Space-filling curve of self-similar sets

(Based on the joint work with
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Classical space-filling curves

- First constructed: G. Peano in 1890.

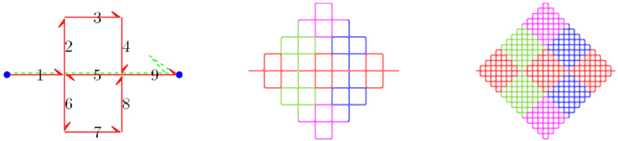
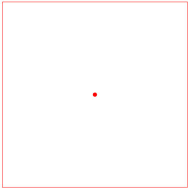


Figure: Peano's construction

- Constructions by D. Hilbert (in 1891), E. H. Moore (in 1900), H. Lebesgue (in 1904), W. Sierpiński (in 1912), G. Pólya (in 1913), ...



Our aims

- To construct space-filling curves (SFC) for self-similar sets
- The SFC has 'nice' properties.

Optimal parametrization

Definition

Let K be a connected, compact set with positive Hausdorff measure. An onto mapping $\psi : [0,1] \to K$ is called an *optimal parametrization* of K if it is

- almost one to one;
- measure preserving (Lebesgue measure vs Hausdorff measure); *i.e.*

$$\mathcal{H}^s(\psi(F)) = c\mathcal{L}(F) \text{ and } \mathcal{L}(\psi^{-1}(B)) = c^{-1}\mathcal{H}^s(B),$$

  for any Borel sets $F \subset [0,1]$ and $B \subset K$, where $c = \mathcal{H}^s(K)$.
- $1/s$-Hölder continuous, that is, there is a constant $c' > 0$ such that

$$|\psi(x) - \psi(y)| \leq c'|x-y|^{\frac{1}{s}} \quad \text{for all } x, y \in [0,1],$$

  where $s = \dim_H K$.

Self-similar set: our object

- Self-similar set K: if there exist similitudes $S_1, \ldots, S_N : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$K = \bigcup_{n=1}^{N} S_n(K).$$

We call it an iterated function system (IFS). (M. Hata, 1985).

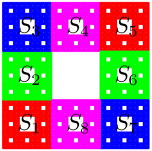


Figure: Sierpiński carpet

- Open set condition (OSC): there exists a nonempty open set U such that

$$\bigcup_{n=1}^{N} S_n(U) \subset U \text{ and } S_i(U) \cap S_j(U) = \emptyset \text{ for } i \neq j.$$

Hata graph

Let K be the invariant set of $\{S_j\}_{j=1}^N$. Let $A \subset K$. The Hata graph $H(A)$ is defined as

- Vertex set: $\{S_1, \ldots, S_N\}$;
- Edge set: there is an edge between S_i and S_j if and only if $S_i(A) \cap S_j(A) \neq \emptyset$.

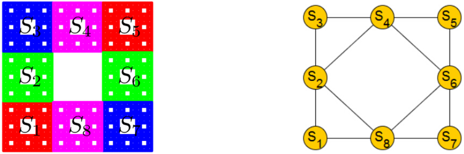


Figure: Hata graph of Sierpiński carpet

Skeletons

A finite subset A of K is called a skeleton, if

- the Hata graph $H(A)$ is connected;
- $A \subset \bigcup_{i=1}^{N} S_i(A)$.

Figure: Skeletons of the Sierpiński carpet.

Main result

Theorem (X. R. Dai, H. Rao, Zhang, Submitted)

Let K be a self-similar set possessing skeletons and satisfying the open set condition. If K is not a Jordan curve, then it admits an optimal parametrization.

To show this, we construct a graph-directed function system from the given self-similar set. Then we can use the following theorem.

Theorem (H. Rao, Zhang, *Nonlinearity*, 2016)

Let $\{E_j\}_{j=1}^N$ be the invariant sets of a linear GIFS satisfying the open set condition with $0 < \mathcal{H}^\delta(E_i) < \infty$ (δ is the similarity dimension). Then all E_j admits optimal parametrizations.

An example: Sierpiński carpet

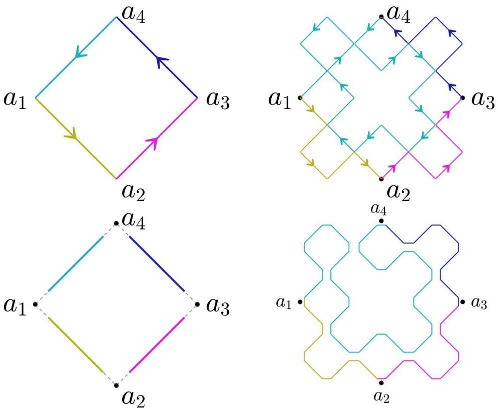


Figure: Sierpiński carpet

An example: Sierpiński carpet (continue)

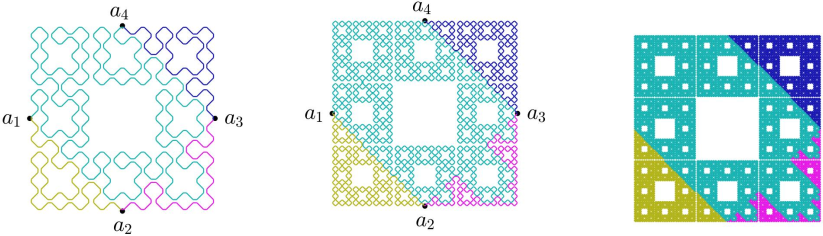


Figure: Sierpiński carpet

Thanks for your attention!