Tiling and Recurrence — CIRM, December 8th, 2017

A set of sequences of complexity $2n + 1$

Julien Cassaigne

Institut de mathématiques de Marseille - CNRS, Marseille, France

julien.cassaigne@math.cnrs.fr

Joint work with Sébastien Labbé (LaBRI - CNRS, Bordeaux, France)
and Julien Leroy (Université de Liège, Belgique).

A set of sequences of complexity $2n + 1$

- Sturmian words and continued fractions
- Looking for a 2-dimensional analogue
- Rauzy graphs
- The new algorithm and associated words
- Link with Selmer algorithm

S-adic representation of Sturmian words

Let $\mathbf{u}$ be a standard Sturmian word on alphabet $A = \{a, b\}$.

Factor complexity of $\mathbf{u}$ is $p(n) = n + 1$.

The word $\mathbf{u}$ contains either aa or bb but not both.

- If $\mathbf{u}$ contains aa, write $\mathbf{u} = \sigma_a(\mathbf{v})$ with $\sigma_a : \ a \mapsto a, b \mapsto ab$.
- If $\mathbf{u}$ contains bb, write $\mathbf{u} = \sigma_b(\mathbf{v})$ with $\sigma_b : \ a \mapsto ba, b \mapsto b$.

Then $\mathbf{v}$ is also standard Sturmian, so we can iterate.

$$\mathbf{u} = \lim_{n \to \infty} \sigma_{d_0} \sigma_{d_1} \ldots \sigma_{d_n}(a)$$

is an S-adic representation of $\mathbf{u}$, where $S = \{\sigma_a, \sigma_b\}$, and (d_n) is the directive sequence of $\mathbf{u}$.

Sturmian words and continued fractions (1)

Let $\mathbf{x} = (x, y) \in \mathbb{R}^2_+$. Define

$$f(x,y) = \begin{cases} (x-y, y) \text{ if } x \geq y \\ (x, y-x) \text{ if } x < y \end{cases}$$

which can be written $f(\mathbf{x}) = M_{d_0}^{-1}\mathbf{x}$ where $d_0 \in A$ and

$$M_a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } M_b = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

are the matrices of σ_a and σ_b.

Iterating, we get $f^n(\mathbf{x}) = M_{d_{n-1}}^{-1} \dots M_{d_0}^{-1}\mathbf{x}$.

Sturmian words and continued fractions (2)

Iterating, we get $f^n(\mathbf{x}) = M_{d_{n-1}}^{-1} \ldots M_{d_0}^{-1}\mathbf{x}$
where $\mathbf{d} = d_0 d_1 d_2 \ldots = a^{a_0} b^{a_1} a^{a_2} b^{a_3} \ldots$
(with $a_0 \geq 0$, $a_1 \geq 1$ for $i \geq 1$).

Then the continued fraction expansion of x/y is $x/y = [a_0; a_1, a_2, \ldots]$.

x/y is irrational if and only if a and b occur infinitely often in $\mathbf{d}$.

If $x + y = 1$, then $\mathbf{d}$ is the directive sequence of a Sturmian word with frequencies $f_a = x$, $f_b = y$.

2-dimensional analogues

Goal: define a continued fraction (CF) algorithm on $\mathbb{R}^3_+$, with good convergence properties, and an associated family of words with nice S-adic representations and low factor complexity.

Billiard in a cube? complexity $n^2 + n + 1$, no S-adic representation, no CF algorithm.

Three-interval exchanges? good complexity $2n + 1$, complicated S-adic construction, complicated CF algorithm.

Arnoux-Rauzy words? good complexity $2n + 1$, nice S-adic representation, CF algorithm defined only on a subset of measure 0.

Jacobi-Perron, Brun, Poincaré, Selmer, Fully substractive, Arnoux-Rauzy-Poincaré, etc. CF algorithms? Words can be associated, complexity and S-adic representations are not nice.

Rauzy graphs

(Rauzy 1983)
For each $n \in \mathbb{N}$, the Rauzy graph Γ_n is the directed graph with

- vertices: $L_n(u)$,
- edges: $L_{n+1}(u)$,
- $x \xrightarrow{z} y$ if x is a prefix of z and y is a suffix of z.

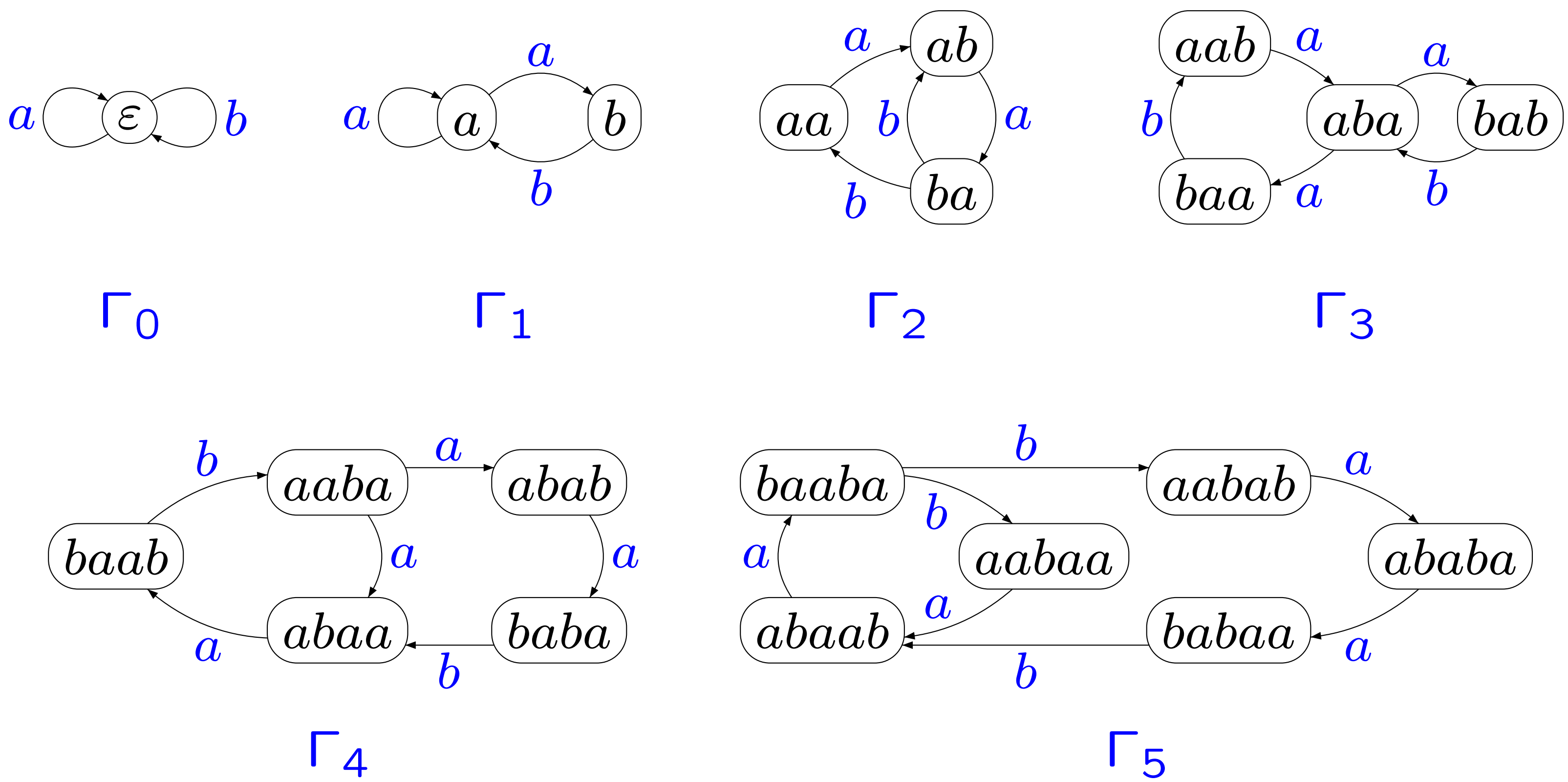
Edges may be labelled in several ways.
Here we choose the first letter of z.

Example: Fibonacci word

Let $u = abaababaabaababaababaabaababaabaab\ldots$ be the Fibonacci word. Its directive sequence is $(ab)^\omega$.

$p(n) = n + 1$ for all n, so Γ_n has $n + 1$ vertices and $n + 2$ edges.

$u = abaababaabaababaababaabaababaabaababaababaabaababaababa\ldots$



Rauzy graphs and special factors

A factor $w \in L(u)$ is right special (for u) if there exist distinct letters a and b such that $wa \in L(u)$ and $wb \in L(u)$.

In Γ_n:
right special factor = vertex with more than one outgoing edge
left special factor = vertex with more than one incoming edge.

On a binary alphabet:
the number of right special factors is $s(n) = p(n+1) - p(n)$;
the number of left special factors is $s(n)$ or $s(n)+1$ (in the case where one vertex has no incoming edge).

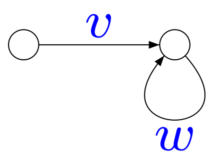
Shape of a Rauzy graph

The shape of a Rauzy graph is the graph obtained by removing all vertices with indegree and outdegree 1. Branches

$$x_0 \xrightarrow{a_1} x_1 \xrightarrow{a_2} x_2 \cdots x_{k-1} \xrightarrow{a_k} x_k$$

are replaced with a single edge $x_0 \xrightarrow{a_1 a_2 \ldots a_k} x_k$ labelled with a word.

If u is eventually (but not purely) periodic, for n large the shape of Γ_n is:



where $u = vw^\omega$.

Rauzy graphs for Sturmian words

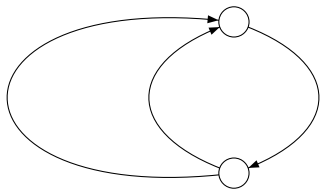
A Sturmian word is a word such that $p(n) = n + 1$ for all n (the smallest possible complexity for a non-periodic word).

Such a word is always recurrent: every factor occurs infinitely often. As a consequence, its Rauzy graphs are strongly connected.

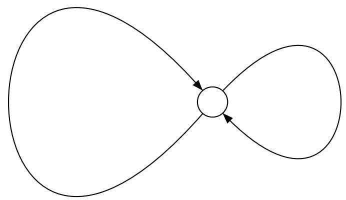
$s(n) = (n+2) - (n+1) = 1$: there is one left special factor l and one right special factor r of length n. Therefore only two shapes are possible for Γ_n:

Rauzy graphs for Sturmian words

$s(n) = (n+2) - (n+1) = 1$: there is one left special factor l and one right special factor r of length n. Therefore only two shapes are possible for Γ_n:



Case 1: $l \neq r$



Case 2: $l = r$

Evolution from Γ_n to Γ_{n+1}

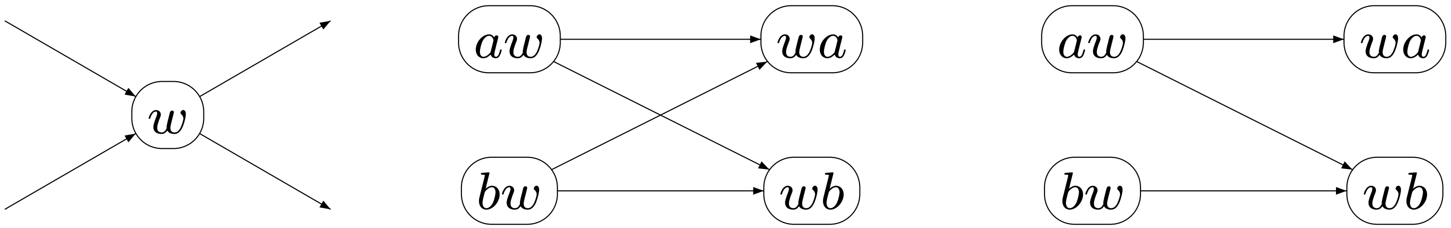
If $G = (V, E)$ is a directed graph, then its line graph is the graph $D(G) = (V', E')$ with $V' = E$ and

$$E' = \{(e_1, e_2) : \text{head}(e_1) = \text{tail}(e_2)\}.$$

Γ_{n+1} is always a subgraph of $D(\Gamma_n)$. Often $\Gamma_{n+1} = D(\Gamma_n)$, in particular when u is recurrent (we assume this from now on) and there is no bispecial factor (a factor that is both left special and right special).

Bispecial factor burst

A bispecial factor is a factor that is both left special and right special. For simplicity assume a binary alphabet $A = \{a, b\}$.



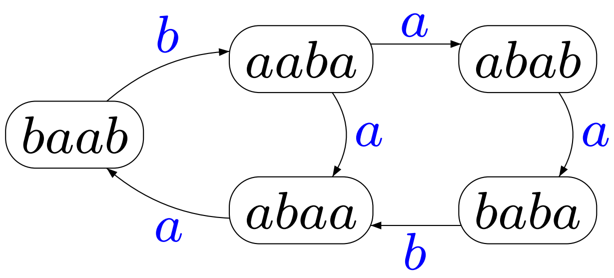
Γ_n (w is bispecial) $D(\Gamma_n)$ (w yields 4 edges) Γ_{n+1} (edges may be deleted)

Evolution without bispecial factor

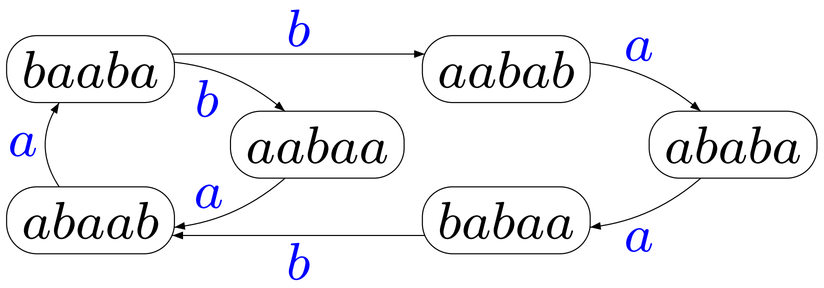
When there is no bispecial factor, $\Gamma_{n+1} = D(\Gamma_n)$ can be deduced from Γ_n without any additional information.

Γ_n and Γ_{n+1} have the same shape. The lengths of branches may increase or decrease by 1. At least one branch shrinks, so eventually a bispecial factor will occur in a later graph.

Example (Fibonacci):



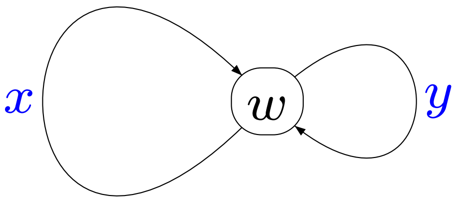
Γ_4



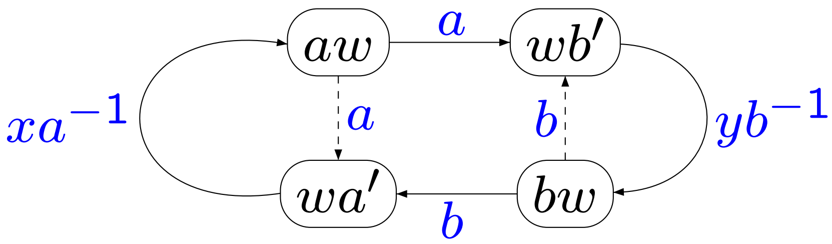
Γ_5

Evolution for Sturmian words

Assume that there is a bispecial factor of length n.

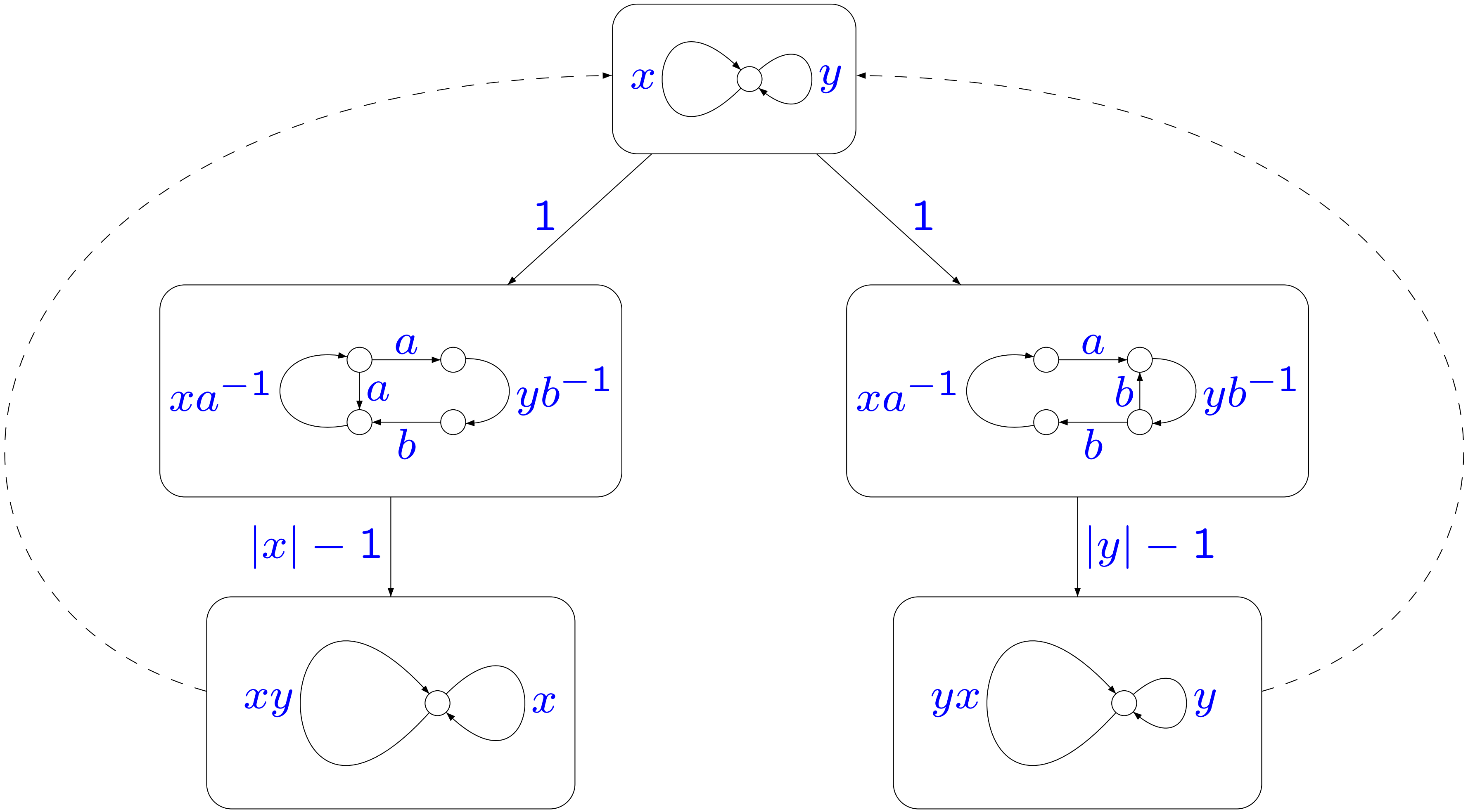


Γ_n



$D(\Gamma_n)$

To obtain Γ_{n+1}, one of the dashed vertical edges has to be removed from $D(\Gamma_n)$ (exactly one to get $p(n+2) = n+3$ edges; and the horizontal edges are needed for strong connectedness). So two evolutions are possible.



x
y
1
1
xa^{-1}
a
a
b
yb^{-1}
xa^{-1}
a
b
b
yb^{-1}
$|x| - 1$
$|y| - 1$
xy
x
yx
y

Loop labels and substitutions

Let n_i be the length of the i-th bispecial factor ($n_0 = 0$).

Let x_i and y_i be the labels of the loops of Γ_{n_i}. Then the labels of the loops of $\Gamma_{n_{i+1}}$ are either $x_{i+1} = x_i$ and $y_{i+1} = x_i y_i$, or $x_{i+1} = y_i x_i$ and $y_{i+1} = y_i$.

Let τ_i be the substitution such that $x_i = \tau_i(a)$ and $y_i = \tau_i(b)$. Then $\tau_{i+1} = \tau_i \circ \sigma_{d_i}$.

Therefore $\tau_i = \sigma_{d_0} \circ \cdots \circ \sigma_{d_{i-1}}$.

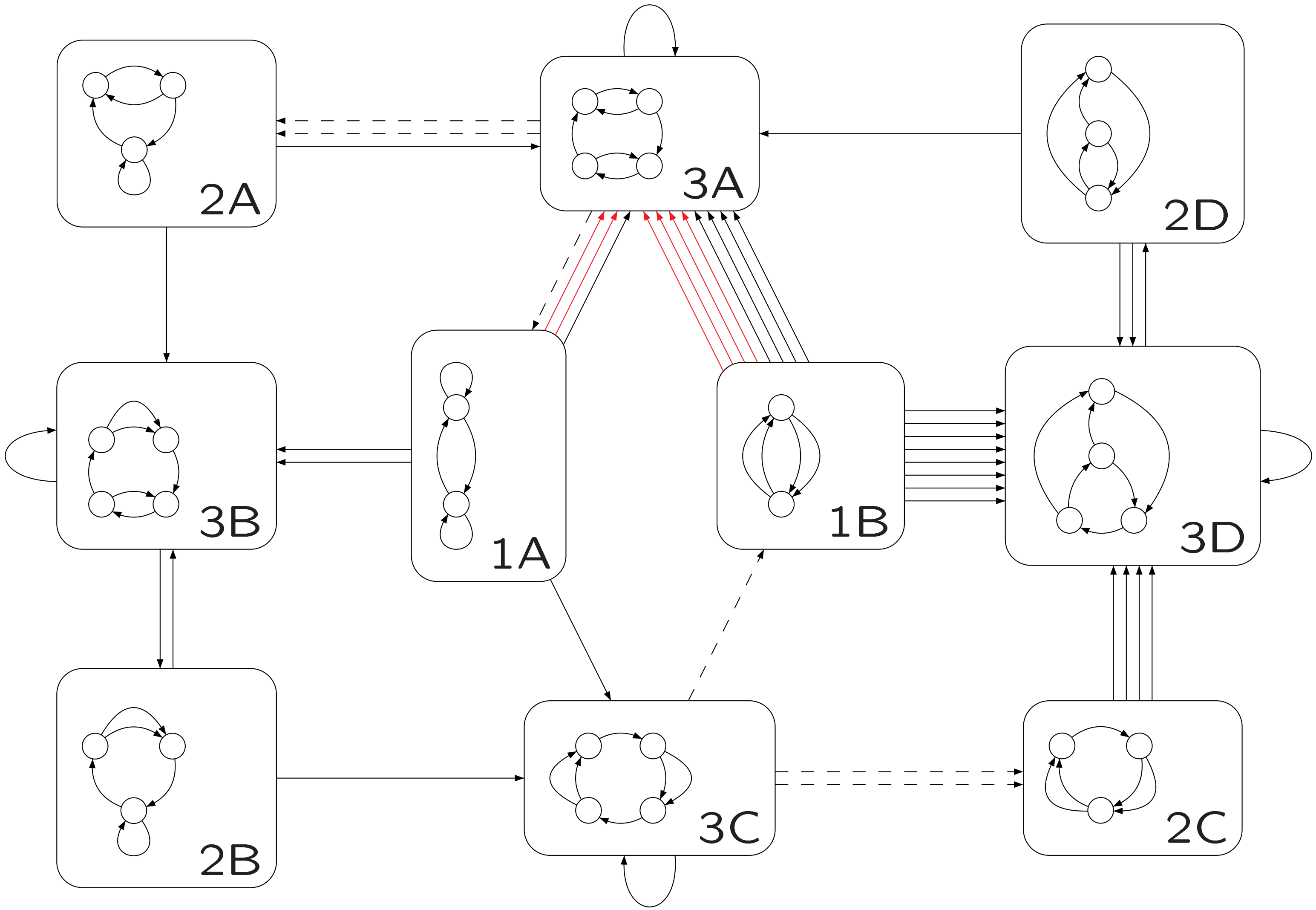
Dynamics of Rauzy graphs for words of complexity $2n$

(Rote 1994, Desideri 2002, Leroy 2012)

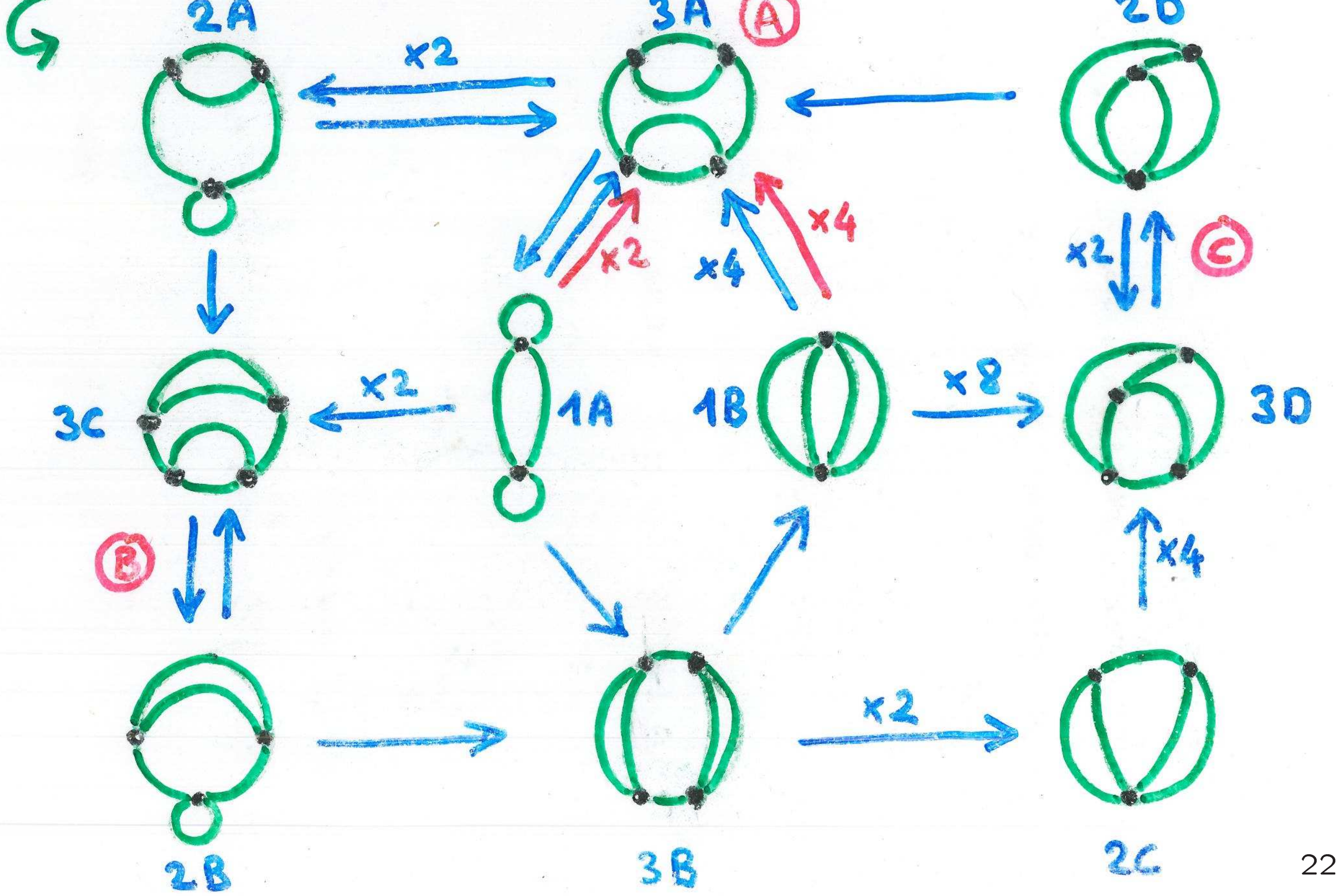
Consider now the class of all recurrent binary words such that $p(n) = 2n$ for all $n \geq 1$.

Goal: find a characterization of the words of this class using substitutions (s-adic with extra conditions).

Method: describe the possible shapes of Rauzy graphs, and the evolutions between them. Encode this with substitutions. Add conditions to ensure that the substitutions are applied only when the evolution is allowed.


3A
2A
2D
1A
1B
3B
3D
2B
3C
2C

Rauzy graphs for sequences of complexity 2n



Our algorithm (1)

Consider words for which all Rauzy graphs have shapes 2D and 3D (for n large enough, or for all n if we allow branches with empty labels).

In the i-th graph of shape 2D, let $\tau_i(a)$, $\tau_i(b)$, $\tau_i(c)$ be the labels of the loops around the bispecial factor:
$\tau_i(a)$ through the right special factor;
$\tau_i(b)$ through the right special and left special factors;
$\tau_i(c)$ through the left special factor.

Then $\tau_{i+1} = \tau_i \circ \gamma_{d_i}$, with $d_i \in \{1, 2\}$ and:
$\gamma_1 : \ a \mapsto a, b \mapsto ac, c \mapsto b$;
$\gamma_2 : \ a \mapsto b, b \mapsto ac, c \mapsto c$.

Our algorithm (2)

This results in the following continued fraction algorithm:

$$F_C(x,y,z) = \begin{cases} (x-z,z,y) \text{ if } x \geq z \\ (y,x,z-x) \text{ if } x < z \end{cases}$$

associated with substitutions $C = \{\gamma_1, \gamma_2\}$:

$\gamma_1 : \; a \mapsto a, b \mapsto ac, c \mapsto b$

$\gamma_2 : \; a \mapsto b, b \mapsto ac, c \mapsto c$

and matrices

$$C_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } C_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

A periodic example

Let $\alpha = \sqrt[3]{2}$ and $\mathbf{x}_0 = (1, \alpha, \alpha^2)$. Then one has:

$$\begin{pmatrix} 1 \\ 1.2599 \\ 1.5874 \end{pmatrix} \xrightarrow{C_2^{-1}} \begin{pmatrix} 1.2599 \\ 1 \\ 0.5874 \end{pmatrix} \xrightarrow{C_1^{-1}} \begin{pmatrix} 0.6725 \\ 0.5874 \\ 1 \end{pmatrix} \xrightarrow{C_2^{-1}} \begin{pmatrix} 0.5874 \\ 0.6725 \\ 0.3275 \end{pmatrix}$$

$$\xrightarrow{C_1^{-1}} \begin{pmatrix} 0.2599 \\ 0.3275 \\ 0.6725 \end{pmatrix} \xrightarrow{C_2^{-1}} \begin{pmatrix} 0.3275 \\ 0.2599 \\ 0.4126 \end{pmatrix} \xrightarrow{C_2^{-1}} \begin{pmatrix} 0.2599 \\ 0.3275 \\ 0.0851 \end{pmatrix} \xrightarrow{C_1^{-1}} \ldots$$

One checks that $\mathbf{x}_{13} = (\alpha - 1)^2 \mathbf{x}_3$ so that $d = 212(1221221121)^\omega$. The corresponding infinite word is $\mathbf{u} = (\gamma_2\gamma_1\gamma_2)(\gamma_1\gamma_2^2\gamma_1\gamma_2^2\gamma_1^2\gamma_2\gamma_1)^\omega(a) =$ $bcbacbcacbacbcbacbacbcacbacbcbacbcacbacbcbacbacbcacbacbcbacbcacbacbc\ldots$

Open problem. Characterize vectors with periodic CF expansion.

Properties

The algorithm F_C is defined for every vector $\mathbf{x}_0$.

Theorem 1. The following are equivalent:
(i) the entries of $\mathbf{x}_0$ are rationally independent
(ii) the C-adic representation (γ_{d_n}) is primitive
(iii) the directive sequence $\mathbf{d}$ does not belong in $\{1,2\}^*\{11,22\}^\omega$.

A sequence of morphisms (τ_n) is primitive if:
$\forall n \in \mathbb{N}, \exists m > n, \forall (i,j) \in A^2, |\tau_n\tau_{n+1}\cdots\tau_{m-1}(j)|_i > 0$.

Properties

If conditions of Theorem 1 are satisfied, then $\mathbf{u}$ has complexity $2n+1$.

Proof: by construction when γ_1 and γ_2 are defined with Rauzy graphs.
A combinatorial proof, using bispecial factors, is also possible.

The continued fraction is always convergent.
Define the cones $\Lambda_n = C_{d_0} C_{d_1} \ldots C_{d_{n-1}} \mathbb{R}^3_+$.
Then their intersection is a half-line: $\cap_{n\in\mathbb{N}} \Lambda_n = \mathbb{R}_+ \mathbf{x}_0$.

Selmer algorithm

Selmer algorithm: subtract the smallest entry from the largest [Ernst Selmer 1956].

Semi-sorted formulation: $F_S : \Gamma \to \Gamma$ where $\Gamma = \{(x,y,z) \in \mathbb{R}^3_+ : \max(y,z) \leq x \leq y+z\}$ and

$$F_S(x,y,z) = \begin{cases} (y, x-z, z) & \text{if } y \geq z \\ (z, y, x-y) & \text{if } y < z \end{cases}$$

Associated matrices are

$$S_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } S_2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Link with Selmer algorithm

Theorem 2. Algorithms F_C and F_S are conjugated.

Let

$$Z = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Then $S_1 Z = Z C_1$ and $S_2 Z = Z C_2$.
The map $z : \mathbb{R}^3_+ \to \Gamma$ defined by $z(\mathbf{x}) = Z\mathbf{x}$ is a homeomorphism, and $z \circ F_C = F_S \circ z$.

Balance

Let $D \in \mathbb{N}$. An infinite word $\mathbf{u} \in A^{\mathbb{N}}$ is D-balanced if for all v and w factors of $\mathbf{u}$ of equal length and all $x \in A$, $||w|_x - |v|_x| \leq D$.
The word $\mathbf{u}$ is finitely balanced if it is D-balanced for some D.

Sturmian words are 1-balanced. Most Arnoux-Rauzy words are finitely balanced, but there exist Arnoux-Rauzy words that are not finitely balanced.

Work in progress: for almost all $\mathbf{x}$, algorithm F_C produces a finitely balanced word $\mathbf{u}$. But there exist $\mathbf{x}$ for which $\mathbf{u}$ is not finitely balanced.

Further work

Characterize $\mathbf{x}$ for which $\mathbf{u}$ is 2-balanced, D-balanced, finitely balanced.

Characterize $\mathbf{x}$ for which $\mathbf{u}$ is morphic, purely morphic, etc.

Construct a similar algorithm in any dimension.
Problem: the Rauzy graph approach does not seem to work.