory for matchbox manifolds*, **Pacific Journal Math**, Vol. 289:91-151, 2017.

S. Hurder and O. Lukina, *Wild solenoids*, **Transactions A.M.S.**, to appear, 2017.

S. Hurder and O. Lukina, *Orbit equivalence for quasi-analytic Cantor actions*, in preparation, 2017.

F. Krieger, *Toeplitz subshifts and odometers for residually finite groups*, **École de Théorie Ergodique**, Sémin. Congr., 20, Soc. Math. France, Paris, 2010.

A. Lubotzky, *Torsion in profinite completions of torsion-free groups*, **Quart. J. Math.** 44:327–332, 1993.

O. Lukina, *Arboreal Cantor actions*, in preparation, 2017.

R.W.K. Odoni, *The Galois theory of iterates and composites of polynomials*, **Proc. London Math. Soc. (3)**, 51:385-414, 1985.

R. Schori, *Inverse limits and homogeneity*, **Transactions A.M.S.**, 124:533–539, 1966.