Tree substitutions and Rauzy fractals

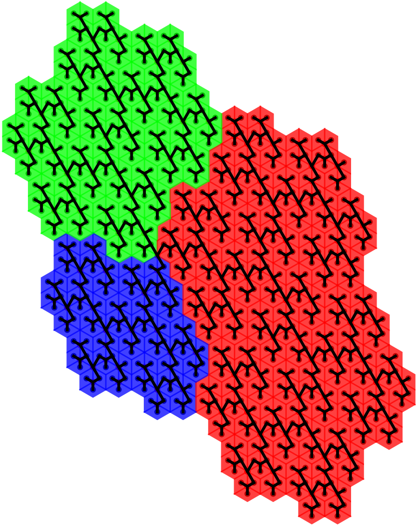
Thierry Coulbois

with Milton Minervino

December 5, 2017

Tree-bonacci

$\sigma : a \mapsto ab,\ b \mapsto ac,\ c \mapsto a$



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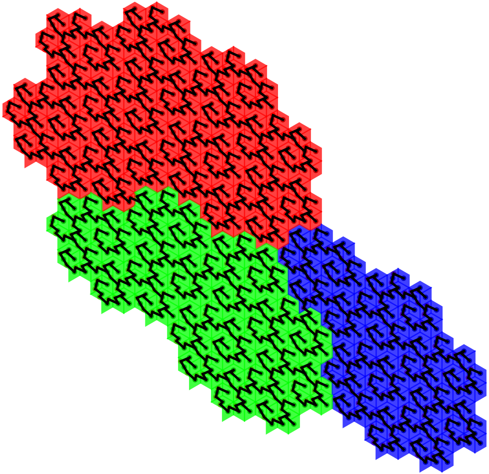

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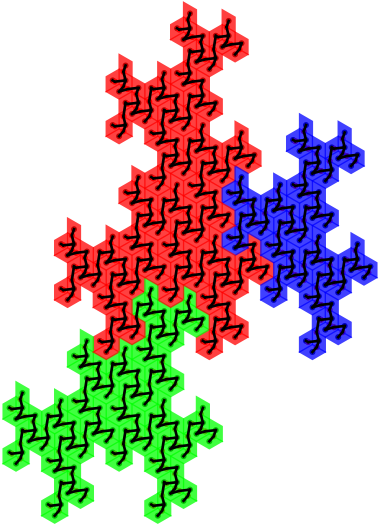

Summary

- Substitutive subshifts
  - ⋆ Singular words, asymptotic pairs, infinite special words
  - ⋆ Share-a-half relation and its quotient
  - ⋆ Trees associated with free group automorphisms
- Self-similarity
  - ⋆ Prefix-suffix expansion and automaton
  - ⋆ Self-similarity of the shift and of the tree
  - ⋆ Prefix-suffix expansion of singular words
- Tree substitution
- Rauzy fractal
  - ⋆ Self-similarity of the Rauzy fractal
  - ⋆ Global picture
  - ⋆ Tree substitution embedded in the Rauzy fractal
  - ⋆ Pruning, covering
- Interval exchange on the circle
  - ⋆ Contour of the tree
  - ⋆ Arnoux-Yoccoz example

Attracting shift

Tribonacci substitution

$$\sigma : a \mapsto ab,\ b \mapsto ac,\ c \mapsto a$$

$$\sigma^\infty(a) = abacabaabacababacabaabac\cdots$$

Attracting shift

$$X_\sigma = \text{set of bi-infinite indexed words in } A^{\mathbb{Z}}$$

whose factors are factors of $\sigma^n(i)$ for $i \in A$ and $n \geq 1$.
Cantor set.

Shift map $S : X_\sigma \to X_\sigma$ (shifting the index).

σ-invariant: $\sigma : X_\sigma \to X_\sigma$

Homeomorphisms.

Singular words

Attracting shift

X_σ = set of bi-infinite indexed words in $A^{\mathbb{Z}}$

$$\left.\begin{array}{ll} w_a: & \sigma^{3\infty}(a) = \cdots abacababacaba \\ w_b: & \sigma^{3\infty}(b) = \cdots abacabaabacab \\ w_c: & \sigma^{3\infty}(c) = \cdots ababacabaabac \end{array}\right\} \cdot abacabaabacabab \cdots = \sigma^{\infty}(a)$$

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$$\begin{array}{lllll} a & \mapsto & ab & \mapsto & abac \\ b & \mapsto & ac & & \\ c & \mapsto & a & & \end{array}$$

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Tribonacci substitution

a	$\mapsto$	ab	$\mapsto$	$abac$	$\mapsto$	$abacaba$
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w_a, w_b and w_c are singular words. Their right part is an infinite left special word. They form asymptotic pairs.

Singular words (continued)

$$w_a : \ \sigma^{3\infty}(a) = \cdots abacababacaba$$
$$w_b : \ \sigma^{3\infty}(b) = \cdots abacabaabacab \quad \longrightarrow \quad \cdot abacabaabacabab \cdots = \sigma^{\infty}(a)$$
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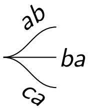
One periodic Nielsen path of period 3:



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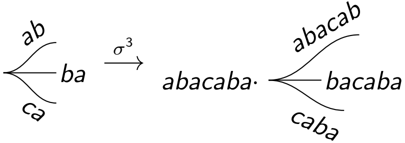

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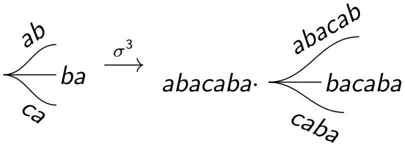

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$\tilde{\sigma} : a \mapsto ba, \ b \mapsto ca, \ c \mapsto a$

Singular words and index

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Index

$$(4-2)+(4-2) = 4 = 2 \times 3 - 2$$

Theorem [Gaboriau-Jaeger-Levitt-Lustig]

σ substitution and free group automorphism, fully irreducible (iwip).
Then, the index of X_σ is at most $2N-2$.

$(N = \mathrm{card}(A))$ fully irreducible (iwip) $\Rightarrow$ primitive.

Share-a-half

Theorem [Gaboriau-Jaeger-Levitt-Lustig]

σ iwip substitution $\Rightarrow$ index of X_σ is at most $2N-2$.

If the index is maximal then σ is parageometric.
Tribonacci is parageometric. Many other examples [Leroy].

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Theorem [GJLL, LL, C-Hilion-Lustig, CH, Kapovich-L]

If σ is iwip and parageometric then Ω_σ is a tree.

$$\Omega_\sigma \subseteq T_{\sigma^{-1}} : \text{ repelling tree}$$

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Repelling tree

$$Q : X_\sigma \to \Omega_\sigma \subseteq T_{\sigma^{-1}}$$

Repelling tree

$T_{\sigma^{-1}}$ is a metric space, with an action of the free group F_A by isometries and a contracting homothety H.

$$Q(S(Z)) = Z_0^{-1} Q(Z) \qquad Q(\sigma(Z)) = H(Q(Z))$$

$$\|u\|_{T_{\sigma^{-1}}} = \lim_{n\to\infty} \frac{\|\sigma^{-n}(u)\|_A}{(\lambda_{\sigma^{-1}})^n}$$

$\lambda_{\sigma^{-1}}$ expansion factor of σ^{-1}.

$$\sigma : \begin{array}{rcl} a & \mapsto & ab \\ b & \mapsto & ac \\ c & \mapsto & a \end{array} \qquad \sigma^{-1} : \begin{array}{rcl} a & \mapsto & c \\ b & \mapsto & c^{-1}a \\ c & \mapsto & c^{-1}b \end{array}$$

Self-similarity

Desubstitution [Mosse]

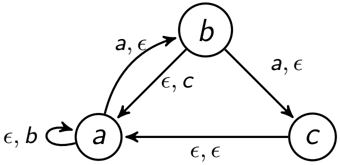
σ primitive. Each $w \in X_\sigma$ can be uniquely written as

$$w = S^{|p|}(\sigma(w')), \quad \text{for some } w' \in X_\sigma, p \in A^*$$

Prefix-suffix graph:

$w_0 \xleftarrow{p,s} w_0' \iff \sigma(w_0') = p w_0 s$

$(p, w_0, s) \in A^* \times A \times A^*$



Iterating we have that each $w \in X_\sigma$ has an infinite desubstitution path:

$$\Gamma : X_\sigma \to \mathcal{P}, \quad w \mapsto \gamma = w_0 \xleftarrow{p_0,s_0} w_0' \xleftarrow{p_1,s_1} w_0'' \xleftarrow{p_2,s_2} \cdots$$

Γ is onto, continuous and 1-to-1 except on the set of periodic points of σ.

Self-similarity (continued)

Cylinders

X_γ bi-inifinite words with prefix-suffix expansion ending with γ

Self-similar decomposition (σ primitive)

$$X_\sigma = \bigsqcup_{|\gamma|=n} X_\gamma$$

$$X_{\gamma\gamma'} = S^{|p(\gamma)|}(\sigma^{|\gamma|}(X_{\gamma'}))$$

Self-similarity of the repelling tree (σ iwip parageometric)

$$\Omega_\gamma = Q(X_\gamma) \qquad \text{a subtree (i.e. connected)}$$

$$\Omega_\sigma = \bigcup_{|\gamma|=n} \Omega_\gamma, \qquad \#(\Omega_\gamma \cap \Omega_{\gamma'}) \leq 1 \quad \text{singular words in } X_\gamma$$

$$\Omega_{\gamma\gamma'} = p(\gamma)^{-1} H^{|\gamma|}(\Omega_{\gamma'})$$

Desubstitution of singular words

Theorem [Queffelec, Holton-Zamboni]

- A primitive σ has finitely many singular bi-infinite words (left-special or right-special).
- Each singular word $w \in X_\sigma$ has an eventually periodic desubstitution path γ.

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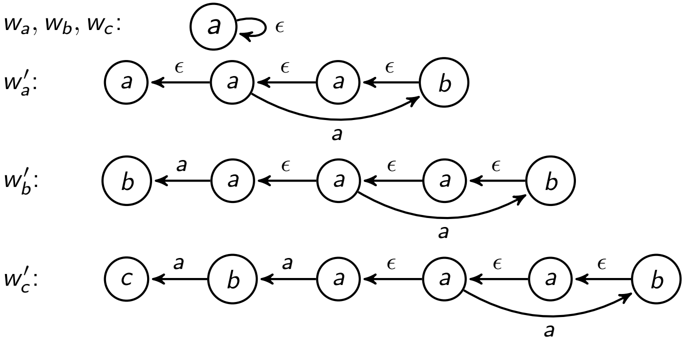
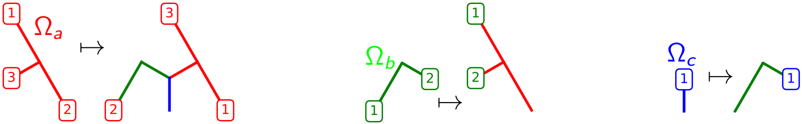
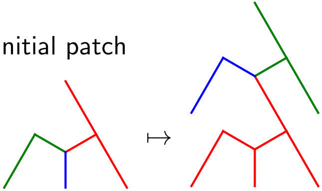


Abstract tree substitution

- Decomposition of Ω_σ in subtrees Ω_γ
- Ω_γ and $\Omega_{\gamma'}$ translate of one-another if they start at the same letter
- $\#\left(\Omega_\gamma \cap \Omega_{\gamma'}\right) \leq 1$, gluing points given by singular words.
- Finitely many gluing points in each prototile Ω_a



Initial patch



Iterating the tree substitution

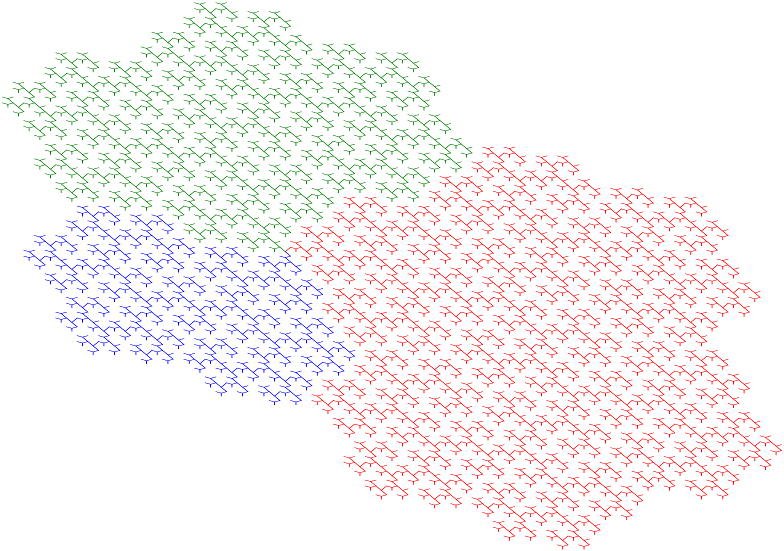
Theorem 1 [C.-Minervino]

Let σ be a primitive substitution on a finite alphabet A and a parageometric iwip automorphism of the free group F_A.
There exists a tree substitution τ such that the iterations of τ on the initial tree renormalized by the ratio of the contracting homothety converge to the repelling tree of the automorphism σ.
We provide an algorithm to construct this tree substitution.

Tree substitution τ [Jullian, Bressaud-Jullian]

- Prototiles $(W_a)_{a\in A}$: trees with gluing points
- Initial patch $W = \bigsqcup W_a/\sim$ some gluing points are identified
- $\tau(W_a)$ union of copies of the prototiles with some gluing points identified
- τ maps gluing points of W_a to gluing points of the tiles of $\tau(W_a)$

Twelfth iterate for Tribonacci (prunned)



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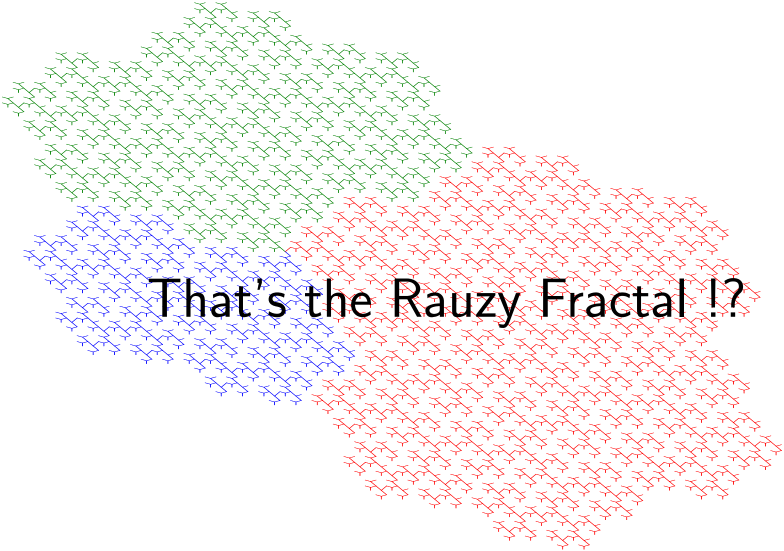

Rauzy Fractal

- Incidence matrix M_σ and its characteristic polynomial χ_σ

$$M_\sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \chi_\sigma(x) = x^3 - x^2 - x - 1$$

- σ primitive $\iff \exists n,\ M_\sigma^n > 0$
- dominant eigenvalue λ_σ (expansion factor)
- positive Perron-Frobenius eigenvector u
- λ_σ is a Pisot number, its Galois conjugates are < 1 in modulus
- σ irreducible Pisot
  - χ_σ is irreducible and λ_σ is Pisot
  - Contracting hyperplane E_c, $\mathbb{R}^A = E_c \oplus \mathbb{R}u$

Rauzy Fractal and the map φ

For any infinite path in the prefix-suffix automaton:

$$\gamma = a_0 \xleftarrow{p_0,s_0} a_1 \xleftarrow{p_1,s_1} a_2 \xleftarrow{p_2,s_2} \cdots$$

We have a converging series:

$$\varphi(\gamma) = \sum_{i=0}^{\infty} \pi_c(M_\sigma^i \, \ell(p_i)) \in E_c$$

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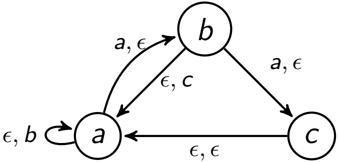
Prefix-suffix graph:



$\mathcal{P}$: infinite prefix-suffix expansions

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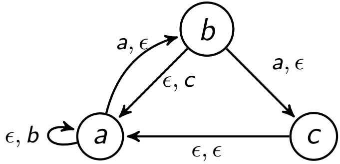


$\mathcal{P}$: infinite prefix-suffix expansions

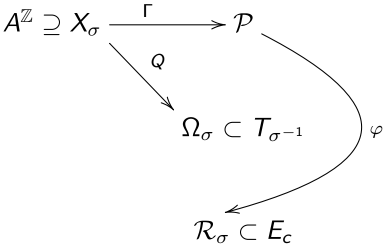
Rauzy fractal [Rauzy, Arnoux-Ito]

$$\mathcal{R}_\sigma = \varphi(\mathcal{P}) \subseteq E_c, \qquad \varphi : \mathcal{P} \to \mathcal{R}_\sigma \quad \text{continuous}$$

$$\mathcal{R}_\gamma = \{\varphi(\gamma') \mid \gamma' \text{ ends with } \gamma\} \qquad \mathcal{R}_\sigma = \bigsqcup_{|\gamma|=n} \mathcal{R}_\gamma$$

Global picture

σ irreducible Pisot and parageometric iwip



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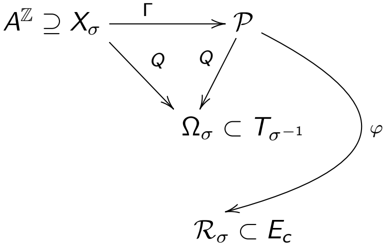

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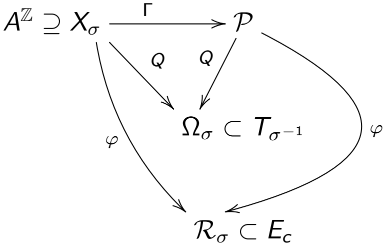

Global picture

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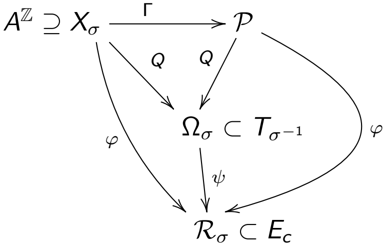

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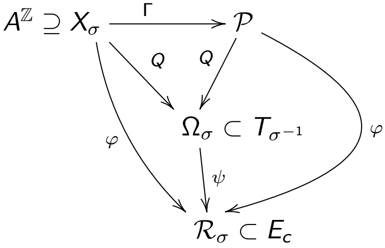

$\psi(\Omega_\sigma)$ is a tree inside the Rauzy fractal (Peano tree?)

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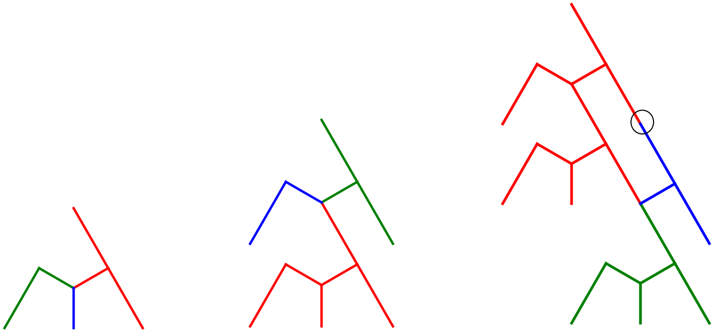
Theorem 2 [C.-Minervino]

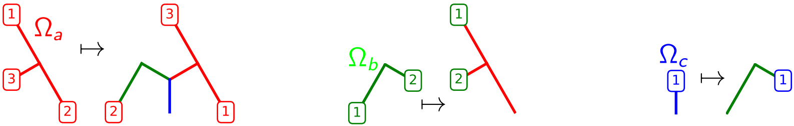
σ irreducible Pisot and parageometric iwip.
Then, the tree substitution τ can be realized inside the contracting space E_c of σ and the renormalized iterated images $M_\sigma^n \tau^n(W)$ converge to the Rauzy fractal $\mathcal{R}_\sigma$.
In all our examples we provide a covering tree substitution which yields trees inside the contracting space.

Pruning and covering

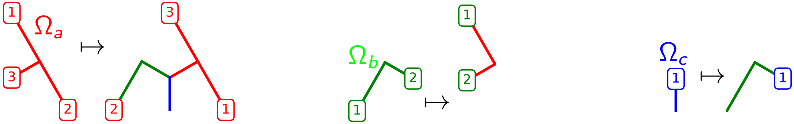


The problem is the non-injectivity of the map ψ.

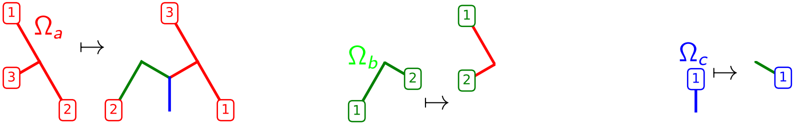
Pruning and covering (continued)



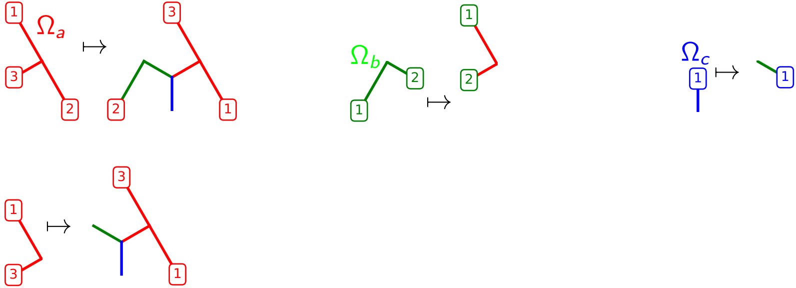
Pruning and covering (continued)



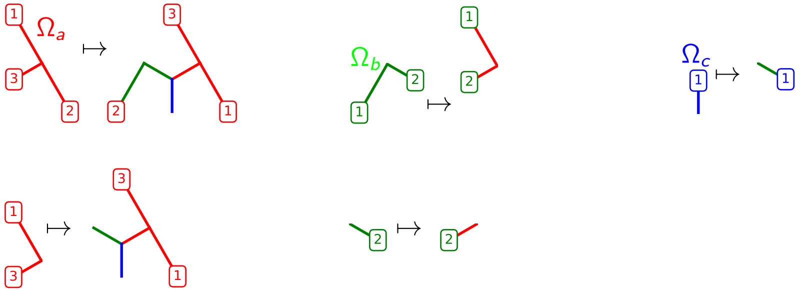
Pruning and covering (continued)



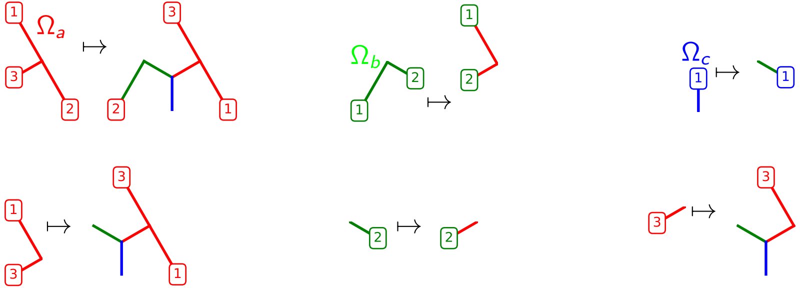
Pruning and covering (continued)



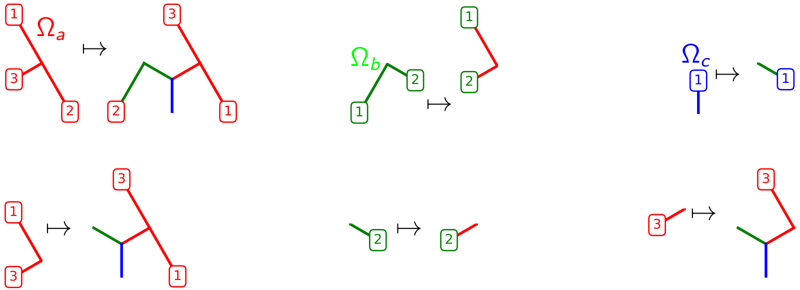
Pruning and covering (continued)



Pruning and covering (continued)



Pruning and covering (continued)

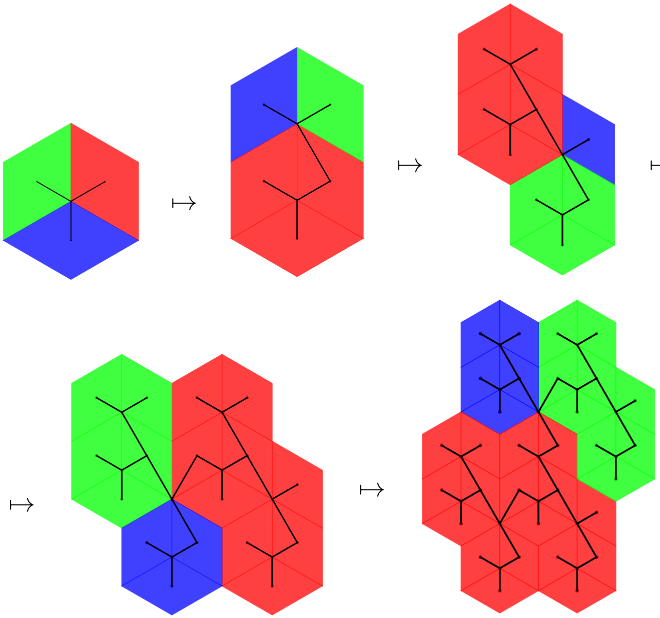


Bigger alphabet (more prototiles), but now its injective in the Rauzy fractal.

Conjecture

Up to covering the initial tree substitution, it is always possible to embed it in the Rauzy fractal.

Inside the dual substitution



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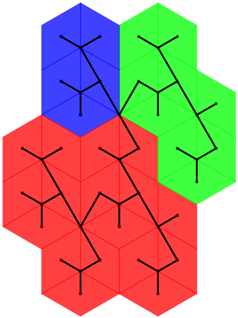

Question: Adjacency of the tiles

If the tiles Ω_γ and $\Omega_{\gamma'}$ have a common point, is it true that the corresponding tiles of the dual substitution have a common point?

And more

Application [Bressaud-Jullian, C.-Minervino]

Contour of the tree: piecewise exchange on the circle

Perspectives

- Non parageometric iwip substitutions
- Parageometric $\Rightarrow$ Rauzy fractal arcwise connected. Disk-like?
- Spectrum of the contour IET versus spectrum of X_σ?
- Pisot conjecture?

And more

Application [Bressaud-Jullian, C.-Minervino]

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Thank You !