Tilings associated to the nearest integer complex continued fractions over imaginary quadratic fields

Hiromi EI (Hirosaki Univ.)

Joint work with

Hitoshi NAKADA and Rie NATSUI

2017.12 Marseille

Aim

We see tilings asscociated to the nearest integer complex continued fractions over imaginary quadratic fields.

- What kind of complex continued fractions over imaginary quadratic fields do we consider?
- Why and how do we construct the tilings?

What kind of complex continued fractions over quadratic field?

FACT:

The unique factorization property only holds for $\mathbb{Q}(\sqrt{-d})$ with $d = 1, 2, 3, 7, 11, 19, 43, 67, 163$.

FACT:

Even among them, the Euclidean algorithm does not work for $d = 19, 43, 67, 163$.

$\rightarrow$ In the case of the imaginary quadratic field, the Euclidean algorithm works only for $d = 1, 2, 3, 7, 11$.

The case of $d = 1, 3$ was studied by A.Hurwitz and the case of $d = 2, 7, 11$ by R. B. Lakein.

FACT:

We CAN NOT consider the naive simple complex continued fraction transformation except $d = 3$ (Shiokawa, Kaneiwa, Tamura).

But we CAN consider the nearest integer complex continued fraction transformation for $d = 1, 2, 3, 7, 11$.

$\rightarrow$ We consider the nearest integer complex continued fractions (N.I.C.F) over imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ with $d = 1, 2, 3, 7, 11$.

Why and how do we construct the tilings?

Recall the case of simple continued fraction transformation of $\mathbb{R}$.
Define the map G on $I := [0,1)$ by

$$G(x) = \frac{1}{x} - \left[\frac{1}{x}\right].$$

Then it is known that an absolutely continuous invariant ergodic probability measure is given by

$$\frac{1}{\log 2}\frac{1}{1+x}dx.$$

How do we get this invariant measure?
H. Nakada, S. Tanaka and S.Ito gave one answer.

Define

$$\hat{I} = [0,1) \times (-\infty, -1],$$

$$\hat{G}(x,y) = \left(\frac{1}{x} - \left[\frac{1}{x}\right], \frac{1}{y} - \left[\frac{1}{x}\right]\right) \text{ for } (x,y) \in \hat{I}.$$

Then $\hat{G}$ on $\hat{I}$ is 1-1 and onto except for a set of Lebesgue measure 0 and

$$\frac{1}{\log 2} \frac{dxdy}{(x-y)^2}$$

gives an invariant measure for $(\hat{I}, \hat{G})$. Then we get

$$\frac{1}{\log 2} \frac{1}{1+x} dx = \left(\int_{-\infty}^{-1} \frac{1}{\log 2} \frac{1}{(x-y)^2} dy\right) dx.$$

How do we determine $\hat{I} = [0,1) \times (-\infty, -1]$?

Take $(x, -\infty) \in [0,1) \times [-\infty, -1]$ with

$$x = \frac{1|}{|a_1} + \frac{1|}{|a_2} + \frac{1|}{|a_3} + \cdots .$$

Then,

$$\hat{G}(x, -\infty) = \left(\frac{1|}{|a_2} + \frac{1|}{|a_3} + \frac{1|}{|a_4} + \cdots , -a_1 \right)$$

$$\hat{G}^2(x, -\infty) = \left(\frac{1|}{|a_3} + \frac{1|}{|a_4} + \cdots , -\left(a_2 + \frac{1|}{|a_1} \right) \right)$$

By induction, we have

$$\hat{G}^n(x, -\infty) = \left(\frac{1}{|a_{n+1}} + \frac{1}{|a_{n+2}} + \cdots, -\left(a_n + \frac{1}{|a_{n-1}} + \cdots + \frac{1}{|a_1} \right) \right)$$

By the set of the reversed sequences of $\{a_n(x)\}$,
we obtain the domain

$$\left\{ -\left(a_n(x) + \frac{1}{|a_{n-1}(x)} + \cdots + \frac{1}{|a_1(x)} \right) : \begin{array}{l} x \in (0,1) \\ n \in \mathbb{N} \end{array} \right\}.$$

$$= (-\infty, -1].$$

$\rightarrow$ We will see that in the case of the nearest integer complex continued fractions over imaginary quadratic fields, we get tilings on this domain.

N.I.C.F over $\mathbb{Q}(\sqrt{-d})$ for $d = 1, 2, 3, 7, 11$

$d = 1, 2, 3, 7, 11.$

The set of algebraic integers $\mathfrak{o}(\sqrt{-d})$ of $\mathbb{Q}(\sqrt{-d})$ is

$\mathfrak{o}(\sqrt{-d}) =$

$$\left\{n + m\sqrt{-d} : n, m \in \mathbb{Z}\right\} \qquad \text{if } d = 1, 2$$

$$\left\{n\left(\frac{-1+\sqrt{-d}}{2}\right) + m\left(\frac{+1+\sqrt{-d}}{2}\right) : n, m \in \mathbb{Z}\right\} \qquad \text{if } d = 3, 7, 11$$

$U_d :=$

$$\left\{z = x + yi \ : \ -\frac{1}{2} \leq x < \frac{1}{2},\ -\frac{\sqrt{d}}{2} \leq y < \frac{\sqrt{d}}{2}\right\} \qquad \text{if } d = 1, 2$$

(Rectangle)

$$\left\{z = x + yi \ : \ -\frac{1}{2} \leq x < \frac{1}{2},\ \frac{1}{\sqrt{d}}x - \frac{d+1}{4\sqrt{d}} \leq y < \frac{1}{\sqrt{d}}x + \frac{d+1}{4\sqrt{d}},\right.$$
$$\left. -\frac{1}{\sqrt{d}}x - \frac{d+1}{4\sqrt{d}} \leq y < -\frac{1}{\sqrt{d}}x + \frac{d+1}{4\sqrt{d}}\right\} \qquad \text{if } d = 3, 7, 11$$

(Hexagon)

Let us define $T_d : U_d \to U_d$ by

$$T_d(z) := \begin{cases} \frac{1}{z} - \left[\frac{1}{z}\right]_d & \text{if} \quad z \neq 0 \\ 0 & \text{if} \quad z = 0 \end{cases},$$

where $[w]_d = a \in \mathfrak{o}(\sqrt{-d})$ if $w \in a + U_d$.

$$a_n(z) = a_{d,n}(z) = \left[\frac{1}{T_d^{n-1}(z)}\right]_d$$

if $T_d^{n-1}(z) \neq 0$ and $a_n(z) = 0$ if $z = 0$.

Then we get the continued fraction expansion of $z \in U_d$:

$$z = \frac{1}{|a_1(z)} + \frac{1}{|a_2(z)} + \cdots + \frac{1}{|a_n(z)} + \cdots .$$

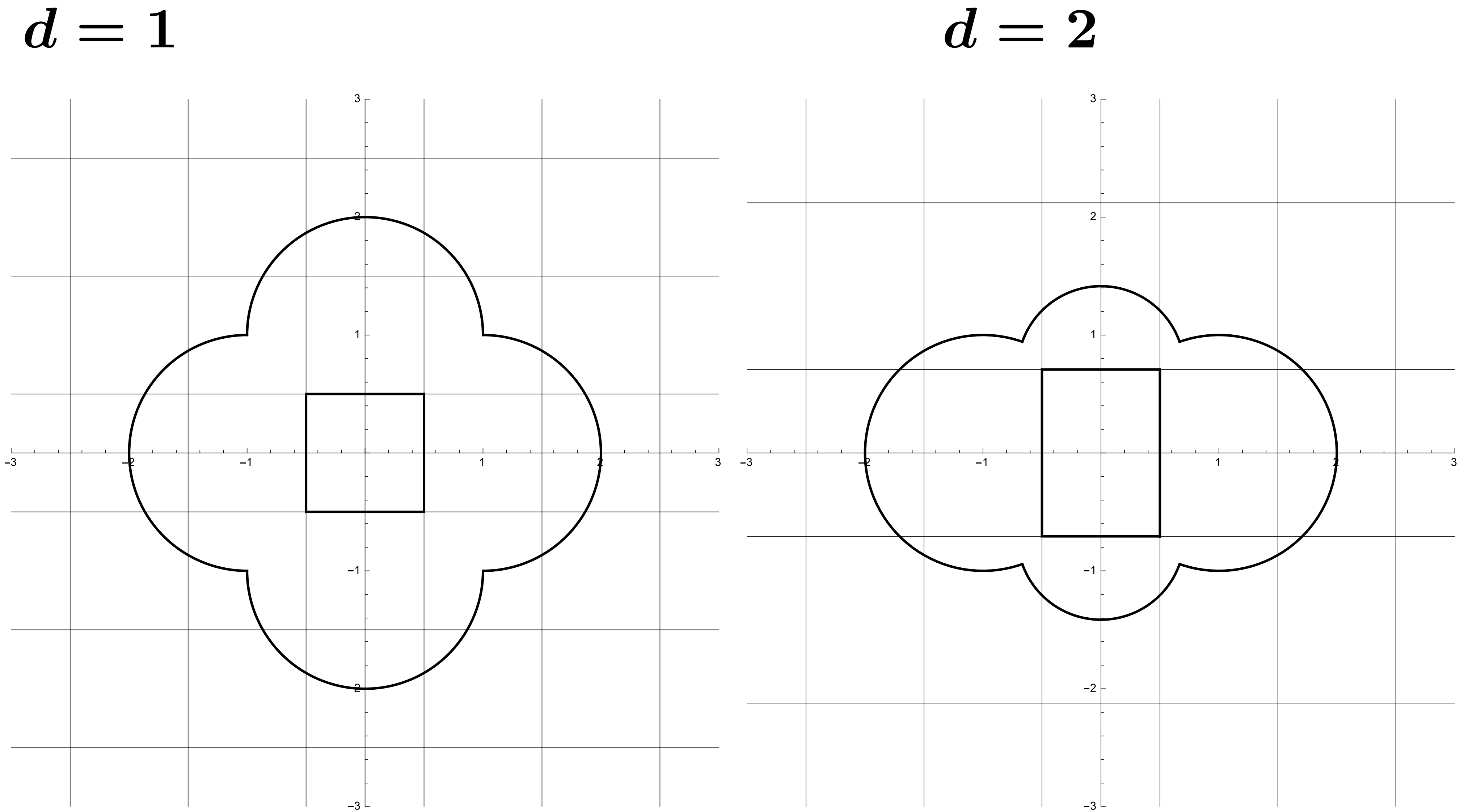


Fig. U_d **and** $T_d(U_d)$

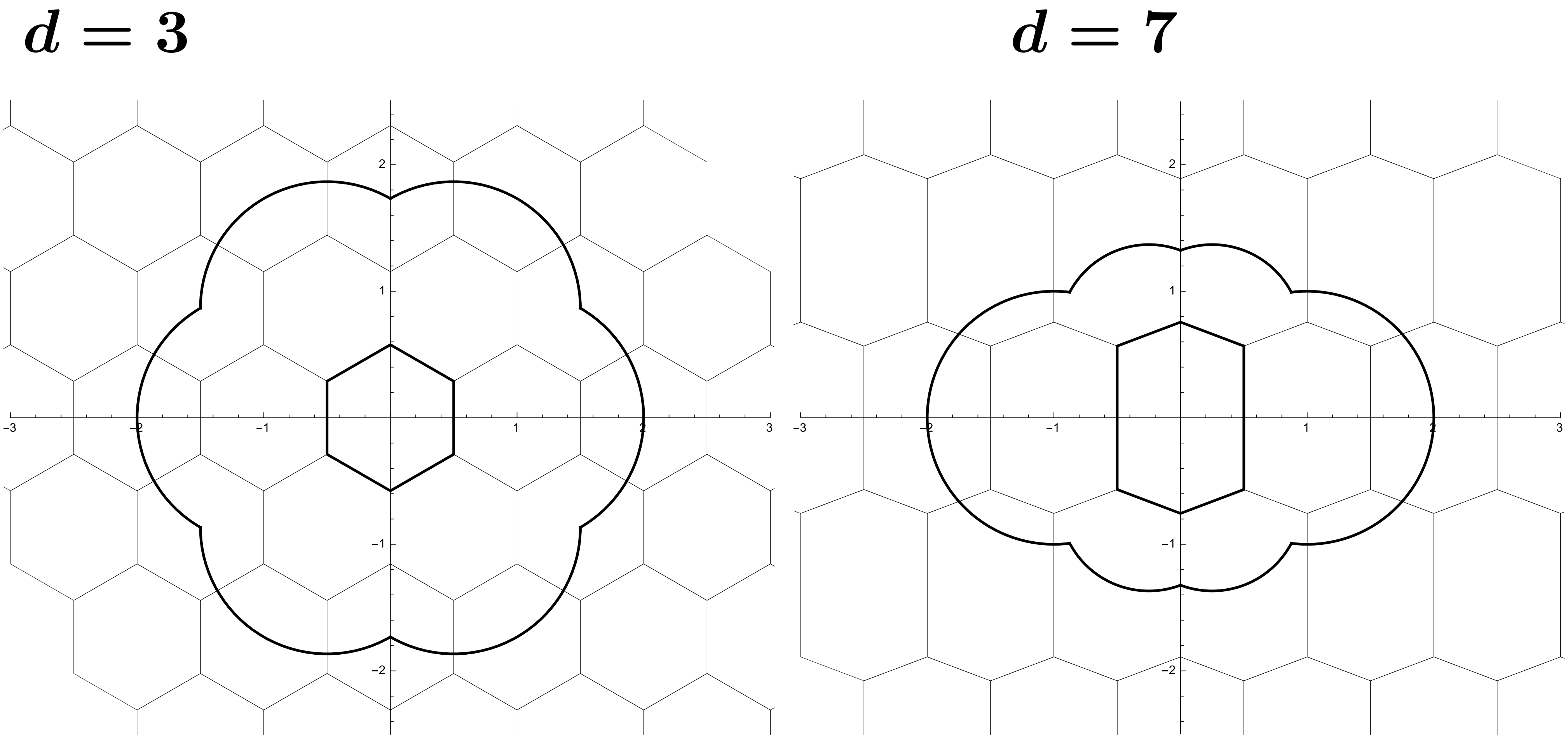
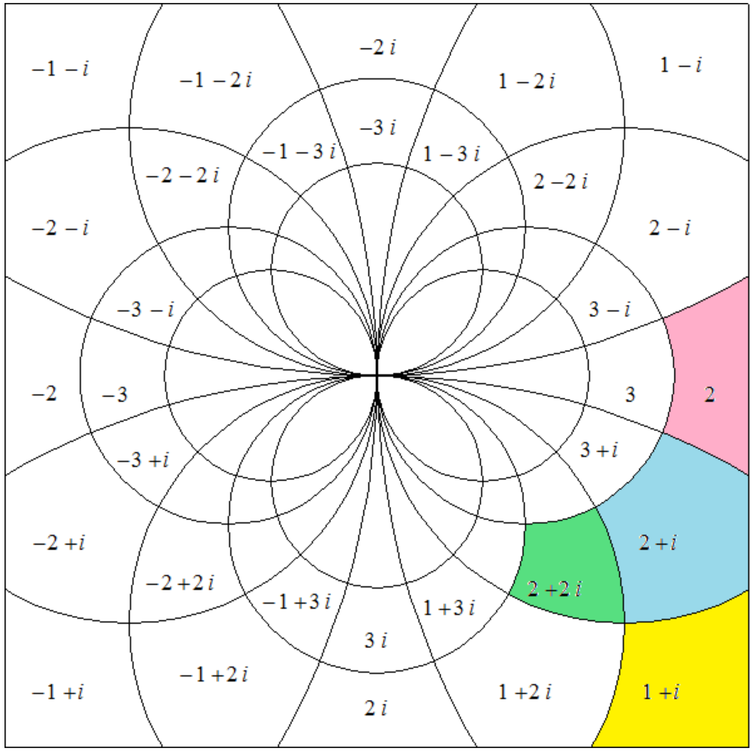
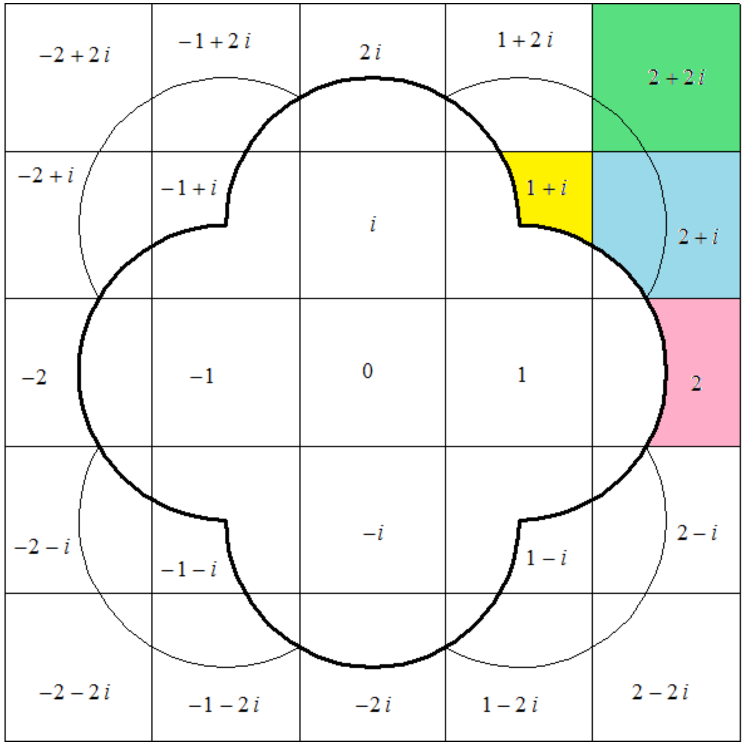


Fig. U_d and $T_d(U_d)$

In the case of $d = 1$ (Hurwitz C.F)





$$\mapsto$$

$$\frac{1}{z}$$

Fig. The partition of U_d

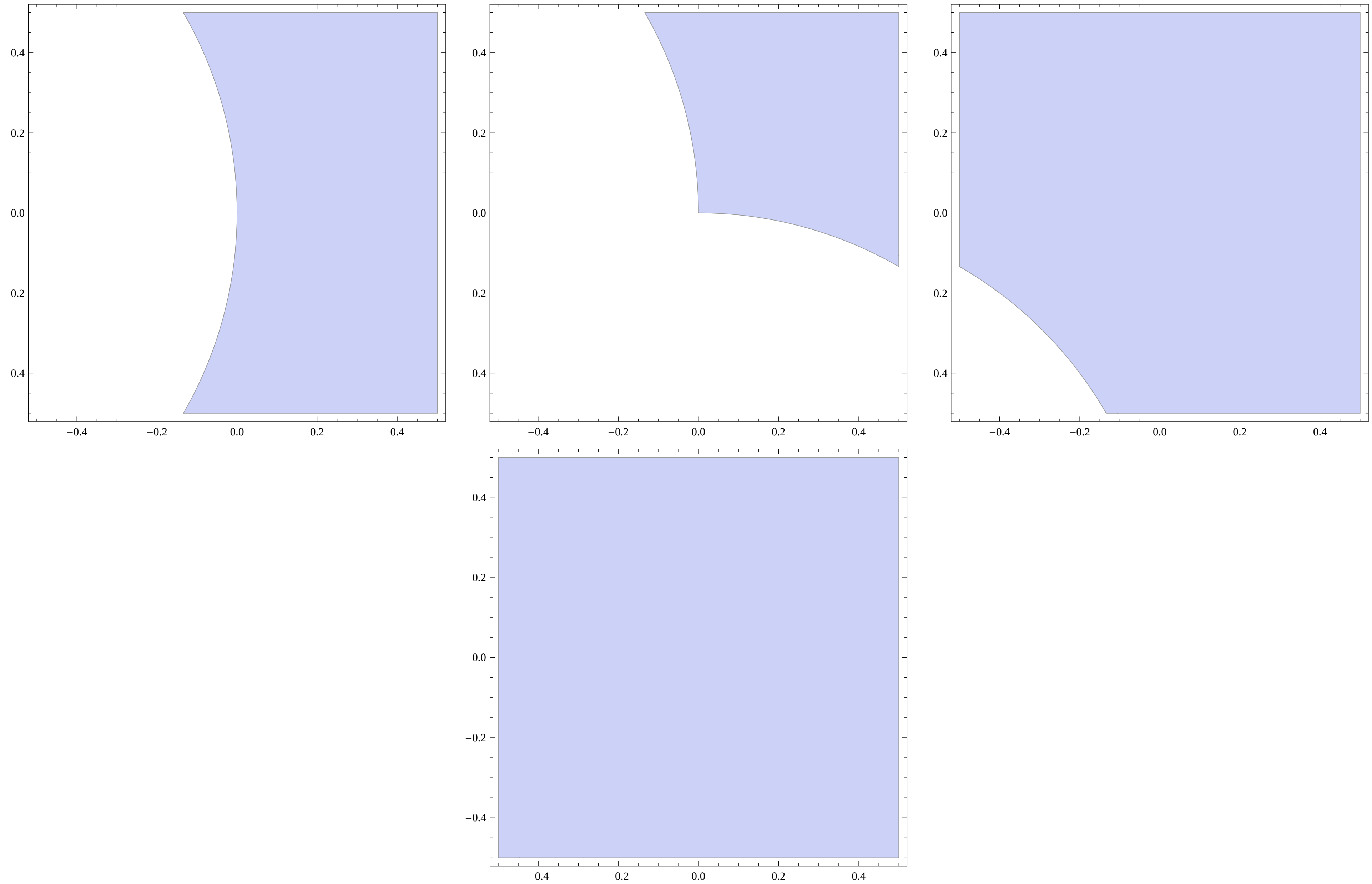


Fig.: The domains by $T_d\langle a\rangle$

In the case of simple continued fraction of real number

$\rightarrow a_n$ (resp. a_{n+1}) is not restricted from a_{n+1} (resp. a_n).

In the case of Hurwitz continued fraction of complex number

$\rightarrow a_n$ (resp. a_{n+1}) is restricted from a_{n+1} (resp. a_n).

$\rightarrow$ We decompose U_d and get the following partition $\{V_k\}$ which is a Markov partition of T_d:

$V_1 = \{z \in U : |z+i| > 1, |z-i| > 1, Re\, z > 0\}$

$V_2 = \{z \in U : |z-1| < 1, |z-i| < 1, |z-(1+i)| > 1\}$

$V_3 = \{z \in U : |z-(1+i)| < 1\}$

$\cdots$

V_{12}

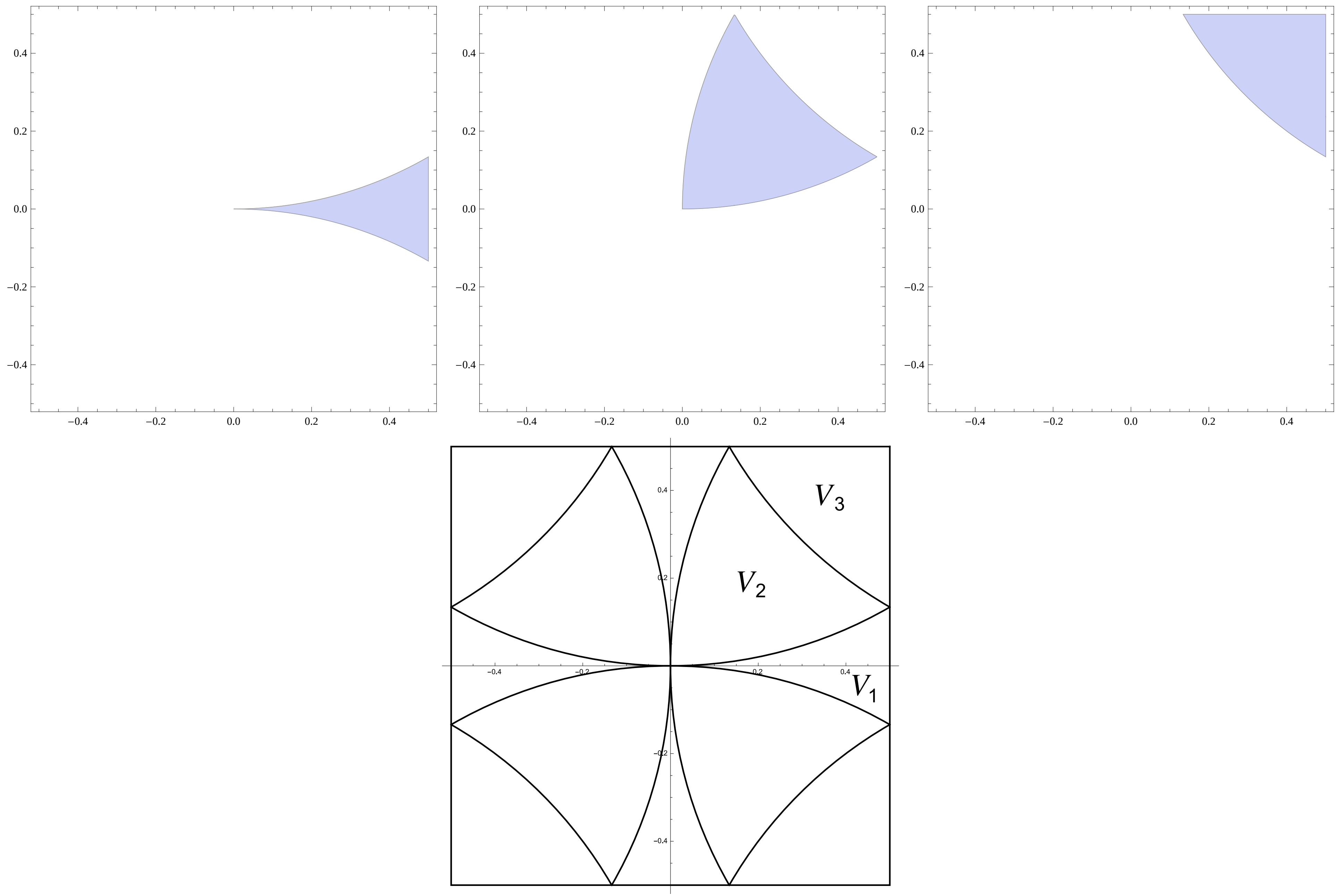


Fig. V_1, V_2, V_3 and the partition of U_d for $d = 1$

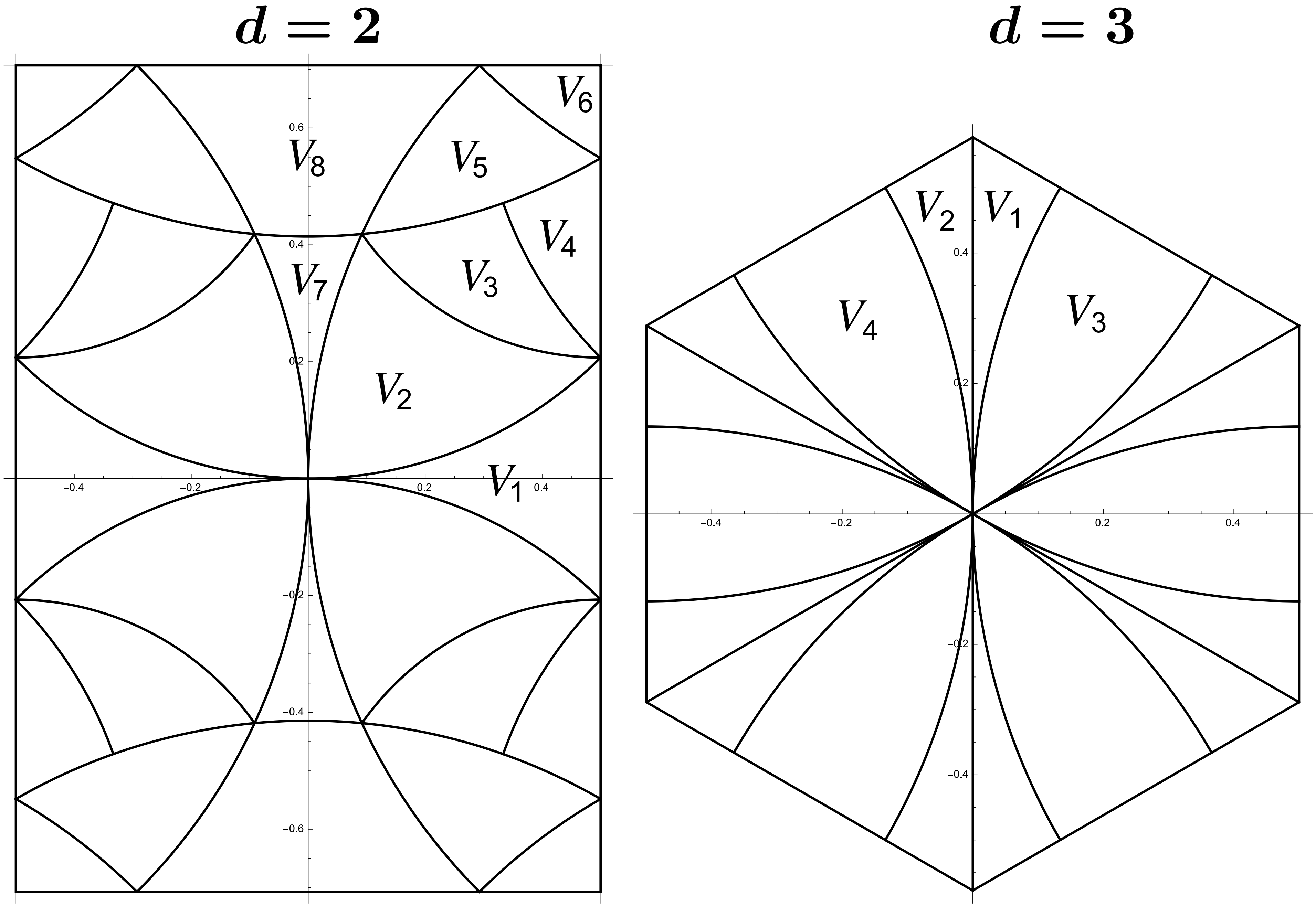
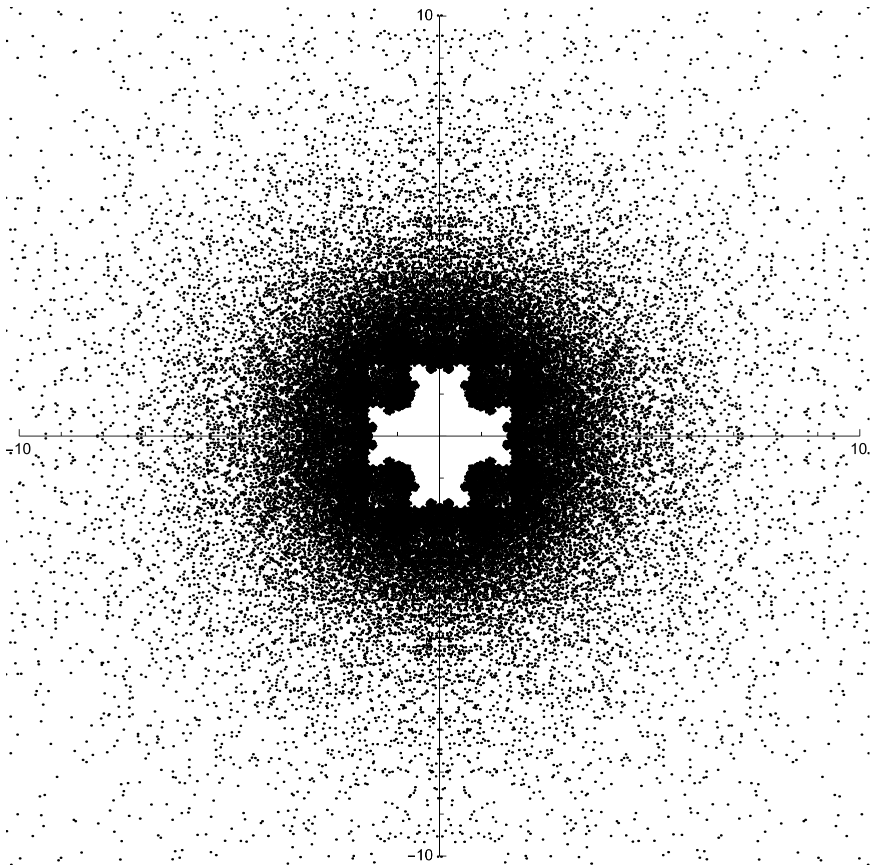


Fig. 3: The partition of U_d for $d = 2, 3$

Construction of the domain by reversed seqence of $\{a_n\}$

Computer experience by Shunji ITO for d=1 (Kokyuroku **496** (1983).)



$\mapsto \dfrac{1}{z}$

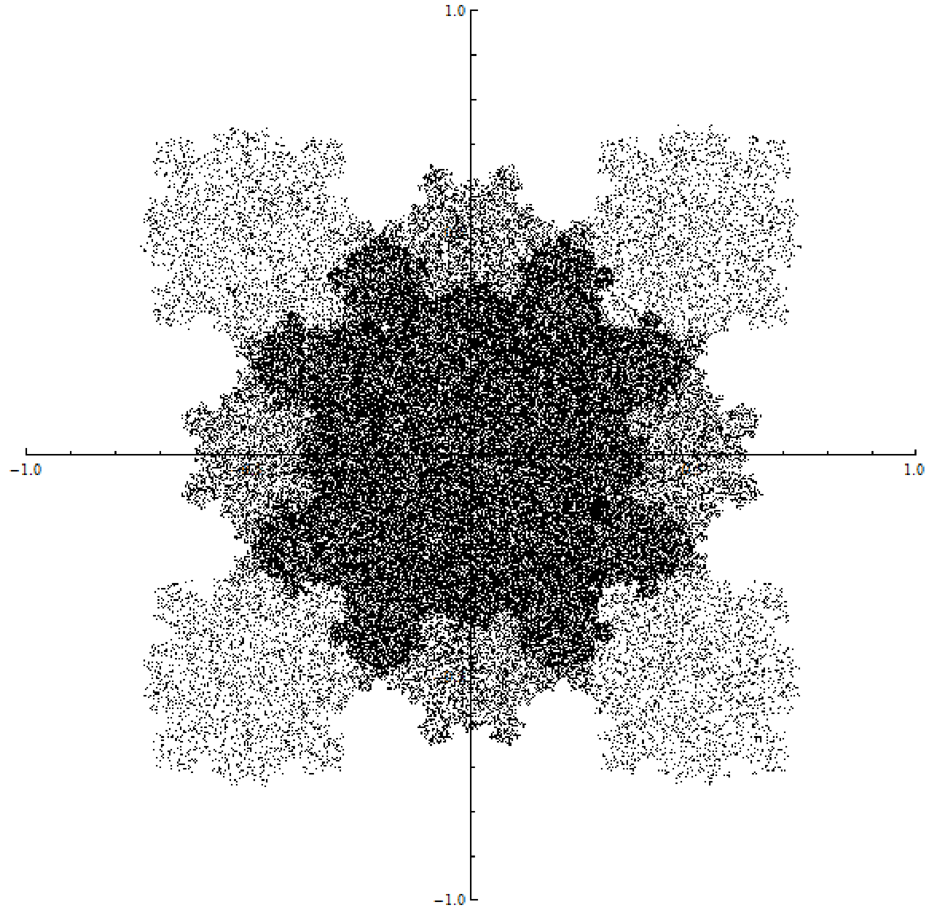


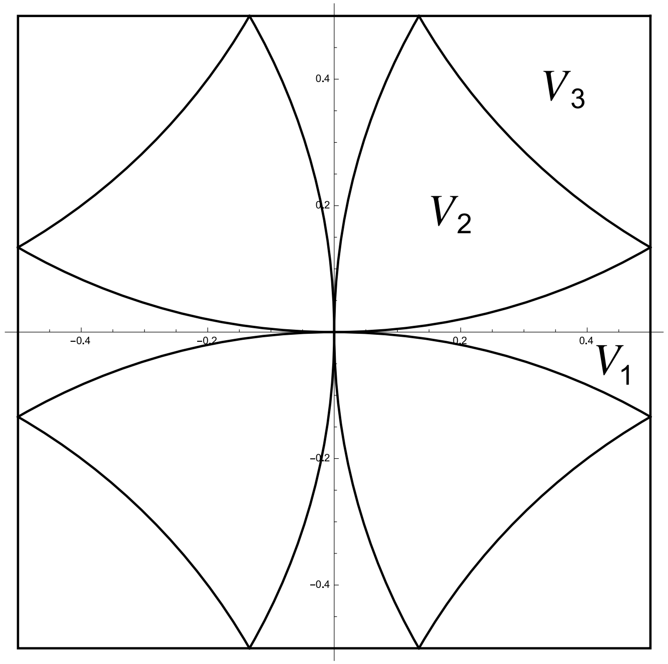
Fig.: $V^* = \overline{\left\{ -\left(a_n(z) + \frac{1}{|a_{n-1}(z)} + \cdots + \frac{1}{|a_1(z)} \right) : \begin{array}{c} z \in U, \\ n \in \mathbb{N} \end{array} \right\}}$ and X

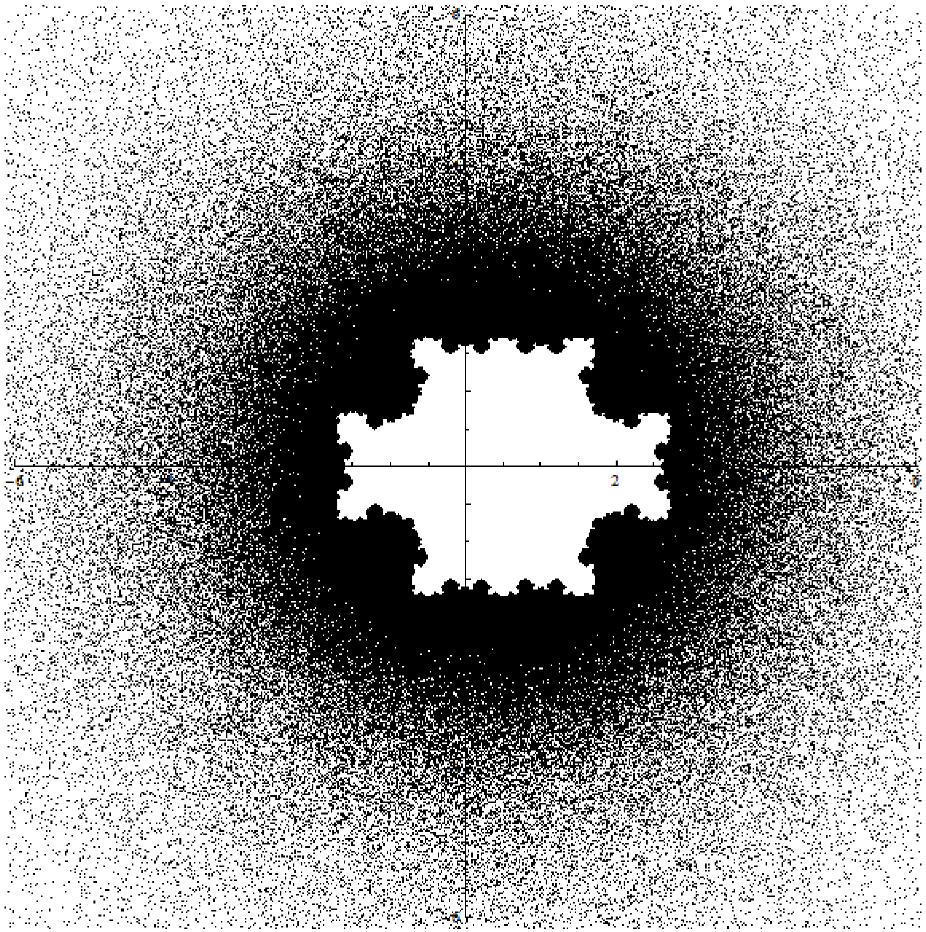
We define

$$V_k^* = \overline{\bigcup_{n=1}^{\infty} \left\{ -\left(a_n(z) + \frac{1}{|a_{n-1}(z)} + \cdots + \frac{1}{|a_1(z)} \right) \; : \; \begin{array}{l} z \in U, \\ T^n(z) \in V_k \end{array} \right\}}$$

$$X_k = \left\{ \frac{1}{w} : \; w \in V_k^* \right\}$$

for $1 \leq k \leq 12$.





$\mapsto \frac{1}{z}$

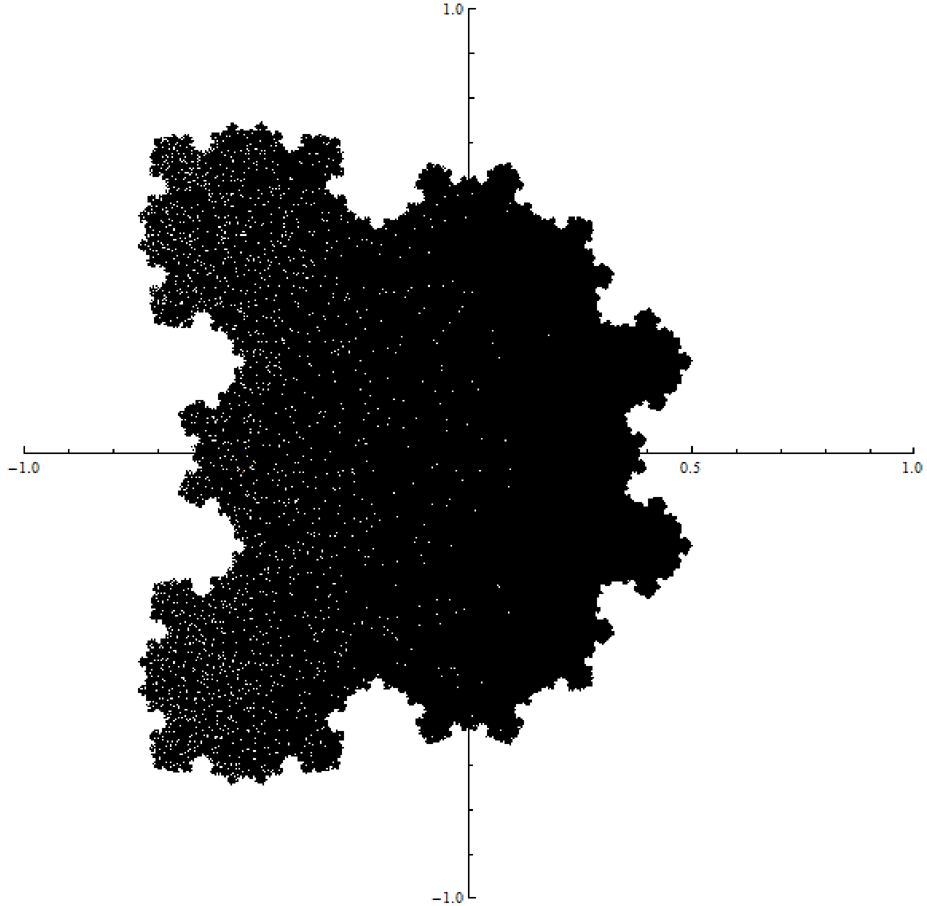
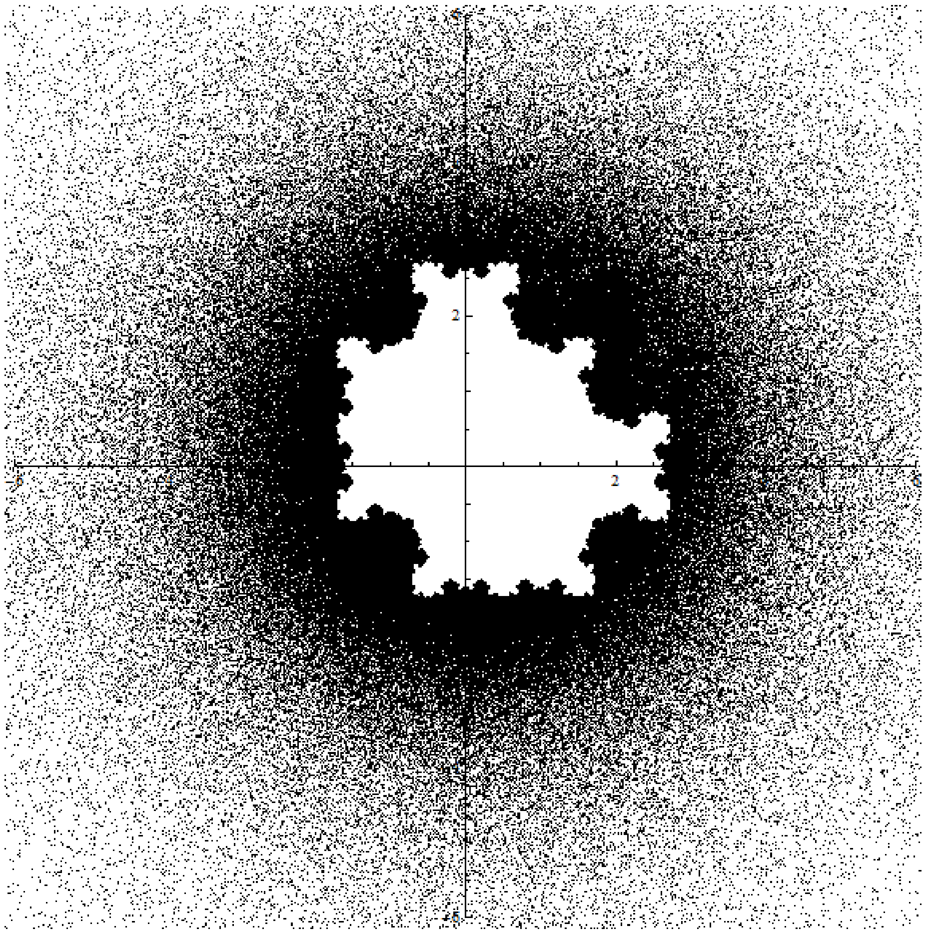


Fig.: V_1^* and X_1 (Gremlin)



$\mapsto \frac{1}{z}$

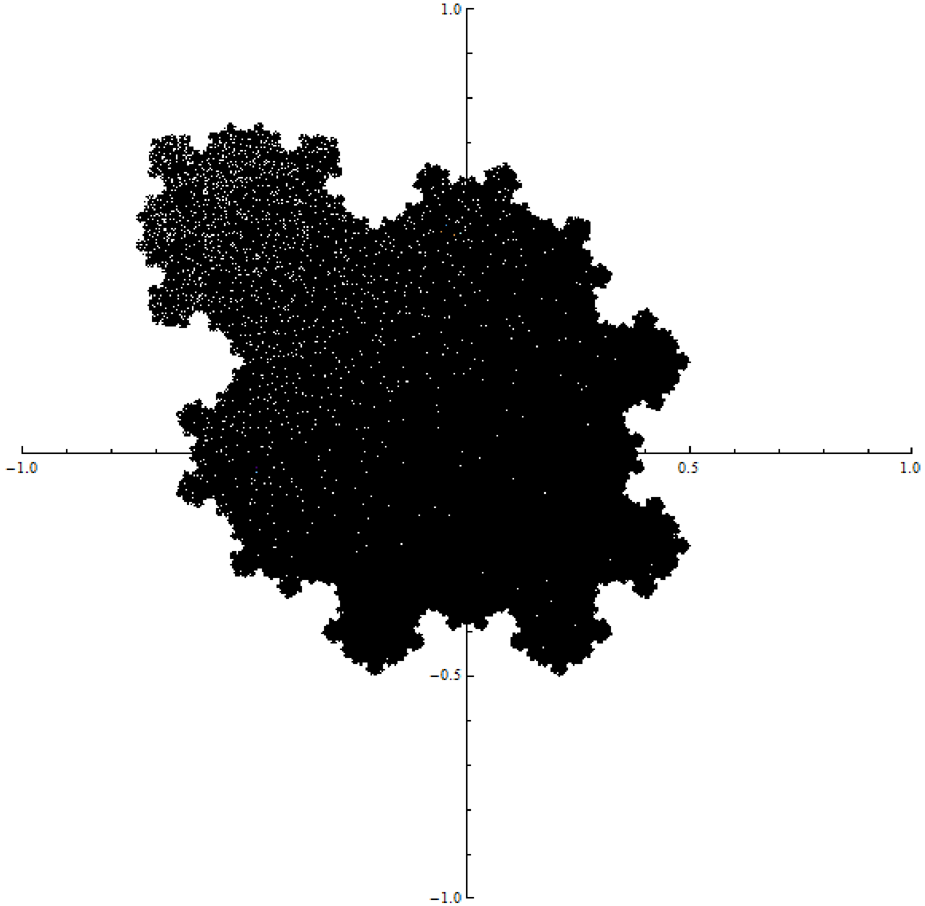
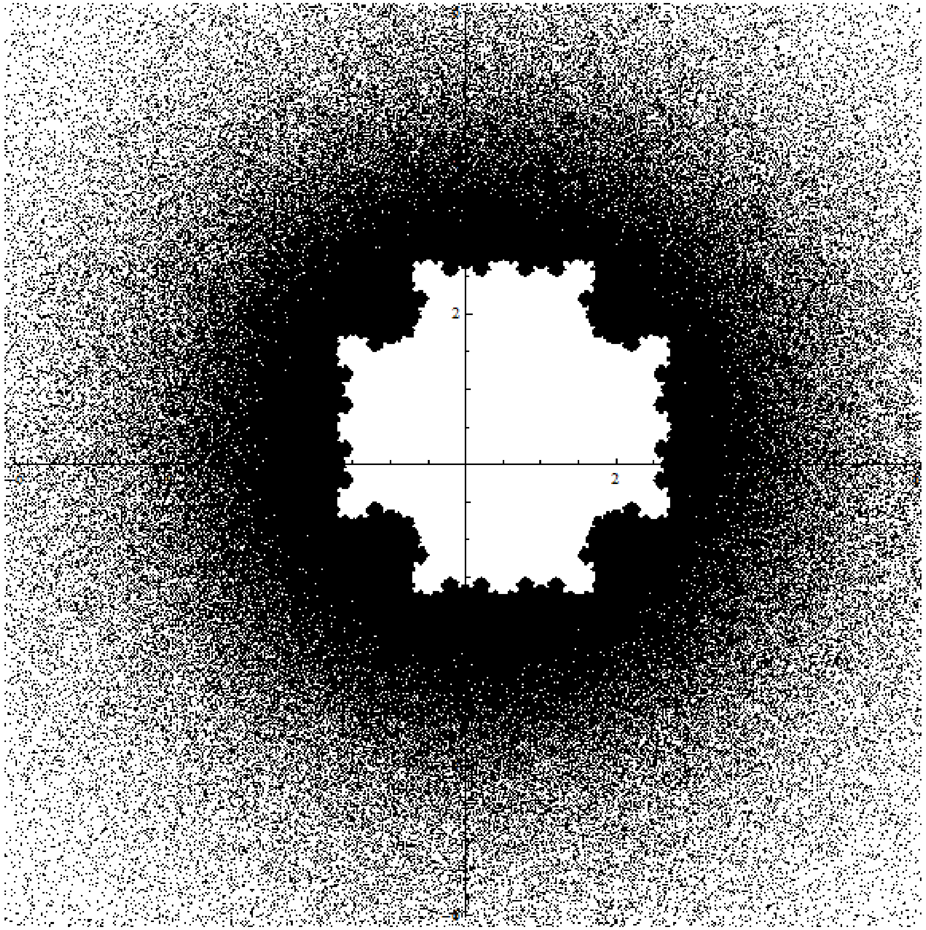


Fig.: V_2^* and X_2 (Turtle +)



$\mapsto \dfrac{1}{z}$

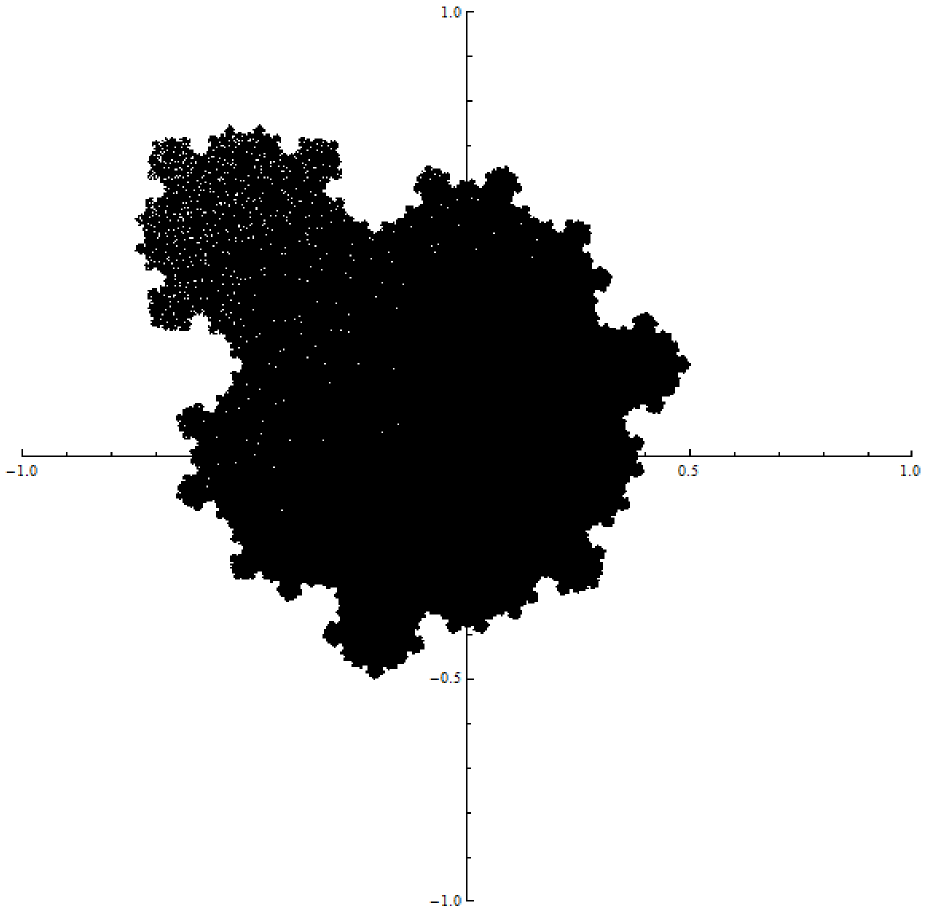


Fig.: V_3^* and X_3 (Turtle)

We put

$$\hat{U} = \bigcup_{k=1}^{12} V_k \times V_k^*$$

and define

$$\hat{T}(z, w) = \left(\frac{1}{z} - a, \ \frac{1}{w} - a\right) = \left(\frac{-aiz + i}{iz}, \ \frac{-aiw + i}{iw}\right)$$

for $(z, w) \in \hat{U}$ where $a = [1/z]$.

We define a measure $\hat{\mu}$ on $\mathbb{C} \times \mathbb{C}$ as follows

$$d\hat{\mu} = \frac{dx_1 dx_2 dw_1 dw_2}{|z - w|^4}$$

for $(z, w) \in \mathbb{C} \times \mathbb{C}$ with $z = x_1 + ix_2$ and $w = w_1 + iw_2$.

Theorem 1 (For $d = 1, 2, 3$)

1. $\hat{U}$ has positive 4-dimensional Lebesgue measure.

2. $\hat{T}$ is 1-1 and onto except for a set of 4-dimensional Lebesgue measure 0.

3. $\hat{\mu}$ is $\hat{T}$-invariant measure.
   i. e. $(\hat{U}, \hat{T}, \hat{\mu})$ is a natural extension of (U, T, μ) where μ is an absolutely continuous invariant measure which is unique.

Corollary (For $d = 1, 2, 3$)

The measure $d\mu$ defined by

$$d\mu(z) = \left(\int_{V_k^*} \frac{1}{|z - w|^4} dw_1 dw_2 \right) dx_1 dx_2$$

for $z \in V_k$ is an inveriant mesure for T_d defined on U_d.

Tilings

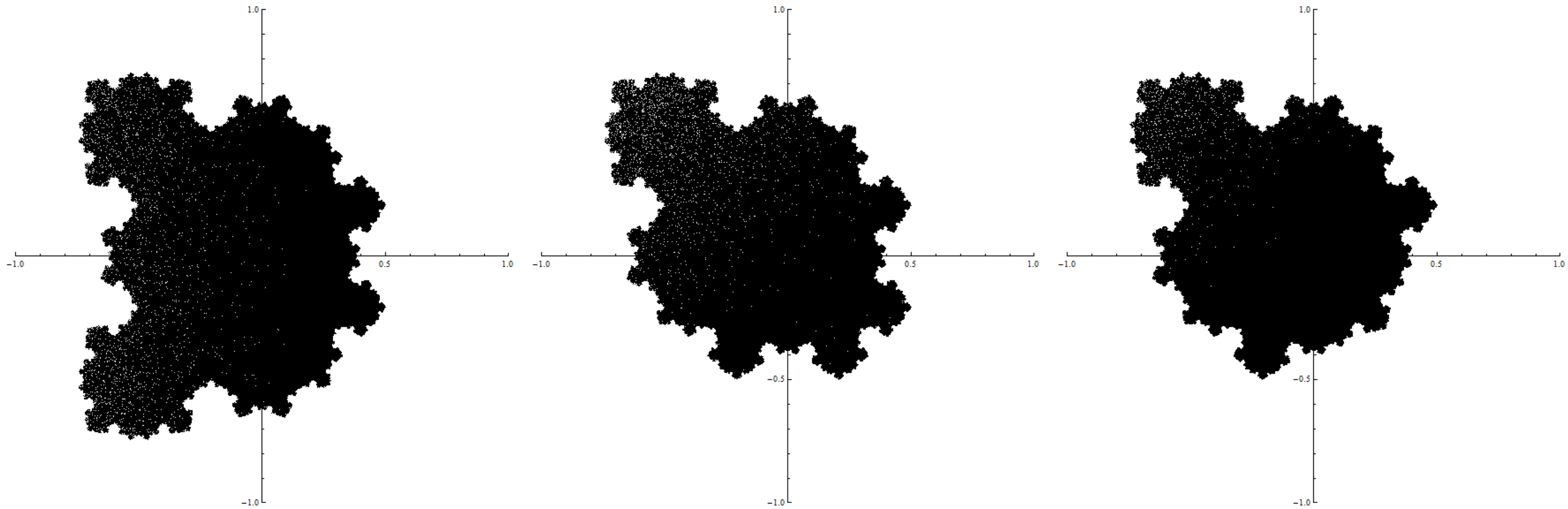


Fig.: The prototiles X_1, X_2, X_3

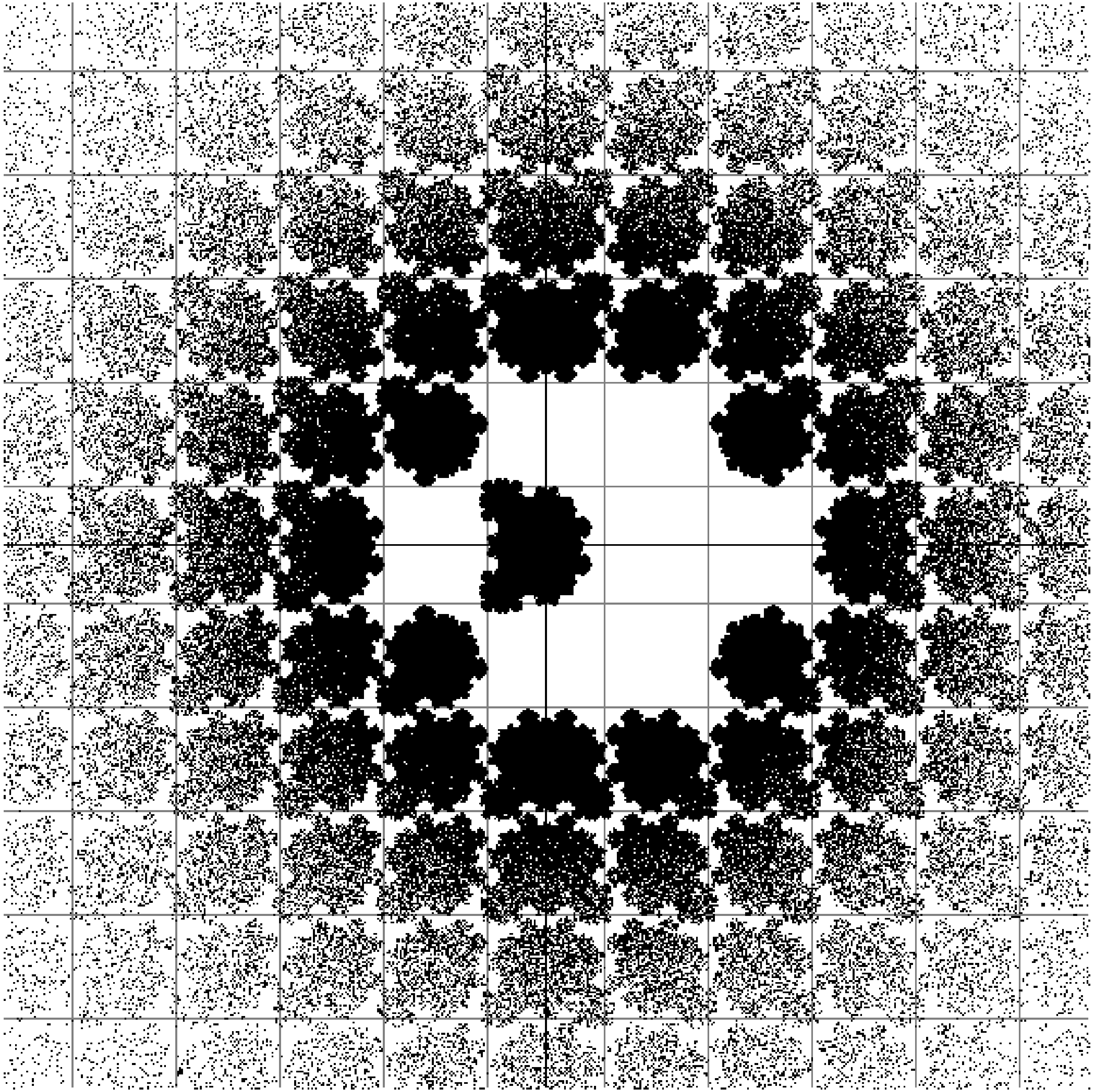


Fig.: Tiling of V_1^* (The original picture was found by S. Ito.)

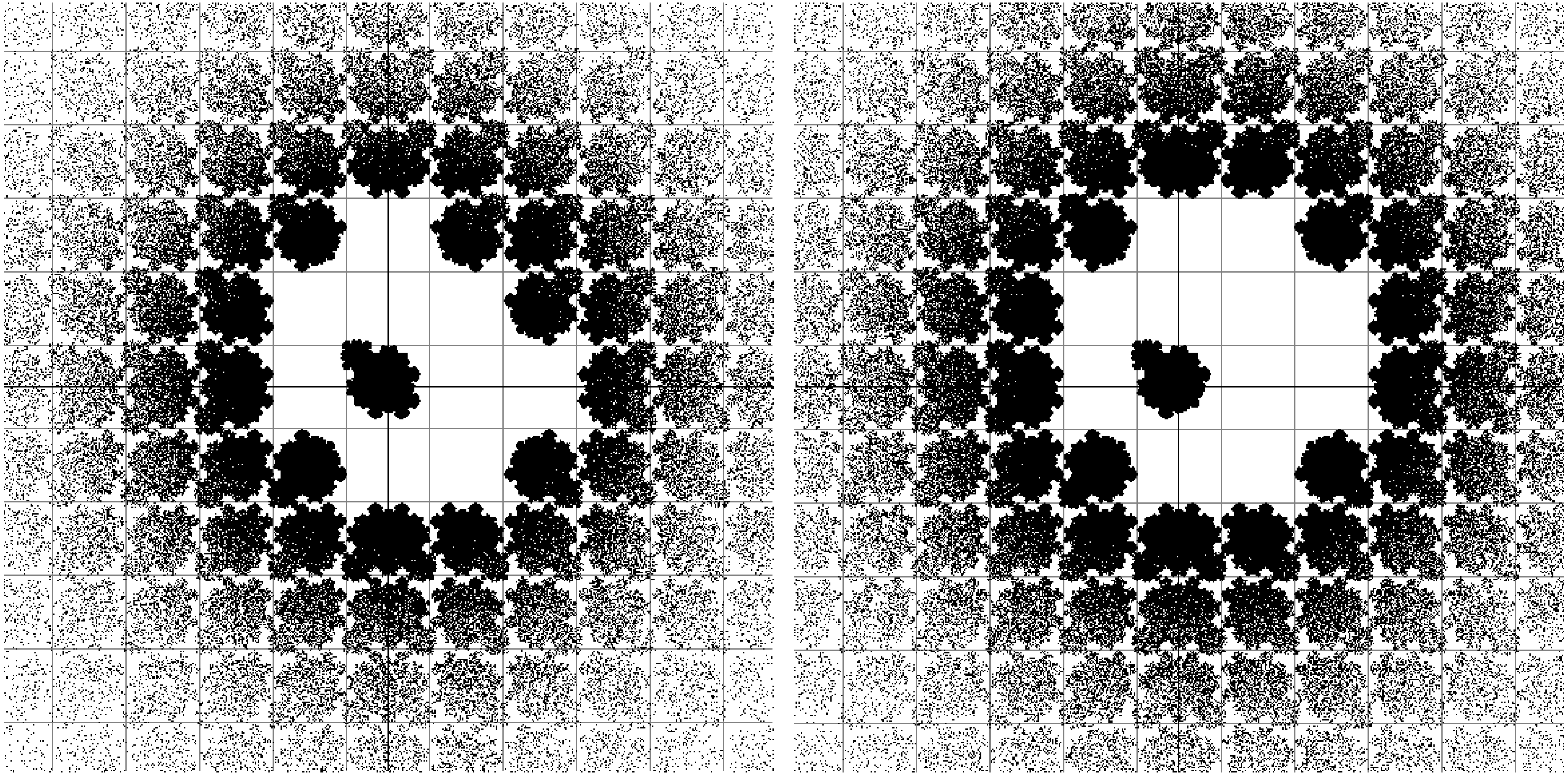


Fig. 10: Tiling of V_2^* and V_3^*

Theorem 2 (For $d = 1, 2, 3$)

1. V_k^* is tiled by $\{X_k : k = 1, 2, \ldots, 12\}$.
   Concretely for any $1 \leq k_0 \leq 12$,

$$V_{k_0}^* = \bigcup_{k=1}^{12} \bigcup_{a \in D_{k_0,k}} (X_k - a)$$

   where

$$D_{k_0,k} = \left\{ a \in \mathfrak{o}(\sqrt{-1}) : \begin{array}{l} \text{there exists } w \in \langle a \rangle \cap V_k \\ \text{such that } Tw \in V_{k_0} \end{array} \right\}.$$

2. The boundary of X_k is a Jordan curve and has 2-dimensional Lebesgue measure 0.
   $\to X_k$ is a topological disk.

Reversed C.F. expansion for Hurwitz C. F.

We can define a reversed continued fraction transformation on the domain with a fractal boundary for Hurwitz C. F. Define

$$\begin{aligned} V^* &= \overline{\left\{ -\left(a_n(z) + \frac{1}{|a_{n-1}(z)} + \cdots + \frac{1}{|a_1(z)} \right) : \begin{array}{l} z \in U, \\ n \in \mathbb{N} \end{array} \right\}} \\ &= \bigcup_{k=1}^{12} \bigcup_{a \in D_k} (X_k - a), \\ X &= \left\{ \frac{1}{z} : z \in V^* \right\} \end{aligned}$$

where $D_k = \bigcup_{k_0=1}^{12} D_{k_0,k}$.

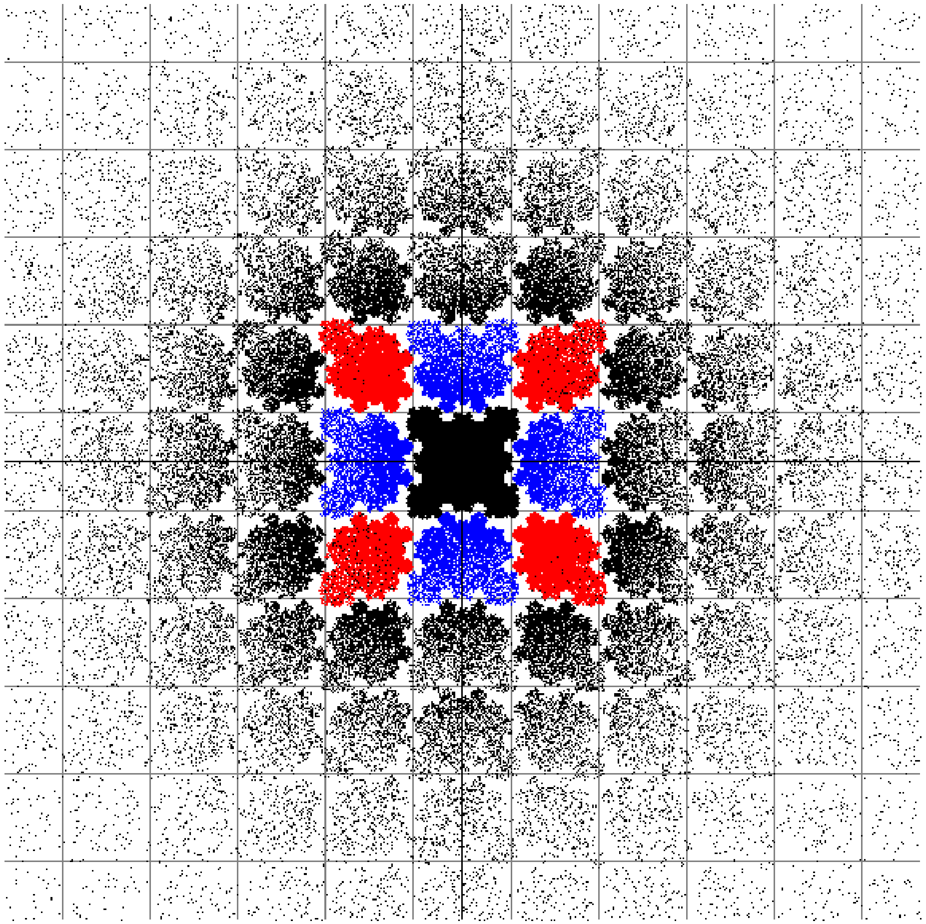


Fig.: Tiling of $\mathbb{R}^2$ with tilis X_k and X

Theorem 3 (For $d = 1, 2$)

Define T_d^* on X by

$$T_d^*(z) = \frac{1}{z} - \left[\frac{1}{z}\right]_*$$

where $[z]_* = -a$ if $a \in D_k$ and $z \in X_k - a$.

Then T_d^* is well-defined and it gives a reversed continued fraction expansion for Hurwitz C. F.

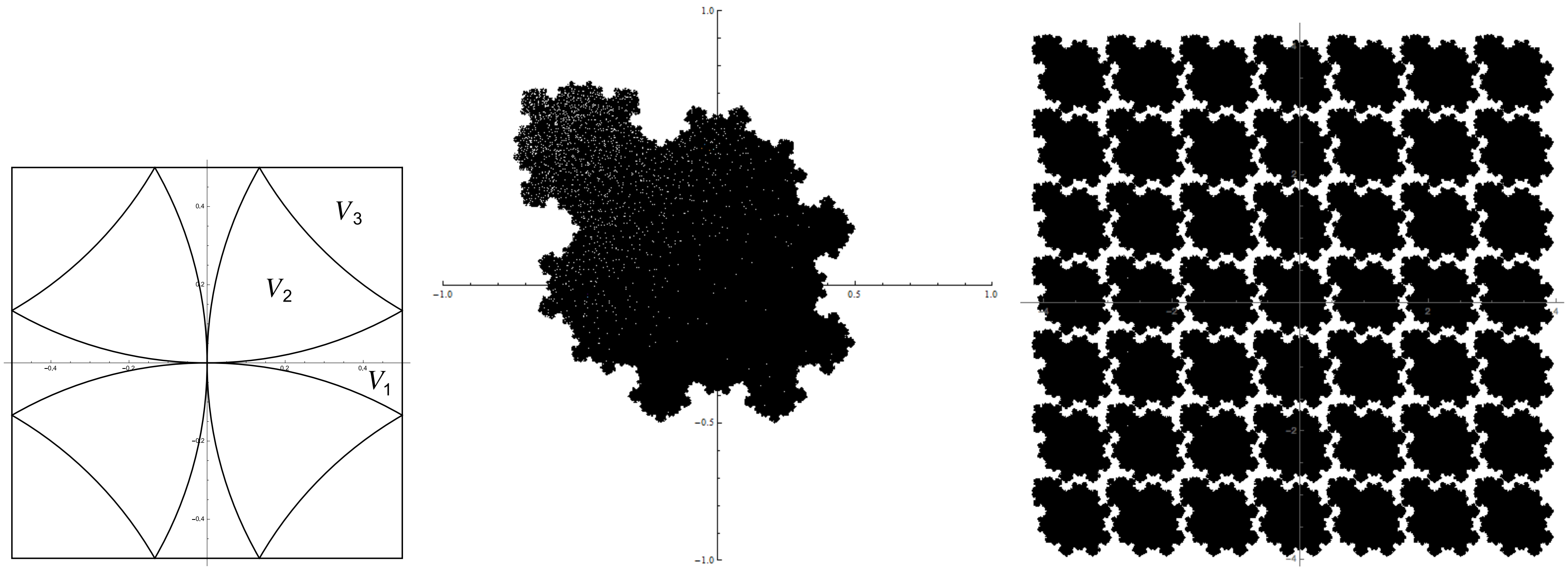


Fig.: The periodic tiling by X_2

The other cases

In the case of $d = 2$

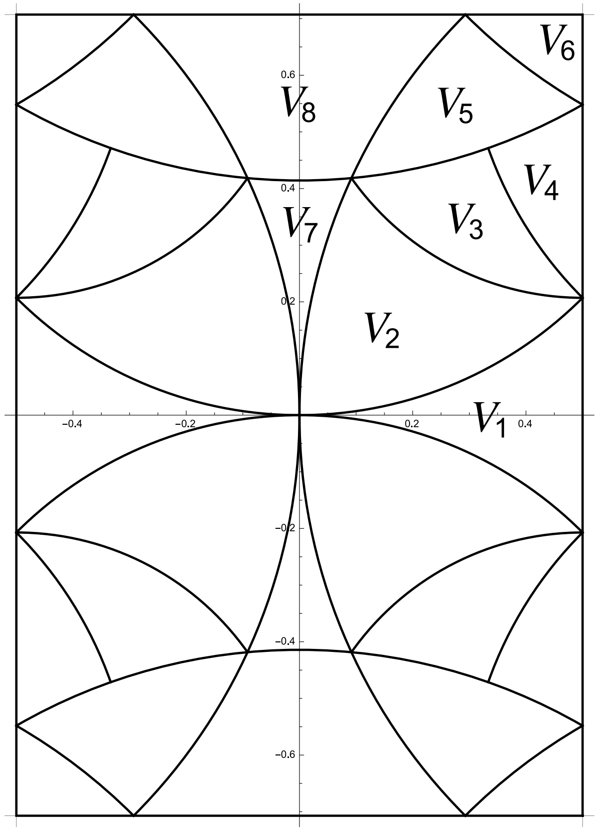


Fig.: The partition of U_d

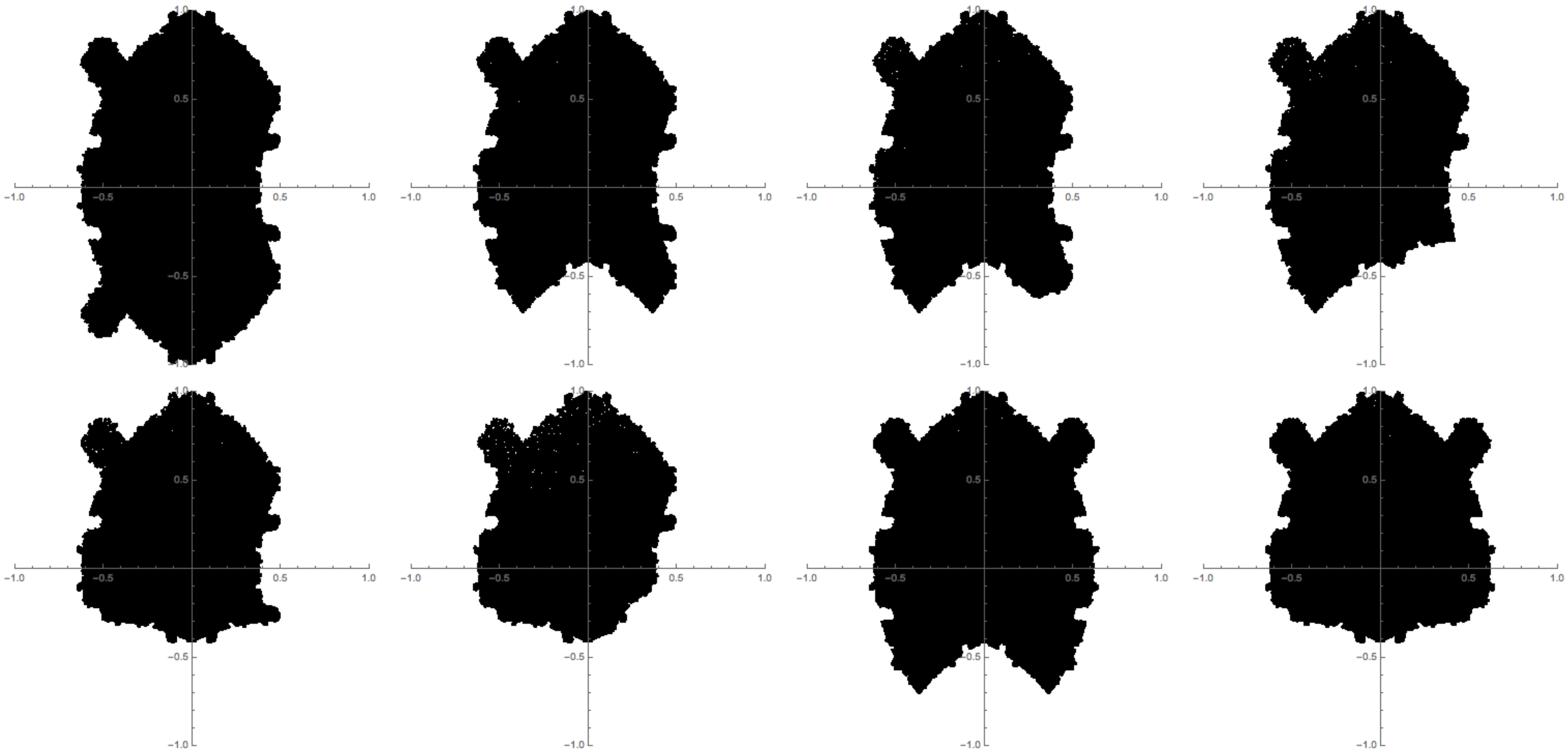


Fig.: The prototiles

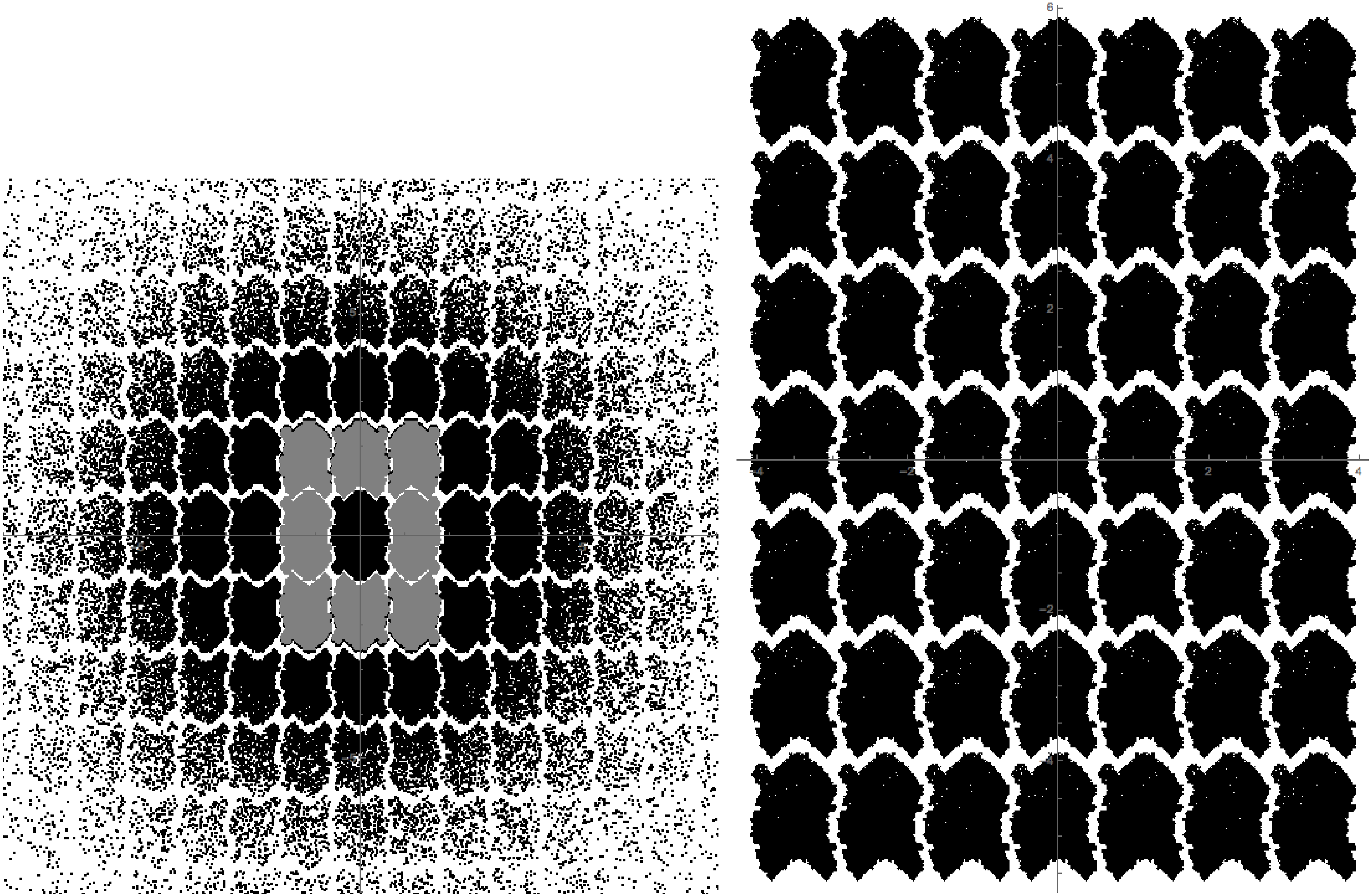


Fig.: The tilings

In the case of $d = 3$

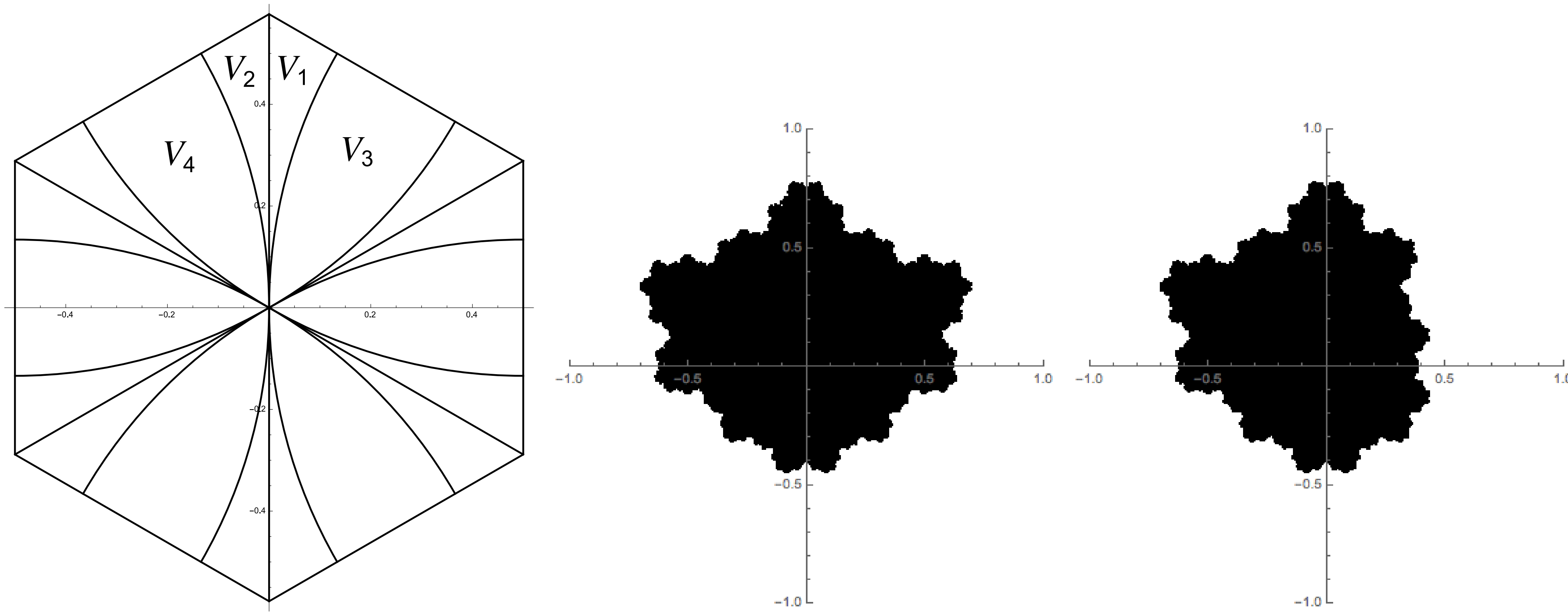


Fig.: The partition of U_d and the tiles X_1 and X_3

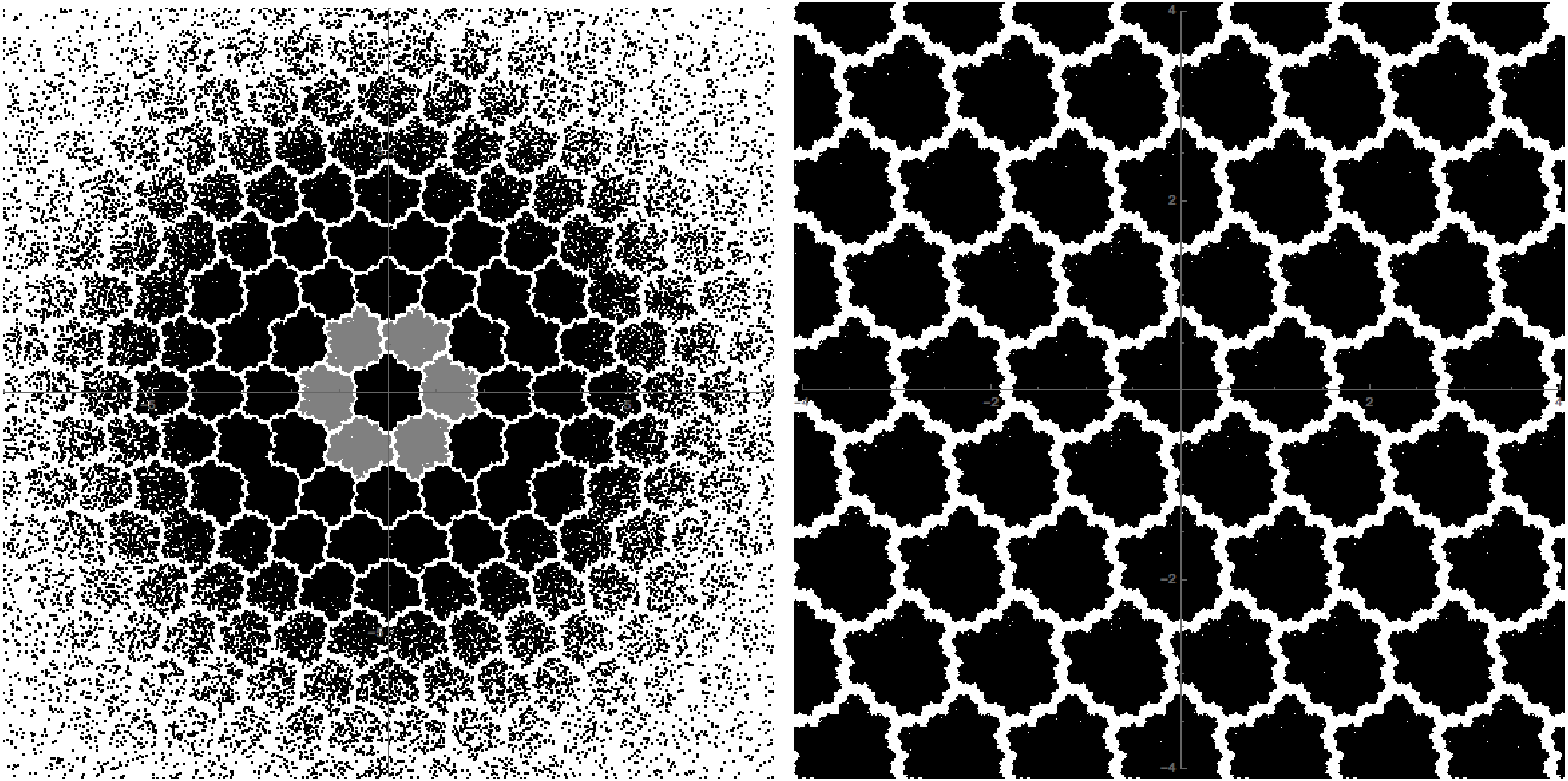


Fig.: The tilings

In the case where the domain is a rectangle for $d = 3$

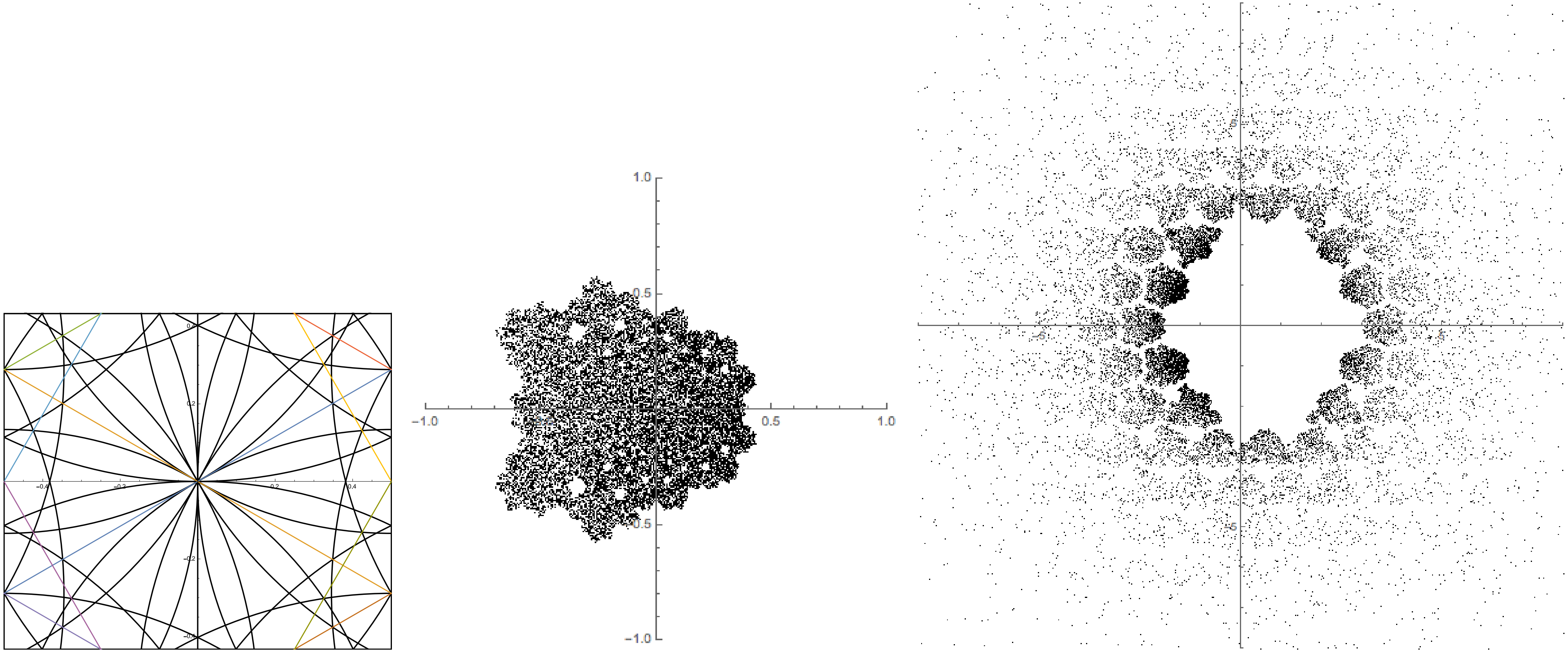


Fig.: The partition of U_d and some tile

Thank you very much.

The other cases

There are some other nearest type complex continued fractions for $-2, -7$ and -11. However, they do not have the best approximation property.

The best approximation property: p/q is a best approximation to x if

$$|q'| < |q| \Longrightarrow |q'x - p'| > |qx - p|.$$