Thermodynamic formalism On Aperiodic Linearly Repetitive Tilings

Adnene Besbes

Institut Préparatoire aux Études d'Ingénieurs de Bizerte

Chaire Jean Morlet CIRM from 4-8 December 2017

Outline

- Lattice gas models on Aperiodic Self-Similar tilings
  (by Geerse and Hof)
- Generalization for Aperiodic Linearly repetive (LR) tilings

Self-similar tilings

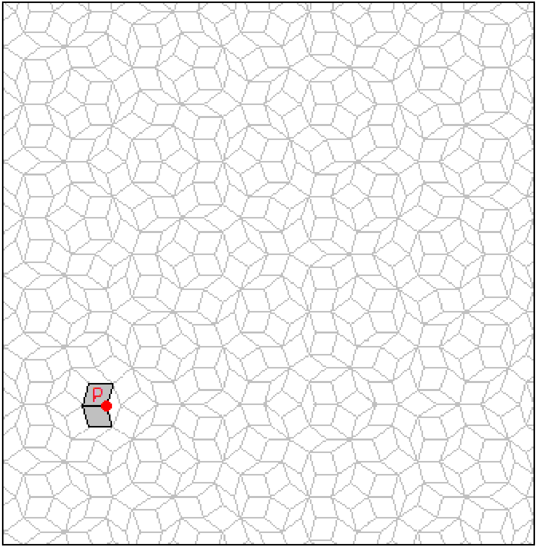
Self-similar tilings

they are tilings build by primitive substitution which consists to enlarge every tile by a factor $\lambda > 1$ and then to segment it by tiles of initial size.

Self-similar tilings

(LR) tilings

A tiling T is (LR) if there is $C > 1$ such that for every patch P any ball of radius $C\,\mathrm{diam}(P)$ contains a translate copy of P.



Quasi-crystallographic tilings

An aperiodic tiling is said quasi-crystallographic if it satisfies the following assumptions :

- **(Repetitivity)** for all patch P there exits a radius $R(P)$ such that every ball of radius $R(P)$ contains a translate copy of P.
- **(Finite local complexity)** for every $D > 0$ the set of possibly patches (up translation) with diameter smaller than D is finite.
- **(Uniform Patch Frequency)** for all patch P there exists a positif number f_P such that for every sequence of balls B_n which satisfies $\lim_{n\to\infty} r_n = +\infty$;

$$\lim_{n\to\infty} \frac{\#_P(B_n)}{\mathrm{Vol}(B_n)} = f_P.$$

Vertices of the tiling

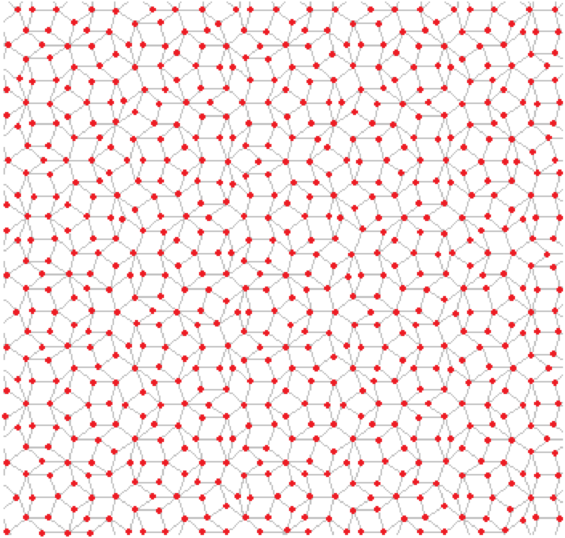


FIGURE – L : lattice of the vertices

Lattice gas models

- Consider a probability space (E, ξ, λ), where E is compact metric space and ξ it's Borel $\sigma-$algebra. To each vertex $i \in L$ we associate a copy (E_i, ξ_i, λ_i) of this probability space.
- an interaction Φ assigns to every finite set $X \subset L$ a continuous function on E^X $(E^X = \bigotimes_{i \in X} E_i)$.
- An interacion Φ belongs to $\mathcal{B}_0$ if it is :

  1) *vertexpattern invariant* if $\Phi(X) \simeq \Phi(X')$ whenever $X' = X + t$ for some $t \in \mathbb{R}^d$.

  2) *of finite range* if there is $D > 0$ s.t $\Phi(X) = 0$ whenever $\text{diam}(X) > D$.

Thermodynamic functions

The Hamiltonian in a bounded $Q \subset \mathbb{R}^d$ is :

$$H_Q^\Phi = \sum_{X \subset Q^*} \Phi(X), \quad \text{where } Q^* = Q \cap L$$

- the pressure is defined by

$$\mathbf{P}_Q(\Phi) = \log \int_{E^{Q^*}} \exp\big(-H_Q^\Phi(u)\big) d\lambda^Q(u). \quad \text{where } \lambda^Q = \bigotimes_{i \in Q \cap L} \lambda_i$$

For a probability measure ρ on E^L we define :

- the energy by

$$\rho(H_Q^\Phi) = \int_{E^{Q^*}} H_Q^\Phi(u) d\rho(u).$$

- and the entropy by

$$S_Q(\rho) = -\int_{E^{Q^*}} f_{\rho^Q} \log\big(f_{\rho^Q}\big) d\lambda^Q(u), \quad \text{where } f_{\rho^Q} \text{ is the density of } \rho_Q.$$

Case of self-similar tilings (Geerse and Hof)

Existence of thermodynamic functions

For every $\Phi \in \mathcal{B}_0$ and $\rho \in \mathcal{P}_B(E^L)$, $\exists\, \mathbf{P}(\Phi),\ e^{\Phi}(\rho),\ s(\rho) \in \mathbb{R}$, s.t forall sequence of cubes (Q_n) with $\mathsf{L}(Q_n) \to \infty$, we have :

$$\mathbf{P}(\Phi) = \lim_{n\to\infty} \mathrm{Vol}(Q_n)^{-1}\mathbf{P}_{Q_n}(\Phi)$$

$$e^{\Phi}(\rho) = \lim_{n\to\infty} \mathrm{Vol}(Q_n)^{-1}\ \rho(H^{\Phi}_{Q_n}),$$

$$s(\rho) = \lim_{n\to\infty} \mathrm{Vol}(Q_n)^{-1}\ S_{Q_n}(\rho).$$

Variational principle

For every $\Phi \in \mathcal{B}_0$

$$\mathbf{P}(\Phi) = \sup_{\rho\in\mathcal{P}_B(E^L)} s(\rho) - e^{\Phi}(\rho)$$

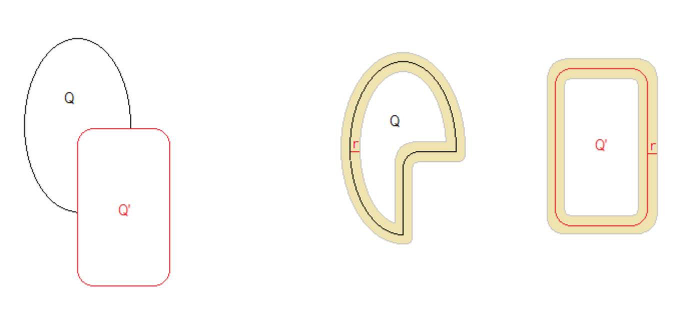
(sub)additive functions

additive and subadditive functions

a function $F : B(\mathbb{R}^d) \to \mathbb{R}$ is called **additive** if it satisfies :

- $\left| F(Q \cup Q') - F(Q) - F(Q') \right| \leq b_0 \big(\text{Vol}(\partial^r Q) + (\partial^r Q')\big)$

  where $Q,\ Q'$ have disjoint interiors.



F is called **subadditive** if it satisfies :

- $F(Q \cup Q') - F(Q) - F(Q') \leq 0,$ where $Q \cap Q' = \varnothing$.

Self-similarity

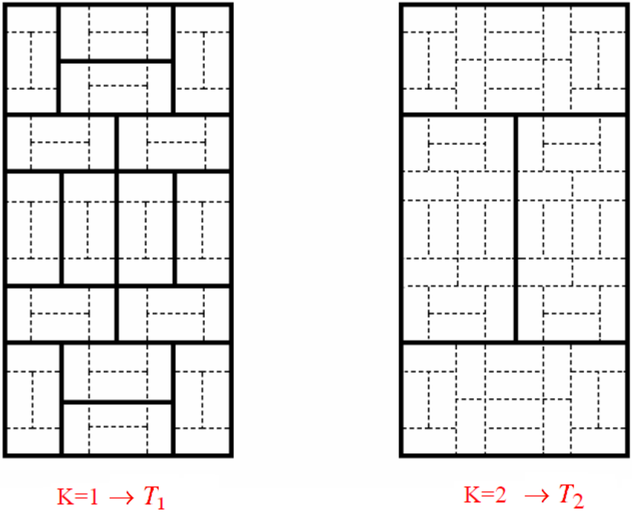


FIGURE – illustration of the self-similarity

Self-similarity

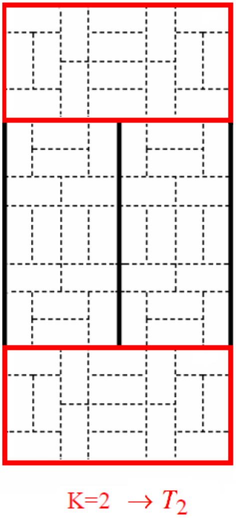


FIGURE – Equivalent supertiles in T_2

unique composition with supertiles

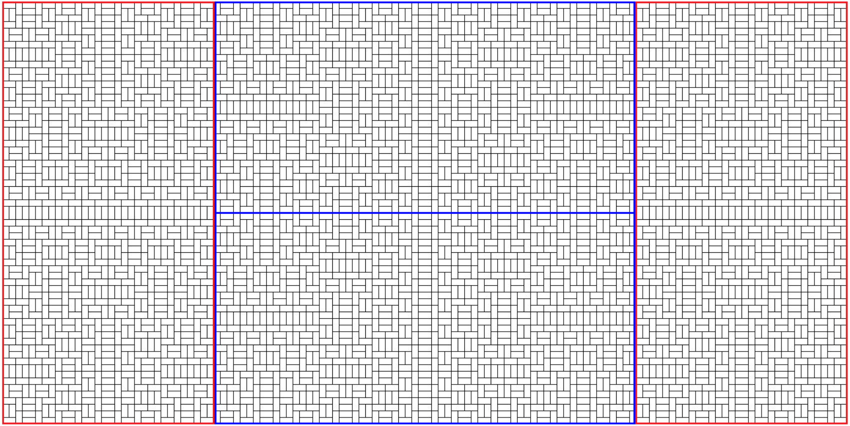


FIGURE – Equivalent supertiles

Ergodic theorems

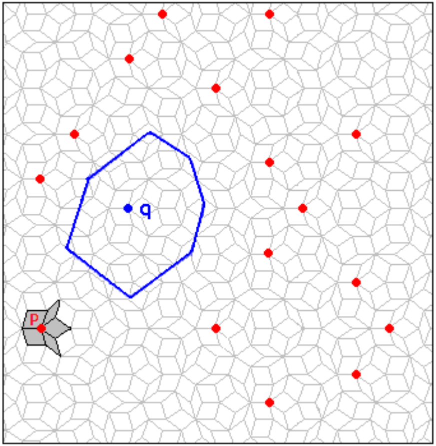
If F is a (sub)additive function satisfying

$$(*) \lim_{k\to\infty} \sup_{M\sim M'\in T_k} \left\{ \frac{|F(M)-F(M')|}{\mathrm{Vol}(M)} \right\} = 0$$

Then $\lim_{n\to\infty} \mathrm{Vol}^{-1}(Q_n)F(Q_n) = \overline{F}$

Voronoï construction for (LR) case

For a locator point q of some patch P. A Voronoï tile V_q is the region formed by points closer to q to the other locators of P.



Return tile

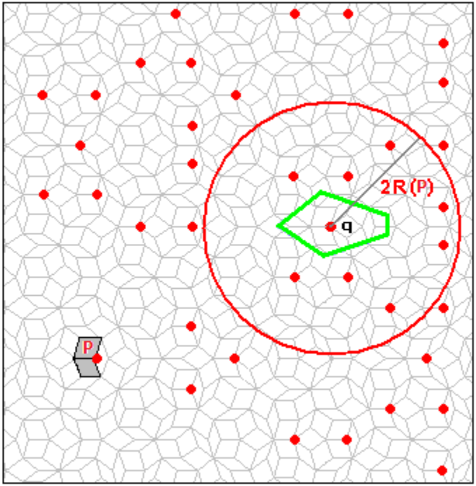


FIGURE – Return tile $M_q = \left(V_q, [B(q, 2R(P))]^T\right)$

Equivalence for return tile

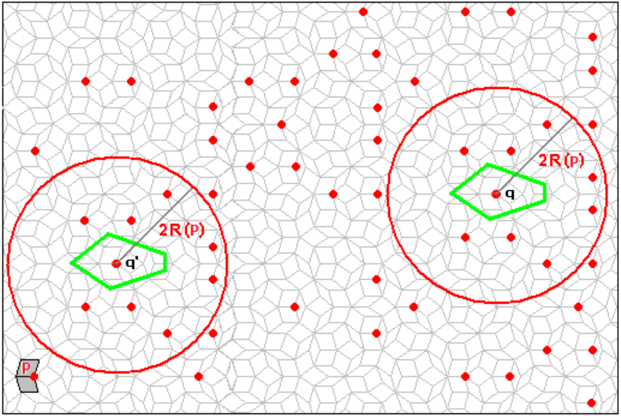


FIGURE – Equivalent return tiles

Hierarchical structure or Week form of self-similarity

we consider the sequence of patches $C_k = [B(0, d_M k)]^T \; \forall\, k \in \mathbb{N} \setminus \{0\}$. Denote by T_k the tiling formed by return tiles associated to the patch C_k.

- By (FLC) the set of possibly return tiles (up translation) in T_k is finite.

Derived Voronoï tiling

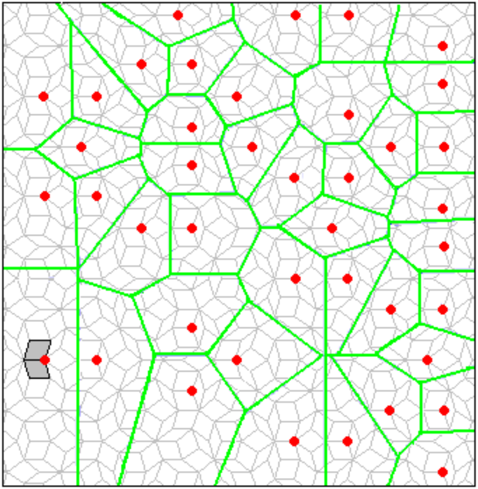


FIGURE – Derived Voronoï tiling for $k = 1$

Derived Voronoï tiling

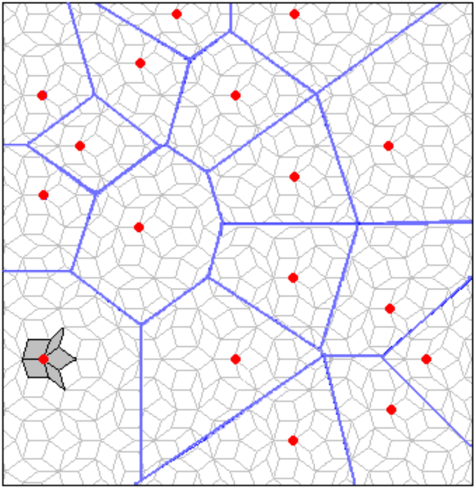
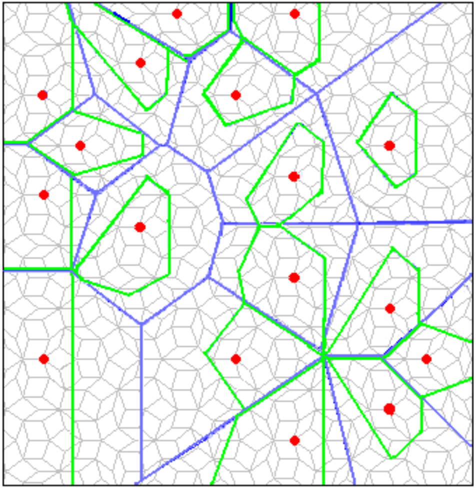


FIGURE – Derived Voronoï tiling for $k = 2$

Derived Voronoï tiling



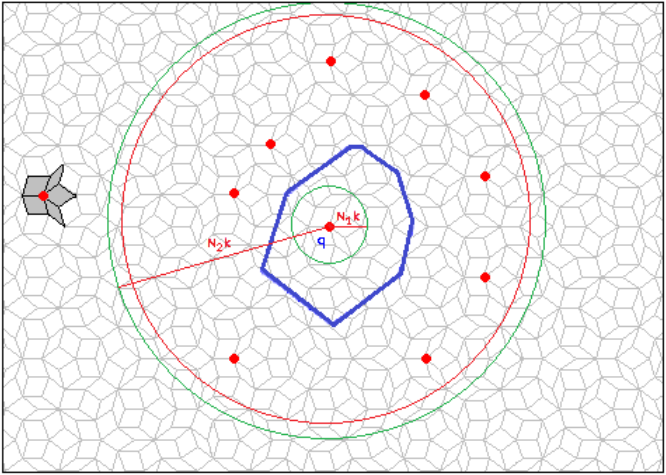
K=1 K=2

FIGURE – Comparaison between tiles from T_1 and T_2

good properties for aperiodic (LR) case

Let T an aperiodic (LR) tiling then $\exists\, N_1 > 0$ and $N_2 > 0$ s.t for all $k \in \mathbb{N}^*$

- $r_{in}(M_q) \geq N_1 k$, (Repulsive property for aperiodic (LR) case)
- $R_{out}(M_q) \leq N_2 k$



Ergodic theorems aperiodic (LR) case

Ergodic theorems

Let T an aperiodic (LR) tiling, and F is a (sub)additive function satisfying

$$(*) \lim_{k\to\infty} \sup_{M_q \sim M'_q \in T_k} \left\{ \frac{\left| F(V_q) - F(V'_q) \right|}{\mathrm{Vol}(V_q)} \right\} = 0$$

Then $\lim_{n\to\infty} \mathrm{Vol}^{-1}(Q_n) F(Q_n) = \overline{F}$

Thank you !