Dimension groups and recurrence for tree subshifts

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Tilings and recurrence, CIRM

Consider the shifts generated by

- a uniformly recurrent Arnoux-Rauzy word
- and a coding of a three-interval exchange

having the same letter frequencies

Consider the shifts generated by

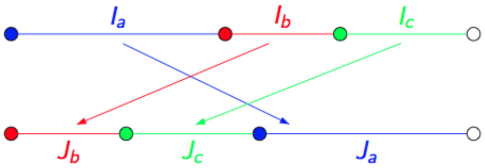
- a uniformly recurrent Arnoux-Rauzy word

  Tribonacci word $\sigma : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$

  The frequencies of letters in $\sigma^\infty(1)$ are $(\alpha, \alpha^2, \alpha^3)$

$$\alpha + \alpha^2 + \alpha^3 = 1$$

- and a coding of a three-interval exchange



having the same letter frequencies

Consider the minimal and uniquely ergodic shifts generated by

- a uniformly recurrent Arnoux-Rauzy word
- and a coding of a three-interval exchange

having the same letter frequencies

They both have factor complexity $2n + 1$

Are the shifts the same?

- They are not topologically conjugate Asymptotic pairs
- But they are orbit equivalent
- They are even strong orbit equivalent

Outline

- Strong orbit equivalence and dimension group
- Tree shifts and extension graphs
- Tree shifts and return words
- Dimension groups of tree shifts

Orbit equivalence

Topological orbit equivalence

Two topological dynamical systems (X, S) and (Y, T) are orbit equivalent if there exists a homeomorphism $h\colon X \to Y$ that sends orbits to orbits

$$h(\mathcal{O}_T(x)) = \mathcal{O}_S(h(x)) \quad \text{for all } x \in X$$

$$h(\{T^n(x) \mid n \in \mathbb{Z}\}) = \{S^n(h(x)) \mid n \in \mathbb{Z}\}$$

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Cocyle map If (X, S) (and hence (Y, T)) are minimal, there exists a unique map

$$n\colon X \to \mathbb{Z} \quad \text{s.t. for all x} \in X \quad h \circ T(x) = S^{n(x)} \circ h(x)$$

There also exists a unique map m

$$m\colon X \to \mathbb{Z}, \quad h \circ T^{m(x)}x = S^m \circ h(x)$$

Strong orbit equivalence The cocyle maps have just one point of discontinuity

Dye's theorem Any two ergodic automorphisms of a Lebesgue space are orbit equivalent in the measure-theoretic sense

Orbit equivalence and measures

Let (X, S) and (Y, T) be uniquely ergodic Cantor systems
Let μ and ν be the corresponding invariant probability measures

Theorem [Giordano-Putman-Skau] (X, S) is orbit equivalent to (Y, T) if and only if

$$\{\mu(E) \,;\text{E clopen subset of } X\} = \{\nu(E) \,;\text{E clopen subset of } Y\}$$

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Corollary The Tribonacci shift is orbit equivalent to the shift generated by the interval exchange of permutation (321) with lengths $(\alpha, \alpha^2, \alpha^3)$, with $\alpha + \alpha^2 + \alpha^3 = 1$

Proof $\langle \mu[w] \text{ ; w factor of the shift} \rangle = \mathbb{Z}[\alpha] \cap [0, 1]$

Strong orbit equivalence

$=$ dimension group

Ordered group

An ordered group is a pair (G, G^+) consisting of an abelian group G together with a subset G^+, called the positive cone, satisfying

$$G^+ + G^+ \subset G^+, \; G^+ \cap (-G^+) = \{0\}, \; G^+ - G^+ = G$$

Ex
- $(\mathbb{Z}^d, \mathbb{N}^d)$
- Continuous functions $(C(X, \mathbb{Z}), C(X, \mathbb{N}))$ for (X, T) a shift

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- $(\mathbb{Z}^d, \mathbb{N}^d, \mathbf{1})$
- Continuous functions $(C(X, \mathbb{Z}), C(X, \mathbb{N}), \mathbf{1})$ for (X, T) a shift

The relation $\leq$ defined by $g \leq h$ if and only if $h - g \in G^+$ is a partial order compatible with the group operation

An order unit of the ordered group G is a nonnegative element $\mathbf{1}$ such that for every $g \in G^+$ there exists $n \in \mathbb{N}$ such that $g \leq n\mathbf{1}$

Dimension group

Definition [Elliot'76] A dimension group is a direct limit of ordered finitely generated free abelian groups (direct sums of copies of $\mathbb{Z}$ ordered in the usual way)

$$\mathbb{Z}^{d_0} \xrightarrow{M_0} \mathbb{Z}^{d_1} \xrightarrow{M_1} \mathbb{Z}^{d_2} \xrightarrow{M_2} \mathbb{Z}^{d_3}$$

Each map $\xrightarrow{M}$ is the multiplication by M with the matrices being nonnegative

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How to decompose a shift (X, T) as an inverse limit?

- Kakutani-Rohlin towers and Bratteli-Vershik maps [Herman-Putman-Skau]
- Return words [Durand-Host-Skau]
- Desubstitution and S-adic representations

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$$C(X,\mathbb{Z}) ☹ \quad \rightsquigarrow \quad C(X,\mathbb{Z})/\beta C(X,\mathbb{Z}) ☺$$

$$\beta\colon C(X,\mathbb{Z}) \to C(X,\mathbb{Z}),\ f \mapsto f \circ T - f$$

Dimension group of a shift

Let (X, T) be a minimal shift

- Coboundaries $\beta\colon C(X, \mathbb{Z}) \to C(X, \mathbb{Z}),\ f \mapsto f \circ T - f$
- Let $H(X, T) = C(X, \mathbb{Z})/\beta C(X, \mathbb{Z})$
- Let $\mathbf{1}$ be the function that takes constant value 1
- The dynamical dimension group of (X, T) is the ordered group with order unit

$$K^0(X, T) := (H(X, T), H^+(X, T), [\mathbf{1}])$$

Dimension groups are complete invariants for strong orbit equivalence

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Theorem [Giordano-Putman-Skau] Let (X,S) and (Y,T) be two minimal Cantor dynamical systems
(X,S) is strong orbit equivalent to (Y,T) iff

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Infinitesimal subgroup of $K^0(X, T)$

$$\mathrm{Inf}(K^0(X, T)) = \left\{[f] \in K^0(X, T)\,;\ \int f d\mu = 0 \text{ for all } \mu \in \mathcal{M}(X, T)\right\}$$

$\mathcal{M}(X, T)$: set of T-invariant probability measures on X

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Dimension group of the Tribonacci shift

Let (X, S) be the Tribonacci shift

Dimension group

$$K^0(X,S) = (H(X,S,\mathbb{Z}), H^+(X,S,\mathbb{Z}), [\mathbf{1}])$$

$$K^0(X,S) \simeq (\mathbb{Z}^3, \{\mathbf{x} \in \mathbb{Z}^3 \mid \langle \mathbf{x}, \mathbf{f} \rangle \geq 0\}, \mathbf{1})$$

with $\mathbf{f} = (\alpha, \alpha^2, \alpha^3)$

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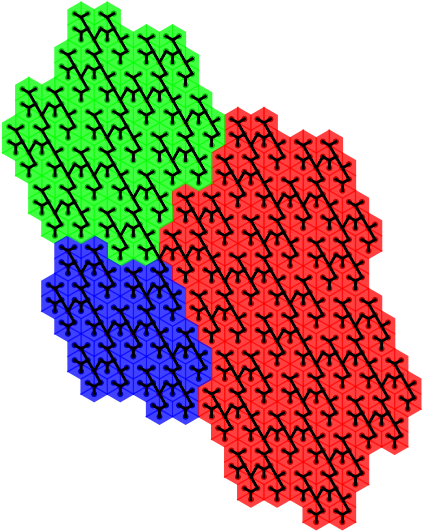
Let (Y, T) be the interval exchange with permutation (321) and lengths $(\alpha, \alpha^2, \alpha^3)$

$$K^0(Y, T) \simeq (\mathbb{Z}^3, \{\mathbf{x} \in \mathbb{Z}^3 \mid \langle \mathbf{x}, \mathbf{f} \rangle \geq 0\}, \mathbf{1})$$

Both shifts are strong orbit equivalent

Tree shifts



Tree words

We study a family of words

tree words

from the viewpoint of word combinatorics

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but we also add more structure

- Topological and measure-theoretic $\rightsquigarrow$ Symbolic dynamics and ergodic theory
- Algebraic $\rightsquigarrow$ From free monoids to free groups Parageometric case [cf. T. Coulbois's lecture]

We want to

- find a common framework for natural generalizations of Sturmian words such as codings of interval exchanges or Arnoux-Rauzy words
- understand the role played by free groups for them
- define generalized continued fraction algorithms [cf. J. Cassaigne's lecture]
  Tool : S-adic expansions via return words

Extension graphs

We consider the set of factors $\mathcal{L}_u$ of $u \in A^{\mathbb{N}}$. Let $w \in \mathcal{L}_u$

$$\ell(w) = \{a \in A \mid aw \in \mathcal{L}_u\}$$

$$r(w) = \{a \in A \mid wa \in \mathcal{L}_u\}$$

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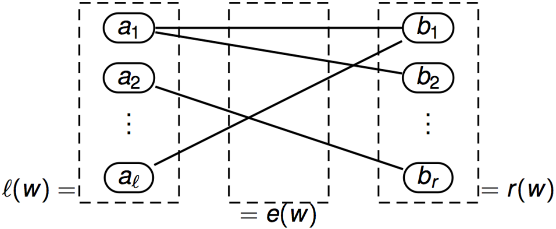
The extension graph of the finite word w is the undirected graph $G(w)$ having

- $\ell(w)$ and $r(w)$ as vertices
- $e(w)$ as edges



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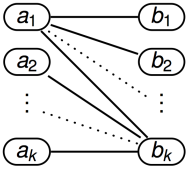

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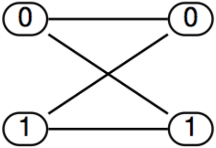
Definition We say that $u \in \mathcal{A}^{\mathbb{N}}$ is a tree word if the graph $G(w)$ is a tree for any $w \in \mathcal{L}_u$

Tree = undirected, acyclic and connected graph

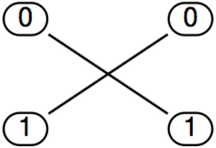
The Thue-Morse word is not a tree word

$$\tau : 0 \mapsto 01,\ 1 \mapsto 10$$

$$u = \tau^{\infty}(0) = 0110100110010110100101100 1\cdots$$



$w = 01$

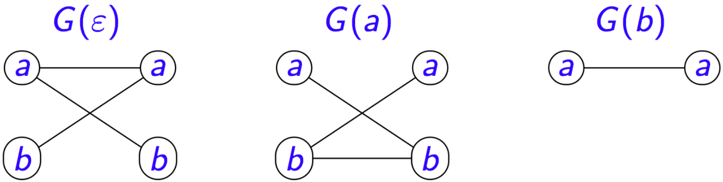


$w = 010$

The Fibonacci word is a tree word

$$\sigma\colon a \mapsto ab,\ \ b \mapsto a$$

$$u = \sigma^{\infty}(a) = abaababaabaababaababaab\cdots$$



The factors of length 2 are aa, ab, ba

Examples of tree words

Theorem [B.,De Felice,Dolce,Leroy, Perrin,Reutenauer,Rindone]
A tree word u on k letters has $(k-1)n+1$ factors of length n

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Sturmian words
and generalizations of Sturmian words on a k-letter alphabet

- Arnoux-Rauzy words Combinatorial generalization

$$l(w) = r(w) = 3$$

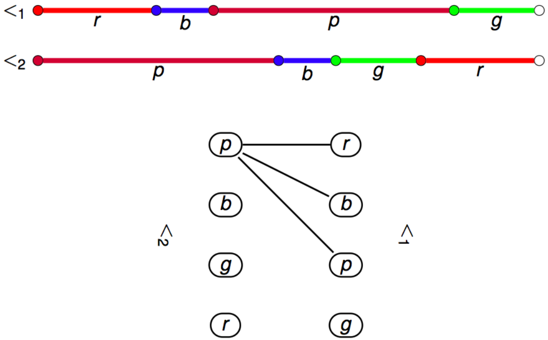
- Codings of interval exchanges Geometric generalization

$l(w) = r(w) = 2$ for w large enough

- Cassaigne-Selmer's substitutions Arithmetic generalization

Interval exchanges are tree words

Extension graph for ε



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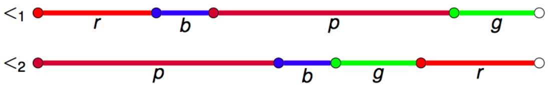

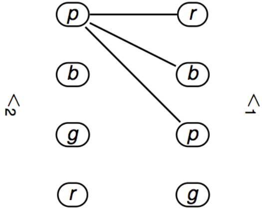

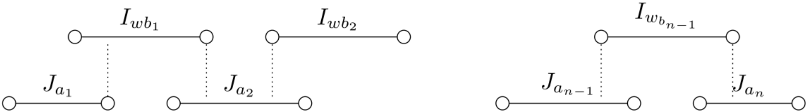

Tree sets and return words

Return words

Let $\mathcal{L}_u$ be the set of factors of $u \in A^{\mathbb{N}}$

We assume u uniformly recurrent

factors occur infinitely often with bounded gaps

Let $w \in \mathcal{L}_u$. A return word to w is a finite word v in $\mathcal{L}_u$ such that wv ends with w and wv contains exactly two occurrences of w

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Example Let $\mathcal{L}_u$ be the set of factors of the Fibonacci word

$a|ba|a|ba|ba|a|ba|a|ba|ba|a|ba|ba|a|ba|a|ba|ba|a|ba|a|ba|ba|a|ba|ba|a\cdots$

a and ba are return words to a

w	a	b	aa	ab	ba
$R_F(w)$	a, ba	ab, aab	$baa, babaa$	ab, aab	ba, aba

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$R_F(aa) = \{baa, babaa\}$ is a basis of the free group on $\{a, b\}$

$$a = (baa)(babaa)^{-1}(baa)$$
$$b = (baa)a^{-1}a^{-1}$$

Tree words and F_d

Let $u \in A^{\mathbb{N}}$ be a uniformly recurrent tree word over an alphabet of cardinality d and let $\mathcal{L}_u$ be the set of its factors.

Theorem [B.,De Felice,Dolce,Leroy,Perrin,Reutenauer,Rindone]
Let w be a factor of u. The set of return words to w is a basis of the free group F_d.

The decoding of a uniformly recurrent tree word u with respect to the return words of a given factor is again a tree word ☺

$\rightsquigarrow$ S-adic expansions

Tree sets are S-adic

Positive automorphism σ of the free group F_A: $\sigma(a) \in A^+, \ \forall a \in A$

Elementary positive automorphisms $\mathcal{S}_e$

- permutations
- $\alpha_{a,b}(a) = ab, \ \alpha_{a,b}(c) = c$ if $c \neq a$
- $\tilde{\alpha}_{a,b}(a) = ba, \ \tilde{\alpha}_{a,b}(c) = c$ if $c \neq a$

Theorem [B.,De Felice,Dolce,Leroy, Perrin,Reutenauer,Rindone] If X is a minimal tree shift over the alphabet A, then it admits a primitive $\mathcal{S}_e$-adic representation

$$u = \lim_{n \to +\infty} \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_n(1), \qquad (\sigma_n)_n \in \mathcal{S}_e^{\mathbb{N}}$$

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Primitive: for all k, there exists ℓ such that $\sigma_k \cdots \sigma_\ell$ is such that the image of any letter contains all the letters

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- Primitive S-adic words u generated by a set of substitutions over a finite alphabet have zero entropy
- But it is possible to construct S-adic words generated by positive elementary automorphisms with high factor complexity among zero entropy words [Cassaigne-Leroy-Pytheas Fogg]

Orbit equivalent tree shifts

Dimension group of a shift

Let (X, T) be a minimal shift

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- Let $H(X, T) = C(X,\mathbb{Z})/\beta C(X,\mathbb{Z})$
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(X,S) is strong orbit equivalent to (Y,T) iff

$$K^0(X,S) \simeq K^0(Y,T)$$

Infinitesimal subgroup of $K^0(X,T)$

$$\mathrm{Inf}(K^0(X,T)) = \left\{[f] \in K^0(X,T)\,;\ \int f d\mu = 0 \text{ for all } \mu \in \mathcal{M}(X,T)\right\}$$

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Image group and infinitesimals

Let $\mathcal{M}(X, T)$ be the set of T-invariant probability measures on X

$$I(X, T) = \bigcap_{\mu \in \mathcal{M}(X,T)} \left\{ \int f d\mu \, ; \, f \in C(X, \mathbb{Z}) \right\}$$

An ordered group with unit: the image subgroup of $K^0(X, T)$

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An ordered group with unit: the image subgroup of $K^0(X,T)$

$$(I(X,T), I(X,T) \cap \mathbb{R}^+, 1)$$

Infinitesimal subgroup of $K^0(X,T)$

$$\text{Inf}(K^0(X,T)) = \left\{ [f] \in K^0(X,T) \, ; \, \int f d\mu = 0 \text{ for all } \mu \in \mathcal{M}(X,T) \right\}$$

If (X,T,μ) is uniquely ergodic

$$I(X,T) = \text{ additive group generated by } \mu[w], \; w \in \mathcal{L}_X$$

$$K^0(X,T)/\text{Inf}(K^0(X,T)) \simeq (I(X,T), I(X,T) \cap \mathbb{R}^+, 1)$$

Infinitesimals of a tree shift: orbit equivalence

Let (X, S, μ) be a uniquely ergodic and minimal tree shift

$$I(X,S) = \left\{ \int f d\mu \,;\, f \in C(X, \mathbb{Z}) \right\}$$

Theorem [B.-Cecchi-Dolce-Durand-Leroy-Perrin-Petite]

$$I(X,S) = \sum_{a \text{ letter in } \mathcal{A}} \mathbb{Z}\mu([a])$$

Proof

- For any $\alpha \in I(X,T) \cap (0,1)$, there exists a clopen set U such that $\alpha = \mu(U)$
- Tree shift and extension graph The measure of any cylinder is in

$$\sum_{a \in \mathcal{A}} \mathbb{Z}\mu([a])$$

Frequencies of letters determine frequencies of factors

$\neq$ Thue-Morse $\mathbb{Z}[1/2]$ dyadic rationals

Tree shifts and invariant measures

Theorem [B.-Cecchi-Dolce-Durand-Leroy-Perrin-Petite] Let (X, T) be a minimal tree shift on a d-letter alphabet $\mathcal{A}$, let μ and μ' be two T-invariant measures on X

If μ and μ' coincide on cylinders associated with letters, then they are equal

$$\mu([a]) = \mu'([a]) \quad \forall a \in \mathcal{A} \Rightarrow \mu(U) = \mu'(U) \quad \forall U \subseteq X \text{ clopen}$$

Dimension group of a tree shift: strong orbit equivalence

Let (X, T) be a minimal tree shift on a d-letter alphabet

Dimension group

$$K^0(X,T) = (H(X,T,\mathbb{Z}), H^+(X,T,\mathbb{Z}), [\mathbf{1}])$$

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- $H(X, T, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^d$
- If moreover (X, T) is uniquely ergodic and has rationally independent letter frequencies, then

$$K^0(X, T) \simeq (\mathbb{Z}^d, \{\mathbf{x} \in \mathbb{Z}^d \mid \langle \mathbf{x}, \mathbf{f} \rangle \geq 0\}, \mathbf{1})$$

f denotes the letter frequency vector

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Proof Description by return words.

Rem For the $\mathbb{Z}^d$ part, one can also use

Theorem [B., De Felice,Dolce,Leroy,Perrin,Reutenauer,Rindone]

For any n, the group described by the Rauzy graph G_n with respect to any vertex is the free group F_A

Orbit equivalent tree shifts

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- Two tree shifts that are minimal and uniquely ergodic with the same additive group of letter frequencies are orbit equivalent
- If the letter frequencies are rationally independent, they are strong orbit equivalent
- All minimal tree shifts on a three-letter alphabet with the same group of letter frequencies are orbit equivalent
- In particular any i.d.o.c. exchange of three intervals is orbit equivalent to any theree-letter Arnoux-Rauzy subshift, or to a two-dimensional Sturmian shift, provided they have all the same letter frequencies

Tree shifts and recognizability

We are given $X \subseteq \mathcal{A}^{\mathbb{Z}}$ and $\sigma : \mathcal{A} \to \mathcal{B}^+$

Centered σ-representation of $y \in \mathcal{B}^{\mathbb{Z}}$

$$y = T^k\sigma(x),\ x \in \mathcal{A}^{\mathbb{Z}},\ 0 \leq k < |\sigma(x_0)|$$

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Theorem [B.-Steiner-Yassawi] Let (X, T) be a minimal tree shift.
Let $(X_{\boldsymbol{\sigma}}, T)$ be a return word S-adic representation of (X, T).
The sequence $\boldsymbol{\sigma}$ is recognizable.
The natural Bratteli-Vershik system associated with $\boldsymbol{\sigma}$ is properly ordered and is topologically conjugate to (X, T). Its topological rank is bounded by the size of the alphabet of X.

Tree shifts

- Minimal tree shifts on a k-letter alphabet have linear factor complexity $(k-1)n+1$
- The class of minimal tree words is closed under decoding with respect to return words
- The set of return words forms a basis of the free group F_A
- S-adic generation by elementary positive automorphisms/ continued fraction algorithms/recognizability
- Orbit equivalence is determined by the group of letter frequencies
- We develop codes and automata theory inside tree languages and provide positive bases of subgroups of the free group F_A.
- Any subgroup of finite index of the free group has a positive basis contained in a tree language
- Find suitable geometric representations for tree shifts cf. [Coulbois-Hilion-Lustig-Minervino] parageometric substitutions/real trees and interval exchanges