Tilings associated to the nearest integer complex continued fractions over imaginary quadratic fields

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Aim

We see tilings associated to the nearest integer complex continued fractions over imaginary quadratic fields.

- What kind of complex continued fractions over imaginary quadratic fields do we consider?
- Why and how do we construct the tilings?
What kind of complex continued fractions over quadratic field?

**FACT:**
The unique factorization property only holds for \( \mathbb{Q}(\sqrt{-d}) \) with \( d = 1, 2, 3, 7, 11, 19, 43, 67, 163 \).

**FACT:**
Even among them, the Euclidean algorithm does not work for \( d = 19, 43, 67, 163 \).

\[ \rightarrow \] In the case of the imaginary quadratic field, the Euclidean algorithm works only for \( d = 1, 2, 3, 7, 11 \).

The case of \( d = 1, 3 \) was studied by A. Hurwitz and the case of \( d = 2, 7, 11 \) by R. B. Lakein.
FACT:
We CAN NOT consider the naive simple complex continued fraction transformation except \( d = 3 \) (Shiokawa, Kaneiwa, Tamura).
But we CAN consider the nearest integer complex continued fraction transformation for \( d = 1, 2, 3, 7, 11 \).

\[ \rightarrow \] We consider the nearest integer complex continued fractions (N.I.C.F) over imaginary quadratic field \( \mathbb{Q}(\sqrt{-d}) \) with \( d = 1, 2, 3, 7, 11 \).
Recall the case of simple continued fraction transformation of $\mathbb{R}$. Define the map $G$ on $I := [0,1)$ by

$$G(x) = \frac{1}{x} - \left[ \frac{1}{x} \right].$$

Then it is known that an absolutely continuous invariant ergodic probability measure is given by

$$\frac{1}{\log 2} \frac{1}{1 + x} dx.$$  

How do we get this invariant measure?

H. Nakada, S. Tanaka and S.Ito gave one answer.
Define

\[ \hat{I} = [0, 1) \times (-\infty, -1], \]

\[ \hat{G}(x, y) = \left( \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, \frac{1}{y} - \left\lfloor \frac{1}{x} \right\rfloor \right) \] for \((x, y) \in \hat{I}.

Then \(\hat{G}\) on \(\hat{I}\) is 1-1 and onto except for a set of Lebesgue measure 0 and

\[ \frac{1}{\log 2} \frac{dxdy}{(x - y)^2} \]

gives an invariant measure for \((\hat{I}, \hat{G})\). Then we get

\[ \frac{1}{\log 2} \frac{1}{1 + x} dx = \left( \int_{-\infty}^{-1} \frac{1}{\log 2} \frac{1}{(x - y)^2} dy \right) dx. \]
How do we determine $\hat{I} = [0, 1) \times (-\infty, -1]$?

Take $(x, -\infty) \in [0, 1) \times [-\infty, -1]$ with

$$x = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots.$$ 

Then,

$$\hat{G}(x, -\infty) = \left( \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \cdots, -a_1 \right)$$

$$\hat{G}^2(x, -\infty) = \left( \frac{1}{a_3} + \frac{1}{a_4} + \cdots, - \left( a_2 + \frac{1}{a_1} \right) \right)$$
By induction, we have

\[ \hat{G}^n(x, -\infty) = \left( \frac{1}{a_{n+1}} + \frac{1}{a_{n+2}} + \cdots, - \left( a_n + \frac{1}{a_{n-1}} + \cdots + \frac{1}{a_1} \right) \right) \]

By the set of the reversed sequences of \( \{a_n(x)\} \), we obtain the domain

\[
\left\{ \left. \left( a_n(x) + \frac{1}{a_{n-1}(x)} + \cdots + \frac{1}{a_1(x)} \right) : \ x \in (0, 1) \right\} \quad n \in \mathbb{N}
\]

= \((-\infty, -1]\).

→ We will see that in the case of the nearest integer complex continued fractions over imaginary quadratic fields, we get tilings on this domain.
N.I.C.F over $\mathbb{Q}(\sqrt{-d})$ for $d = 1, 2, 3, 7, 11$

$d = 1, 2, 3, 7, 11$.

The set of algebraic integers $\mathfrak{o}(\sqrt{-d})$ of $\mathbb{Q}(\sqrt{-d})$ is

$\mathfrak{o}(\sqrt{-d}) =$

\[
\begin{cases}
  \{ n + m\sqrt{-d} : n, m \in \mathbb{Z} \} & \text{if } d = 1, 2 \\
  \{ n \left( \frac{-1+\sqrt{-d}}{2} \right) + m \left( \frac{+1+\sqrt{-d}}{2} \right) : n, m \in \mathbb{Z} \} & \text{if } d = 3, 7, 11
\end{cases}
\]

\[U_d := \]

\[
\begin{cases}
  \{ z = x + yi : -\frac{1}{2} \leq x < \frac{1}{2}, -\frac{\sqrt{d}}{2} \leq y < \frac{\sqrt{d}}{2} \} & \text{if } d = 1, 2 \\
  \{ z = x + yi : -\frac{1}{2} \leq x < \frac{1}{2}, \frac{1}{\sqrt{d}} x - \frac{d+1}{4\sqrt{d}} \leq y < \frac{1}{\sqrt{d}} x + \frac{d+1}{4\sqrt{d}}, \\
  \quad -\frac{1}{\sqrt{d}} x - \frac{d+1}{4\sqrt{d}} \leq y < -\frac{1}{\sqrt{d}} x + \frac{d+1}{4\sqrt{d}} \} & \text{if } d = 3, 7, 11
\end{cases}
\]

(\text{Rectangle})

(\text{Hexagon})
Let us define $T_d : U_d \to U_d$ by

$$T_d(z) := \begin{cases} \frac{1}{z} - \lfloor \frac{1}{z} \rfloor_d & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases},$$

where $[w]_d = a \in o(\sqrt{-d})$ if $w \in a + U_d$.

$$a_n(z) = a_{d,n}(z) = \frac{1}{T^n_d(z)}$$

if $T^{n-1}_d(z) \neq 0$ and $a_n(z) = 0$ if $z = 0$.

Then we get the continued fraction expansion of $z \in U_d$:

$$z = \frac{1}{a_1(z)} + \frac{1}{a_2(z)} + \cdots + \frac{1}{a_n(z)} + \cdots.$$
$d = 1$

$U_d$ and $T_d(U_d)$
$d = 3$

$U_d$ and $T_d(U_d)$
In the case of $d = 1$ (Hurwitz C.F)

Fig. The partition of $U_d$
Fig.: The domains by $T_d\langle \alpha \rangle$
In the case of simple continued fraction of real number

→ $a_n$ (resp. $a_{n+1}$) is not restricted from $a_{n+1}$ (resp. $a_n$).

In the case of Hurwitz continued fraction of complex number

→ $a_n$ (resp. $a_{n+1}$) is restricted from $a_{n+1}$ (resp. $a_n$).

→ We decompose $U_d$ and get the following partition $\{V_k\}$ which is a Markov partition of $T_d$:

$V_1 = \{z \in U : |z + i| > 1, |z - i| > 1, Re z > 0\}$

$V_2 = \{z \in U : |z - 1| < 1, |z - i| < 1, |z - (1 + i)| > 1\}$

$V_3 = \{z \in U : |z - (1 + i)| < 1\}$

...  

$V_{12}$
Fig. $V_1$, $V_2$, $V_3$ and the partition of $U_d$ for $d = 1$
Fig. 3: The partition of $U_d$ for $d = 2, 3$
Construction of the domain by reversed sequence of \( \{a_n\} \)

Computer experience by Shunji ITO for \( d=1 \) (Kokyuroku 496 (1983).)

Fig.: \( v^* = \left\{ -\left( a_n(z) + \frac{1}{a_{n-1}(z)} + \cdots + \frac{1}{a_1(z)} \right) : z \in U, \quad n \in \mathbb{N} \right\} \) and \( X \)
We define

\[ V_k^* = \bigcup_{n=1}^{\infty} \left\{ -\left( a_n(z) + \frac{1}{|a_{n-1}(z)|} + \cdots + \frac{1}{|a_1(z)|} \right) : \begin{array}{l} z \in U, \\
T^n(z) \in V_k \end{array} \right\} \]

\[ X_k = \left\{ \frac{1}{w} : w \in V_k^* \right\} \]

for \( 1 \leq k \leq 12 \).
Fig.: $V_{1}^{*}$ and $X_{1}$ (Gremlin)
Fig.: $V_2^*$ and $X_2$ (Turtle +)
Fig.: $V^*_3$ and $X_3$ (Turtle)
We put

\[ \hat{U} = \bigcup_{k=1}^{12} V_k \times V_k^* \]

and define

\[ \hat{T}(z, w) = \left( \frac{1}{z} - a, \frac{1}{w} - a \right) = \left( \frac{-ai z + i}{iz}, \frac{-ai w + i}{iw} \right) \]

for \((z, w) \in \hat{U}\) where \(a = [1/z]\).

We define a measure \(\hat{\mu}\) on \(\mathbb{C} \times \mathbb{C}\) as follows

\[ d\hat{\mu} = \frac{dx_1 dx_2 dw_1 dw_2}{|z - w|^4} \]

for \((z, w) \in \mathbb{C} \times \mathbb{C}\) with \(z = x_1 + ix_2\) and \(w = w_1 + iw_2\).
Theorem 1

(For $d = 1, 2, 3$)

1. $\hat{U}$ has positive 4-dimensional Lebesgue measure.

2. $\hat{T}$ is 1-1 and onto except for a set of 4-dimensional Lebesgue measure 0.

3. $\hat{\mu}$ is $\hat{T}$-invariant measure.
   i. e. $(\hat{U}, \hat{T}, \hat{\mu})$ is a natural extension of $(U, T, \mu)$ where $\mu$ is an absolutely continuous invariant measure which is unique.
Corollary (For $d = 1, 2, 3$)

The measure $d\mu$ defined by

$$d\mu(z) = \left( \int_{V_k^*} \frac{1}{|z - w|^4} dw_1 dw_2 \right) dx_1 dx_2$$

for $z \in V_k$ is an invariant measure for $T_d$ defined on $U_d$. 
Fig.: The prototiles $X_1$, $X_2$, $X_3$
Fig.: Tiling of $V_1^*$ (The original picture was found by S. Ito.)
Fig. 10: Tiling of $V_2^*$ and $V_3^*$
Theorem 2

(For $d = 1, 2, 3$)

1. $V^*_k$ is tiled by $\{X_k : k = 1, 2, \ldots, 12\}$.
   Concretely for any $1 \leq k_0 \leq 12$,

   $$V^*_{k_0} = \bigcup_{k=1}^{12} \bigcup_{a \in D_{k_0,k}} (X_k - a)$$

   where

   $$D_{k_0,k} = \left\{ a \in \phi(\sqrt{-1}) : \text{there exists } w \in \langle a \rangle \cap V_k \text{ such that } Tw \in V_{k_0} \right\}.$$ 

2. The boundary of $X_k$ is a Jordan curve and has 2-dimensional Lebesgue measure 0.
   $\rightarrow X_k$ is a topological disk.
Reversed C.F. expansion for Hurwitz C. F.

We can define a reversed continued fraction transformation on the domain with a fractal boundary for Hurwitz C. F. Define

\[ V^* = \left\{ -\left( a_n(z) + \frac{1}{a_{n-1}(z)} + \cdots + \frac{1}{a_1(z)} \right) : z \in U, \quad n \in \mathbb{N} \right\} \]

\[ = \bigcup_{k=1}^{12} \bigcup_{a \in D_k} (X_k - a), \]

\[ X = \left\{ \frac{1}{z} : z \in V^* \right\} \]

where \( D_k = \bigcup_{k_0=1}^{12} D_{k_0,k} \).
Fig.: Tiling of $\mathbb{R}^2$ with tilis $X_k$ and $X$
Theorem 3  
(For $d = 1, 2$)

Define $T_d^*$ on $X$ by

$$T_d^*(z) = \frac{1}{z} - \left[ \frac{1}{z} \right]_*$$

where $[z]_* = -a$ if $a \in D_k$ and $z \in X_k - a$.

Then $T_d^*$ is well-defined and it gives a reversed continued fraction expansion for Hurwitz C. F.
Fig.: The periodic tiling by $X_2$
The other cases
In the case of $d = 2$

Fig.: The partition of $U_d$
Fig.: The prototiles
Fig.: The tilings
In the case of $d = 3$

Fig.: The partition of $U_d$ and the tiles $X_1$ and $X_3$
Fig.: The tilings
In the case where the domain is a rectangle for $d = 3$

Fig.: The partition of $U_d$ and some tile
Thank you very much.
The other cases

There are some other nearest type complex continued fractions for $-2, -7$ and $-11$. However, they do not have the best approximation property.

The best approximation property: $p/q$ is a best approximation to $x$ if

$$|q'| < |q| \implies |q'x - p'| > |qx - p|.$$