***S*-adic sequences**
A bridge between dynamics, arithmetic, and geometry

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PART 3

S-adic Rauzy fractals and rotations

Contents

The underlying papers

- Berthé, V., Steiner, W., and Thuswaldner, J., Geometry, dynamics, and arithmetic of S-adic shifts, preprint, 2016 (available at `https://arxiv.org/abs/1410.0331`).

- Arnoux, P., Berthé, V., Minervino, M., Steiner, W., and Thuswaldner, J., Nonstationary Markov partitions, flows on homogeneous spaces, and continued fractions, *in preparation*.

S-adic sequence abd S-adic shift

S-adic sequence: For some $a \in \mathcal{A}$ we have

$$w = \lim_{n\to\infty} \sigma_{[0,n)}(a)$$

(this is related to primitivity).

Definition (S-adic shift)

For an S-adic sequence w Let

$$X_w = \overline{\{\Sigma^k w \,:\, k \in \mathbb{N}\}}.$$

$(X_w, \Sigma) = (X_{\boldsymbol{\sigma}}, \Sigma)$ is the S-adic shift (or S-adic system) generated by w.

A result on the way

Theorem

Let $\boldsymbol{\sigma}$ be a sequence of unimodular substitutions with associated sequence of incidence matrices $\mathbf{M}$. If $\mathbf{M}$ is primitive and recurrent, $(X_{\boldsymbol{\sigma}}, \Sigma)$ is minimal and uniquely ergodic.

An Example: Brun substitutions

Lemma

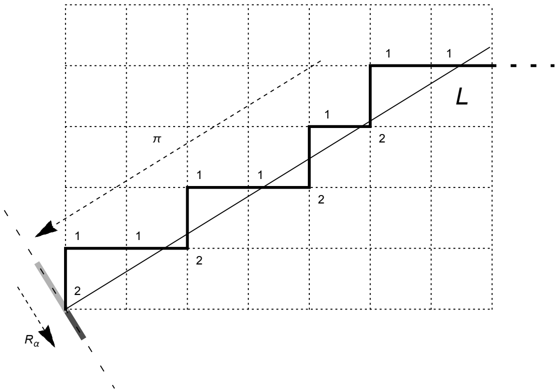
Let $S = \{\sigma_1, \sigma_2, \sigma_3\}$ be the set of Brun substitutions and $\boldsymbol{\sigma} \in S^{\mathbb{N}}$. If $\boldsymbol{\sigma}$ is recurrent and contains the block $(\sigma_3, \sigma_2, \sigma_3, \sigma_2)$ then the associated S-adic system $(X_{\boldsymbol{\sigma}}, \Sigma)$ is minimal and uniquely ergodic.

Proof.

It is immediate that $M_3M_2M_3M_2$ is a strictly positive matrix. Since $\boldsymbol{\sigma}$ is recurrent, it contains the block $(\sigma_3, \sigma_2, \sigma_3, \sigma_2)$ infinitely often. Thus $\boldsymbol{\sigma}$ is primitive and the result follows from the theorem. □

Being recurrent is a generic property.

Looking back to the Sturmian case



- We “see” the rotation on the Rauzy fractal if it has “good” properties.
- It is our aim to establish these properties.

Preparations for the definition

An *S*-adic Rauzy fractal will be defined in terms of a projection to a hyperplane.

- $\mathbf{w} \in \mathbb{R}^d_{\geq 0} \setminus \{\mathbf{0}\}$.
- $\mathbf{w}^\perp = \{\mathbf{x} : \mathbf{x} \cdot \mathbf{w} = 0\}$ orthogonal hyperplane
- $\mathbf{w}^\perp$ is equiped with the Lebesgue measure $\lambda_{\mathbf{w}}$.
- The vector $\mathbf{1} = (1, \ldots, 1)^t$ will be of special interest
- $\mathbf{u}, \mathbf{w} \in \mathbb{R}^d_{\geq 0} \setminus \{\mathbf{0}\}$ noncollinear. Then we denote the projection along $\mathbf{u}$ to $\mathbf{w}^\perp$ by $\pi_{\mathbf{u},\mathbf{w}}$.

S-adic Rauzy fractal

Definition (S-adic Rauzy fractals and subtiles)

Let $\boldsymbol{\sigma}$ be a sequence of unimodular substitutions over the alphabet $\mathcal{A}$ with generalized eigenvector $\mathbf{u} \in \mathbb{R}^d_{>0}$.

Let $(X_{\boldsymbol{\sigma}}, \Sigma)$ be the associated S-adic system.

The S-adic Rauzy fractal (in $\mathbf{w}^{\perp}$, $\mathbf{w} \in \mathbb{R}^d_{\geq 0}$) associated with $\boldsymbol{\sigma}$ is the set

$$\mathcal{R}_{\mathbf{w}} := \overline{\{\pi_{\mathbf{u},\mathbf{w}}\mathbf{l}(p) \ : \ p \text{ is a prefix of a limit sequence of } \boldsymbol{\sigma}\}}.$$

The set $\mathcal{R}_{\mathbf{w}}$ can be naturally covered by the subtiles ($i \in \mathcal{A}$)

$$\mathcal{R}_{\mathbf{w}}(i) := \overline{\{\pi_{\mathbf{u},\mathbf{w}}\mathbf{l}(p) \ : \ pi \text{ is a prefix of a limit sequence of } \boldsymbol{\sigma}\}}.$$

For convenience we set $\mathcal{R}_{\mathbf{1}}(i) = \mathcal{R}(i)$ and $\mathcal{R}_{\mathbf{1}} = \mathcal{R}$.

Illustration of the definition

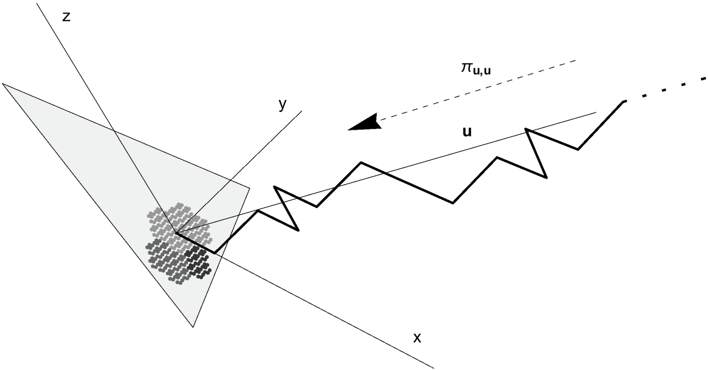


Figure: Definition of $\mathcal{R}_{\mathbf{u}}$ and its subtiles

What we need

We want to “see” the rotation on the Rauzy fractal.

- $\mathcal{R}$ should be bounded.
- $\mathcal{R}$ should be the closure of its interior.
- The boundary $\partial\mathcal{R}$ should have λ_1-measure zero.
- The subtiles $\mathcal{R}(i)$, $i \in \mathcal{A}$, should not overlap on a set of positive measure.
- $\mathcal{R}$ should be the fundamental domain of a lattice, *i.e.*, it can be used as a tile for a lattice tiling.

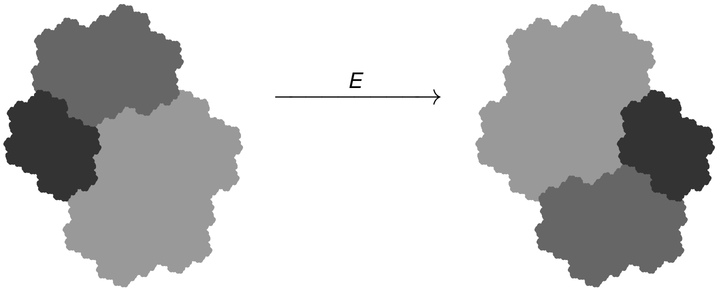
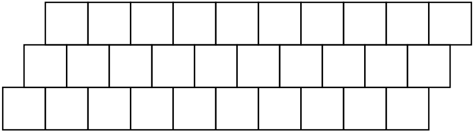


Figure: The domain exchange

Multiple tiling and tiling

Definition (Multiple tiling and tiling)

- Let $\mathcal{K}$ be a collection of subsets of an Euclidean space $\mathcal{E}$.
- Assume that each element of $\mathcal{K}$ is compact and equal to the closure of its interior.
- $\mathcal{K}$ is a multiple tiling if there is $m \in \mathbb{N}$ such that a. e. point (w.r.t. Lebesgue measure) of $\mathcal{E}$ is contained in exactly m elements of $\mathcal{K}$.
- $\mathcal{K}$ is a multiple tiling if $m = 1$.



Discrete hyperplane

- A discrete hyperplane can be viewed as an approximation of a hyperplane by translates of unit hypercubes.
- Pick $\mathbf{w} \in \mathbb{R}^d_{\geq 0} \setminus \{\mathbf{0}\}$ and denote by $\langle \cdot, \cdot \rangle$ the dot product.
- The discrete hyperplanes is defined by

$$\Gamma(\mathbf{w}) = \{[\mathbf{x}, i] \in \mathbb{Z}^d \times \mathcal{A} \ : \ 0 \leq \langle \mathbf{x}, \mathbf{w} \rangle < \langle \mathbf{e}_i, \mathbf{w} \rangle\}$$

(here $\mathbf{e}_i$ is the standard basis vector).

- Interpret the symbol $[\mathbf{x}, i] \in \mathbb{Z}^d \times \mathcal{A}$ as the hypercube or "face"

$$[\mathbf{x}, i] = \left\{ \mathbf{x} + \sum_{j \in \mathcal{A} \setminus \{i\}} \lambda_j \mathbf{e}_j \ : \ \lambda_j \in [0, 1] \right\}.$$

Then the set $\Gamma(\mathbf{w})$ turns into a stepped hyperplane that approximates $\mathbf{w}^{\perp}$ by hypercubes.

Examples of stepped surfaces

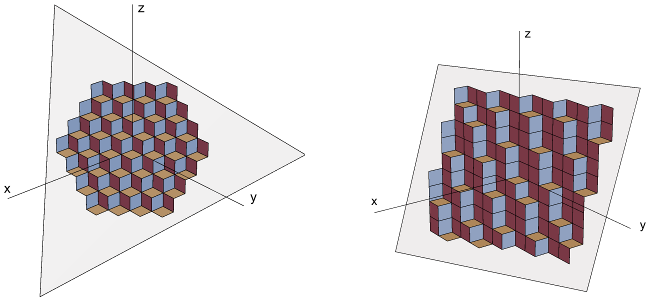


Figure: A subset of a periodic and an aperiodic stepped surface

A finite subset of a discrete hyperplane will be called a patch.

Collections of Rauzy fractals

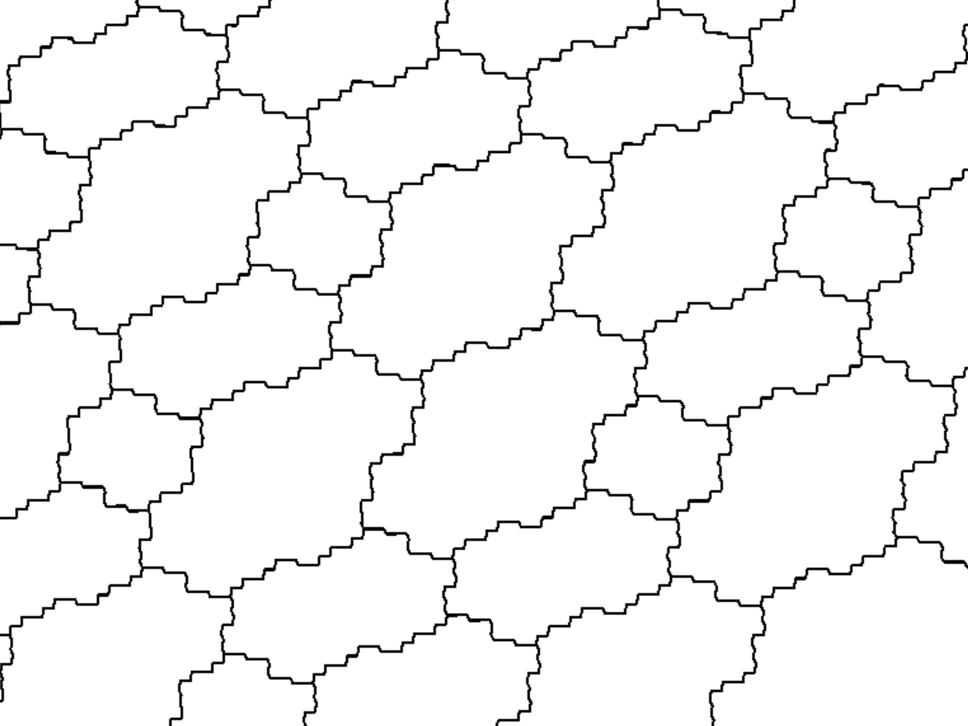
- Using the concept of discrete hyperplane we define the following collections of Rauzy fractals.
- Let $\boldsymbol{\sigma}$ be a sequence of substitutions with generalized eigenvector $\mathbf{u}$ and choose $\mathbf{w} \in \mathbb{R}^d_{\geq 0} \setminus \{\mathbf{0}\}$.

Definition (Collections of Rauzy fractals)

Set

$$\mathcal{C}_\mathbf{w} = \{\pi_{\mathbf{u},\mathbf{w}}\mathbf{x} + \mathcal{R}_\mathbf{w}(i) \ : \ [\mathbf{x}, i] \in \Gamma(\mathbf{w})\}.$$

- We will see that these collections often form a tiling of the space $\mathbf{w}^\perp$.
- A special role will be played by the collection $\mathcal{C}_\mathbf{1}$ which will give rise to a periodic tiling of $\mathbf{1}^\perp$ by lattice translates of the Rauzy fractal $\mathcal{R}$.



Balance

Definition (Balance)

Let $\mathcal{A}$ be an alphabet and consider a pair of words $(u, v) \in \mathcal{A}^* \times \mathcal{A}^*$ of the same length.

- If there is $C > 0$ such that

$$\big||v|_i - |u|_i\big| \leq C$$

  holds for each letter $i \in \mathcal{A}$, the pair (u, v) is called *C-balanced*.
- A language $\mathcal{L}$ is called *C-balanced* if each pair $(u, v) \in \mathcal{L} \times \mathcal{L}$ with $|u| = |v|$ is C-balanced. It is called *finitely balanced* if it is C-balanced for some $C > 0$.
- A sequence $w \in \mathcal{A}^{\mathbb{N}}$ is *C-balanced* if the language $L(w)$ of all finite subwords of w is *C-balanced*.

Balance and boundedness of $\mathcal{R}$

We associate to $\boldsymbol{\sigma} = (\sigma_n)$ a sequence of languages

$$\mathcal{L}_{\boldsymbol{\sigma}}^{(m)} = \{w \in \mathcal{A}^* \ : \ w \text{ is a factor of } \sigma_{[m,n)}(a) \text{ for some } a \in \mathcal{A}, n > m\}$$

and call $\mathcal{L}_{\boldsymbol{\sigma}} = \mathcal{L}_{\boldsymbol{\sigma}}^{(0)}$ the language of $\boldsymbol{\sigma}$.

Lemma

Let $\boldsymbol{\sigma}$ be a primitive and recurrent sequence of unimodular substitution that admits a generalized right eigenvector. Then $\mathcal{R}$ is bounded if and only if $\mathcal{L}_{\boldsymbol{\sigma}}$ is balanced.

Note: $\mathcal{L}_{\boldsymbol{\sigma}}$ is the union of the languages of all limit words of σ. The broken line remains at bounded distance from $\mathbb{R}\mathbf{u}$ if and only if this language is balanced.

Rational independence

- The definition is defined as (the closure of) the projection of some lattice points along the generalized right vector **u**.
- Our goal is to have a Rauzy fractal with nonempty interior.
- If there is an integer vector $\mathbf{z} \in \mathbb{Z}^d$ such that $\langle \mathbf{u}, \mathbf{z} \rangle = 0$, the projection $\pi_{\mathbf{u},\mathbf{w}}(\mathbb{Z}^d)$ is contained in $(d-2)$-dimensional affine subspaces of $\mathbf{w}^\perp$: no hope for nonempty interior

Definition (Rational indepencence)

A vector $\mathbf{u} \in \mathbb{R}^d$ is called rationally independent if the only $\mathbf{z} \in \mathbb{Z}^d$ satisfying $\langle \mathbf{u}, \mathbf{z} \rangle = 0$ is the vector $\mathbf{z} = \mathbf{0}$.

Algebraic irreducibility

We need to exclude generalized right eigenvectors that are rationally dependent. This requires a condition.

Definition (Algebraic irreducibility)

Let $\mathbf{M} = (M_n)$ be a sequence of nonnegative matrices in $GL_d(\mathbb{Z})$. We say that $\mathbf{M}$ is algebraically irreducible if for each $m \in \mathbb{N}$ there is $n > m$ such that the characteristic polynomial of $M_{[m,\ell)}$ is irreducibly for each $\ell \geq n$.
A sequence $\boldsymbol{\sigma}$ of unimodular substitutions is called algebraically irreducible if it has a sequence of incidence matrices which is algebraically irreducible.

Pisot

In our setting these polynomials are even Pisot polynomials. This is related to convergence properties of generalized continued fraction algorithms.

A key lemma...

Algebraic irreducibility yields the desired property.

Lemma

Let σ be an algebraically irreducible sequence of unimodular substitutions with balanced language $\mathcal{L}_\sigma$ that admits a generalized eigenvector $\mathbf{u}$. Then $\mathbf{u}$ has rationally independent coordinates.

A stronger form of convergence

- So far we defined weak convergence.
- We need a stronger form: strong convergence.
- Sturmian case: the cascade of inductions we perform on the interval leads to smaller and smaller intervals whose lengths tend to 0.
- To get an analogous behaviour on S-adic Rauzy fractals we need to introduce a certain subdivision on them whose pieces have a diameter that tends to zero.
- Strong convergence is well-known in the theory of generalized continued fractions.

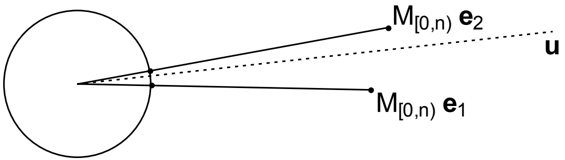
Strong convergence and its consequences

Definition (Strong convergence)

We say that a sequence $\mathbf{M} = (M_n)$ of matrices from $GL_d(\mathbb{Z})$ admit strong convergence to $\mathbf{u} \in \mathbb{R}^d_{\geq 0} \setminus \{0\}$ if

$$\lim_{n\to\infty} \pi_{\mathbf{u},\mathbf{1}} M_{[0,n)} \mathbf{e}_i = \mathbf{0} \quad \text{for all } i \in \mathcal{A}.$$

If σ has a strongly convergent sequence of incidence matrices we say that σ admits strong convergence.



Lemma

Let $\boldsymbol{\sigma}$ be a primitive, algebraically irreducible, and recurrent sequence of substitutions with balanced language $\mathcal{L}_{\boldsymbol{\sigma}}$. Then

$$\lim_{n\to\infty} \sup\{||\pi_{\mathbf{u},\mathbf{1}} M_{[0,n)} \mathbf{l}(v)|| \ : \ v \in \mathcal{L}_{\boldsymbol{\sigma}}^{(n)}\} = 0.$$

By primitivity this implies that $\boldsymbol{\sigma}$ is strongly convergent, i.e.,

$$\lim_{n\to\infty} \sup\{||\pi_{\mathbf{u},\mathbf{1}} M_{[0,n)} \mathbf{e}_i|| \ : \ v \in \mathcal{L}_{\boldsymbol{\sigma}}^{(n)}\} = 0$$

for each $i \in \mathcal{A}$.

Our goal

Theorem (Properties of Rauzy fractals)

Let S be a finite set of unimodular substitutions over a finite alphabet $\mathcal{A}$ and let $\boldsymbol{\sigma} = (\sigma_n)$ be a primitive and algebraically irreducible sequence of substitutions taken from the set S. Assume that there is $C > 0$ such that for every $\ell \in \mathbb{N}$ there exists $n \geq 1$ such that $(\sigma_n, \ldots, \sigma_{n+\ell-1}) = (\sigma_0, \ldots, \sigma_{\ell-1})$ and the language $\mathcal{L}_{\boldsymbol{\sigma}}^{(n+\ell)}$ is C-balanced.

Then each subtile $\mathcal{R}(i)$, $i \in \mathcal{A}$, of the Rauzy fractal $\mathcal{R}$ is a nonempty compact set which is equal to the closure of its interior and has a boundary whose Lebesgue measure λ_1 is zero.

Rauzy fractals of shifted sequences

Definition

For $k \in \mathbb{N}$ let

- projection at level k:

$$\pi^{(n)}_{\mathbf{u},\mathbf{w}} = \pi_{M^{-1}_{[0,n)}\mathbf{u}, M^{t}_{[0,n)}\mathbf{w}},$$

- Subtiles of the shifted sequence of substitutions $(\sigma_{n+k})_{n\in\mathbb{N}}$ projected to $M^{t}_{[0,n)}\mathbf{w}$:

$$\mathcal{R}^{(k)}_{\mathbf{w}}(i) := \overline{\{\pi^{(k)}_{\mathbf{u},\mathbf{w}}(\mathbf{l}p') \ : \ p'j \text{ prefix of a limit word of } (\sigma_{n+k})\}},$$

  Rauzy fractal of the shifted sequence of substitutions $(\sigma_{n+k})_{n\in\mathbb{N}}$

$$\mathcal{R}^{(k)}_{\mathbf{w}} = \bigcup_{i\in\mathcal{A}} \mathcal{R}^{(k)}_{\mathbf{w}}(i).$$

The set equation

Lemma (The set equation)

Let $\boldsymbol{\sigma}$ be a primitive and recurrent sequence of unimodular substitutions with generlaized right eigenvalue $\mathbf{u}$. *Then for each* $[\mathbf{x}, i] \in \mathbb{Z}^d \times \mathcal{A}$ *and every* k, ℓ *with* $k < \ell$ *we have*

$$\pi_{\mathbf{u},\mathbf{w}}^{(k)}\mathbf{x} + \mathcal{R}_{\mathbf{w}}^{(k)}(i) = \bigcup_{[\mathbf{y},j] \in E_1^*(\sigma_{[k,\ell)})[\mathbf{x},i]} M_{[k,\ell)}(\pi_{\mathbf{u},\mathbf{w}}^{(\ell)}\mathbf{y} + \mathcal{R}_{\mathbf{w}}^{(\ell)}(j)),$$

where

$$E_1^*(\sigma)[\mathbf{x}, i] = \{[M_\sigma^{-1}(\mathbf{x} + \mathbf{l}p), j] \ : \ j \in \mathcal{A},\ p \in \mathcal{A}^* \text{ such that } pi \text{ is a prefix of } \sigma(j)\}.$$

$E_1^*(\sigma)[\mathbf{x}, i]$ is the dual of the geometric realization of a substitution.

$E_1^*(\sigma)$

- Let σ be the Tribonacci substitution.

$$\sigma(1) = 12, \quad \sigma(2) = 13, \quad \sigma(3) = 1.$$

Then

$$E_1^*(\sigma)[\mathbf{0}, 1] = \{[\mathbf{0}, 1], [\mathbf{0}, 2], [\mathbf{0}, 3]\},$$
$$E_1^*(\sigma)[\mathbf{0}, 2] = \{[(0, 0, 1)^t, 1]\},$$
$$E_1^*(\sigma)[\mathbf{0}, 3] = \{[(0, 0, 1)^t, 2]\}.$$

together with the obvious fact that
$E_1^*(\sigma)[\mathbf{x}, i] = M_\sigma^{-1}\mathbf{x} + E_1^*(\sigma)[\mathbf{0}, i]$.

- One can extend the definition of $E_1^*(\sigma)$ to subsets of $\mathbb{Z}^d \times \mathcal{A}$ in a natural way. We can then iterate $E_1^*(\sigma)$.

Iterating $E_1^*(\sigma)$

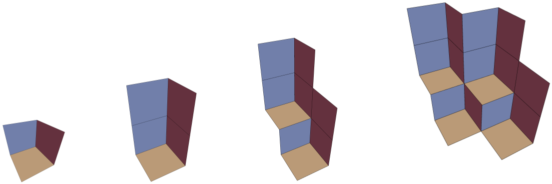
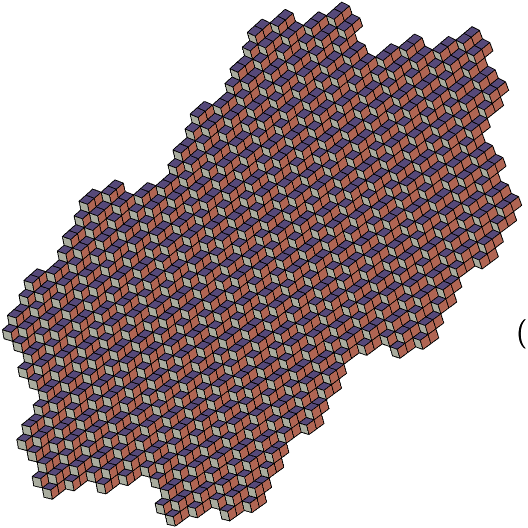


Figure: The iterates $(E_1^*(\sigma))^k[\mathbf{0}, 1]$ for $k \in \{1, 2, 3, 4\}$

$E_1^*(\sigma)$ generates the Rauzy fractal



$(E_1^*(\sigma))^{12}[\mathbf{0}, 1]$

Mapping properties

Set

$$\mathbf{u}^{(n)} = (M_{[0,n)})^{-1}\mathbf{u}, \qquad \mathbf{w}^{(n)} = (M_{[0,n)})^{t}\mathbf{w} \qquad (n \in \mathbb{N}).$$

Lemma

Let $\boldsymbol{\sigma} = (\sigma)_n$ be a sequence of unimodular substitutions. Then for all $k < \ell$ the following assertions hold.

(i) $M_{[k,\ell)}(\mathbf{w}^{(\ell)})^{\perp} = (\mathbf{w}^{(k)})^{\perp}$,

(ii) $E_1^*(\sigma_{[k,\ell)})\Gamma(\mathbf{w}^{(k)}) = \Gamma(\mathbf{w}^{(\ell)})$,

(iii) *for distinct pairs* $[\mathbf{x}, i], [\mathbf{x}', i'] \in \Gamma(\mathbf{w}^{(k)})$ *the images* $E_1^*(\sigma_{[k,\ell)})[\mathbf{x}, i]$ *and* $E_1^*(\sigma_{[k,\ell)})[\mathbf{x}', i']$ *are disjoint.*

Set equation for collections

Set

$$\mathcal{C}_{\mathbf{w}}^{(n)} = \{\pi_{\mathbf{u},\mathbf{w}}\mathbf{x} + \mathcal{R}_{\mathbf{w}}^{(n)}(i) \ : \ [\mathbf{x}, i] \in \Gamma(\mathbf{w}^{(n)})\}.$$

for the collection of the subtiles associated with the shifted sequences of $\boldsymbol{\sigma}$.

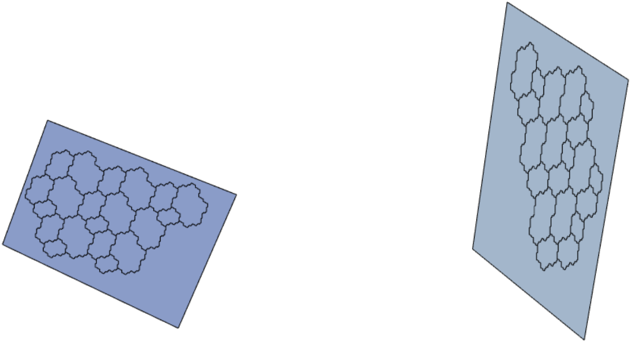
Lemma

Let $\boldsymbol{\sigma}$ be a primitive and recurrent sequence of unimodular substitutions with generlaized right eigenvector $\mathbf{u}$. Then for each $[\mathbf{x}, i] \in \mathbb{Z}^d \times \mathcal{A}$ and every $k, \ell \in \mathbb{N}$ with $k < \ell$ we have

$$\bigcup_{[\mathbf{y},j]\in\Gamma(\mathbf{w}^{(k)})} \pi_{\mathbf{u},\mathbf{w}}\mathbf{x} + \mathcal{R}_{\mathbf{w}}^{(k)}(i) = \bigcup_{[\mathbf{y},j]\in\Gamma(\mathbf{w}^{(\ell)})} M_{[k,\ell)}(\pi_{\mathbf{u},\mathbf{w}}^{(\ell)}\mathbf{y} + \mathcal{R}_{\mathbf{w}}^{(\ell)}(j)).$$

The collection $M_{[k,\ell)}\mathcal{C}_{\mathbf{w}}^{(\ell)}$ is a refinement of $\mathcal{C}_{\mathbf{w}}^{(k)}$ in the sense that each element of the latter is a finite union of elements of the former.

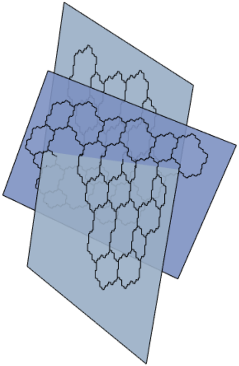
Illustration of the set equation I



(a) (b)

Figure: An illustration of the set equation. (a) shows a patch P_0 of the collection $\mathcal{C}_{\mathbf{w}} = \mathcal{C}_{\mathbf{w}}^{(0)}$, (b) contains a patch P_1 of $\mathcal{C}_{\mathbf{w}}^{(1)}$.

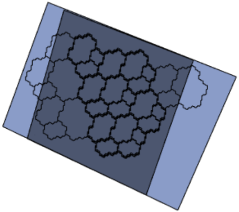
Illustration of the set equation II



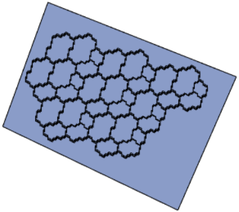
(c)

Figure: An illustration of the set equation. In (c) P_0 an P_1 are drawn together to illustrate that they lie in different planes.

Illustration of the set equation III



(d)



(e)

Figure: An illustration of the set equation. In (d) the matrix M_0 is applied to P_1: the image $M_0 P_1$ is located in the same plane as P_0 and, according to the set equation, forms a subdivision of some tiles of P_0. The subdivision of P_0 in patches from $M_0 \mathcal{C}_{\mathbf{w}}^{(1)}$ is shown in (e) for the whole patch P_0.

Closure of interior

Lemma

Let $\boldsymbol{\sigma}$ be a sequence of unimodular substitutions and $\mathbf{w} \in \mathbb{R}_{\geq 0} \setminus \{\mathbf{0}\}$. If $\boldsymbol{\sigma}$ is primitive, recurrent, algebraically irreducible, and has balanced language $\mathcal{L}_{\boldsymbol{\sigma}}$ then $\mathcal{C}_{\mathbf{w}}^{(n)}$ covers $(\mathbf{w}^{(n)})^{\perp}$ with finite covering degree. The covering degree of $\mathcal{C}_{\mathbf{w}}^{(n)}$ increases monotonically with n.

Lemma

Let $\boldsymbol{\sigma}$ be a sequence of unimodular substitutions over the alphabet $\mathcal{A}$ and $\mathbf{w} \in \mathbb{R}_{\geq 0} \setminus \{\mathbf{0}\}$. If $\boldsymbol{\sigma}$ is primitive, recurrent, algebraically irreducible, and has balanced language $\mathcal{L}_{\boldsymbol{\sigma}}$. Then $\mathcal{R}(i)$ is the closure of its interior for each $i \in \mathcal{A}$.

Generalized left eigenvector

- The collections $\mathcal{C}_{\mathbf{w}}^{(n)}$ lie in different hyperplanes
- By recurrence there is a sequence (n_k) and a generalized left vector $\mathbf{v}$ such that the following properties hold.

Lemma

- *$\mathcal{R}_{\mathbf{v}}$ and $\mathcal{R}_{\mathbf{v}}^{(n_k)}$ have the same subdivision structure for a long time.*
- *The hyperplanes $\mathbf{v}^{\perp}$ and $(\mathbf{v}^{(n_k)})^{\perp}$ with $\mathbf{v}^{(n_k)} = M_{[0,n_k)}^t \mathbf{v}$ are close to each other for large k.*
- *$\mathcal{R}_{\mathbf{v}}^{(n_k)}$ tends to $\mathcal{R}_{\mathbf{v}}$ in Hausdorff metric.*
- *$\mathcal{C}_{\mathbf{w}}$ and $\mathcal{C}_{\mathbf{w}}^{(n)}$ have the same covering degree for each $n \in \mathbb{N}$.*

Measure of the boundary I

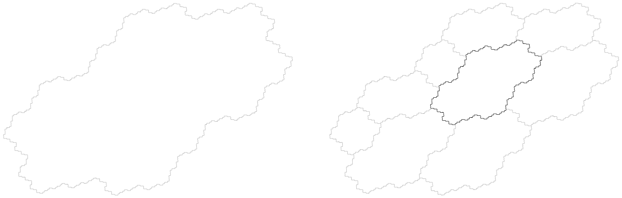
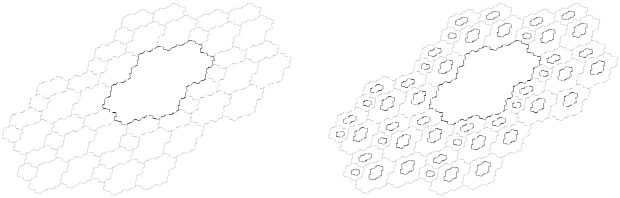

(a) (b)

Figure: Illustration of the proof of measure zero of $\partial\mathcal{R}$. In (a) the subtile $\mathcal{R}(i)$, $i \in \mathcal{A}$, is shown. In (b) we see the ℓ-th subdivision of $\mathcal{R}(i)$. The level ℓ subtile contained in $\mathrm{int}(\mathcal{R}(i))$ has black boundary.

Measure of the boundary II



(c) (d)

Figure: Illustration of the proof of measure zero of $\partial\mathcal{R}$. In (c) all other level ℓ subtiles are further subdivided in level n_k subtiles. Each of them contains a level $n_k + \ell$ subtile in its interior. These level $n_k + \ell$ subtiles, which *a fortiori* are also contained in $\mathrm{int}(\mathcal{R}(i))$, are depicted in (d) also with black boundary.

$\partial\mathcal{R}$ has measure 0

Lemma

Let $\boldsymbol{\sigma}$ be a sequence of unimodular substitutions over the alphabet $\mathcal{A}$ and $\mathbf{w} \in \mathbb{R}_{\geq 0} \setminus \{\mathbf{0}\}$. If $\boldsymbol{\sigma}$ is primitive, recurrent, algebraically irreducible, and has balanced language $\mathcal{L}_{\boldsymbol{\sigma}}$. Then $\lambda_1(\partial\mathcal{R}(i)) = 0$ for each $i \in \mathcal{A}$.

Remember what we want:

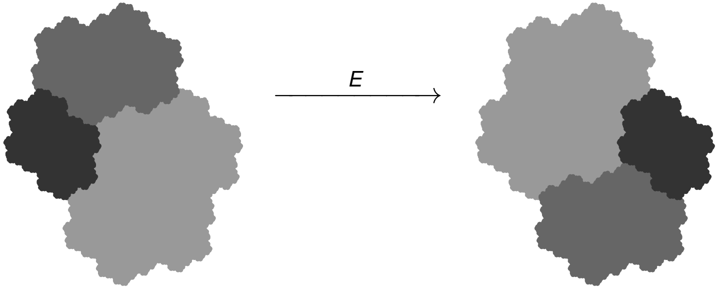


Figure: The domain exchange and rotation

Three kinds of tilings

We have three unions which we want to be disjoint in measure:

(i) The unions of subtiles on the right hand side of the set equation

$$\pi_{\mathbf{u},\mathbf{w}}^{(k)}\mathbf{x} + \mathcal{R}_{\mathbf{w}}^{(k)}(i) = \bigcup_{[\mathbf{y},j]\in E_1^*(\sigma_{[k,\ell)})[\mathbf{x},i]} M_{[k,\ell)}(\pi_{\mathbf{u},\mathbf{w}}^{(\ell)}\mathbf{y} + \mathcal{R}_{\mathbf{w}}^{(\ell)}(j)),$$

(ii) The union

$$\mathcal{R} = \mathcal{R}(1) \cup \cdots \cup \mathcal{R}(d).$$

(iii) The union

$$\mathbf{1}^{\perp} = \bigcup_{\mathbf{x}\in\mathbb{Z}^d\,:\,\mathbf{x}\cdot\mathbf{1}=0} \pi_{\mathbf{u},\mathbf{1}}\mathbf{x} + \mathcal{R} = \bigcup_{[\mathbf{x},i]\in\Gamma(\mathbf{1})} \pi_{\mathbf{u},\mathbf{1}}\mathbf{x} + \mathcal{R}(i).$$

For (i) we can prove the disjointness. For (ii) and (iii) we derive it from (i) by additional coincidence conditions.

Disjointness of unions in the set equation

Lemma

Let $\boldsymbol{\sigma}$ be a primitive and algebraically irreducible sequence of unimodular substitutions. Assume that there is $C > 0$ such that for every $\ell \in \mathbb{N}$ there exists $n \geq 1$ such that $(\sigma_n, \ldots, \sigma_{n+\ell-1}) = (\sigma_0, \ldots, \sigma_{\ell-1})$ and the language $\mathcal{L}_{\boldsymbol{\sigma}}^{(n+\ell)}$ is C-balanced.
Then the unions in the set equation

$$\pi_{\mathbf{u},\mathbf{w}}^{(k)}\mathbf{x} + \mathcal{R}_{\mathbf{w}}^{(k)}(i) = \bigcup_{[\mathbf{y},j]\in E_1^*(\sigma_{[k,\ell)})[\mathbf{x},i]} M_{[k,\ell)}(\pi_{\mathbf{u},\mathbf{w}}^{(\ell)}\mathbf{y} + \mathcal{R}_{\mathbf{w}}^{(\ell)}(j)),$$

are disjoint in measure.

Follows because all collections $\mathcal{C}_{\mathbf{w}}^{(n)}$ have the same covering degree.

Definition (Strong coincidence condition)

A sequence $\boldsymbol{\sigma}$ of substitutions over an alphabet $\mathcal{A}$ satisfies the *strong coincidence condition* if there is $\ell \in \mathbb{N}$ such that for each pair $(j_1, j_2) \in \mathcal{A}^2$ there are $i \in \mathcal{A}$ and $p_1, p_2 \in \mathcal{A}^*$ with $\mathbf{l}p_1 = \mathbf{l}p_2$ such that $\sigma_{[0,\ell)}(j_1) \in p_1 i \mathcal{A}^*$ and $\sigma_{[0,\ell)}(j_1) \in p_2 i \mathcal{A}^*$.

Example

Coincidence for $\boldsymbol{\sigma} = (\sigma)$ with $\sigma(1) = 121$, $\sigma(2) = 21$.

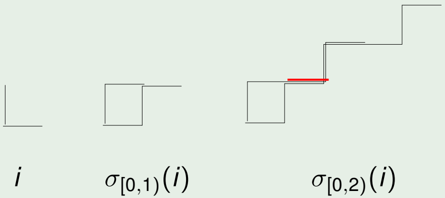


Figure: Coincidence is indicated by the red line

Measure disjointness of the subtiles $\mathcal{R}(i)$

Lemma

Let $\boldsymbol{\sigma}$ be a primitive and algebraically irreducible sequence of unimodular substitutions. Assume that there is $C > 0$ such that for every $\ell \in \mathbb{N}$ there exists $n \geq 1$ such that $(\sigma_n, \ldots, \sigma_{n+\ell-1}) = (\sigma_0, \ldots, \sigma_{\ell-1})$ and the language $\mathcal{L}_{\boldsymbol{\sigma}}^{(n+\ell)}$ is C-balanced.
If the strong coincidence condition holds then $\mathcal{R}(i)$, $i \in \mathcal{A}$, are disjoint in measure.

Geometric coincidence and geometric finiteness

Definition (Geometric coincidence and geometric finiteness)

A sequence $\boldsymbol{\sigma}$ of unimodular substitutions over an alphabet $\mathcal{A}$ satisfies

- the geometric coincidence condition if the following is true. For each $R > 0$ there is $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ the set $E_1^*(\sigma_{[0,n)})[\mathbf{0}, i_n]$ contains a ball of radius R of the discrete hyperplane $\Gamma(M_{[0,n)}^t \mathbf{1})$ for some $i_n \in \mathcal{A}$.
- the geometric finiteness condition if the followng is true. For each $R > 0$ there is $\ell \in \mathbb{N}$ such that $\bigcup_{i \in \mathcal{A}} E_1^*(\sigma_{[0,n)})[\mathbf{0}, i]$ contains the ball $\{[\mathbf{x}, i] \in \Gamma({}^t(M_{[0,n)})\,\mathbf{1}) : \|\mathbf{x}\| \leq R\}$ for all $n \geq \ell$.

A tiling result

Lemma

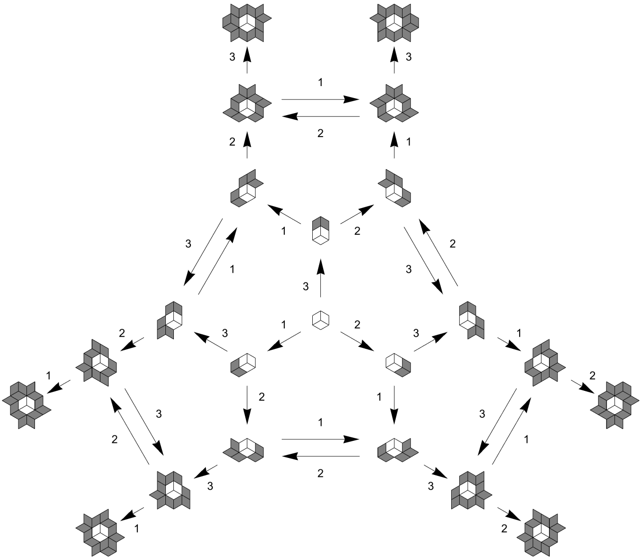
Let $\boldsymbol{\sigma}$ be a primitive and algebraically irreducible sequence of unimodular substitutions. Assume that there is $C > 0$ such that for every $\ell \in \mathbb{N}$ there exists $n \geq 1$ such that $(\sigma_n, \ldots, \sigma_{n+\ell-1}) = (\sigma_0, \ldots, \sigma_{\ell-1})$ and the language $\mathcal{L}_{\boldsymbol{\sigma}}^{(n+\ell)}$ is C-balanced. Then the following assertions are equivalent.

(i) *The collection $\mathcal{C}_{\mathbf{1}}$ forms a tiling of $\mathbf{1}^{\perp}$.*

(ii) *The sequence $\boldsymbol{\sigma}$ satisfies the geometric coincidence condition.*

(iii) *The sequence $\boldsymbol{\sigma}$ satisfies the strong coincidence condition and for each $R > 0$ there exists $n_0 \in \mathbb{N}$ such that $\bigcup_{i \in \mathcal{A}} E_1^*(\sigma_{[0,n)})[\mathbf{0}, i]$ contains a ball of radius R of $\Gamma({}^t(M_{[0,n)})\,\mathbf{1})$ for all $n \geq n_0$.*

How to check this? (Berthe, Jolivet, Siegel 2012)



Known classes with geometric finiteness

- For Arnoux-Rauzy sequences σ which contain each of the three Arnoux-Rauzy substitutions infinitely often.
- For Brun sequences σ which contain each of the three Brun substitutions infinitely often.
- For certain sequences σ of substitutions related to the Jacobi-Perron continued fraction algorithm.

We refer to Berthé, Bourdon, Jolivet, and Siegel (2012,1016)

Main theorem: assumptions

Assumptions

- Let S be a finite set of unimodular substitutions over a finite alphabet $\mathcal{A}$.
- Let $\boldsymbol{\sigma} = (\sigma_n)$ be a primitive and algebraically irreducible sequence of substitutions taken from the set $S^{\mathbb{N}}$.
- Assume that there is $C > 0$ such that for every $\ell \in \mathbb{N}$ there exists $n \geq 1$ such that $(\sigma_n, \ldots, \sigma_{n+\ell-1}) = (\sigma_0, \ldots, \sigma_{\ell-1})$ and the language $\mathcal{L}_{\boldsymbol{\sigma}}^{(n+\ell)}$ is C-balanced.
- Assume that the collection $\mathcal{C}_{\mathbf{1}}$ forms a tiling of $\mathbf{1}^{\perp}$.

Main theorem: Assertions

Assertions

1. The S-adic shift $(X_{\sigma}, \Sigma, \mu)$ is measurably conjugate to a translation T on the torus $\mathbb{T}^{d-1}$.
2. Each $\omega \in X_{\sigma}$ is a natural coding of the toral translation T with respect to the partition $\{\mathcal{R}(i) : i \in \mathcal{A}\}$.
3. The set $\mathcal{R}(i)$ is a bounded remainder set for the toral translation T for each $i \in \mathcal{A}$.

"Proof"

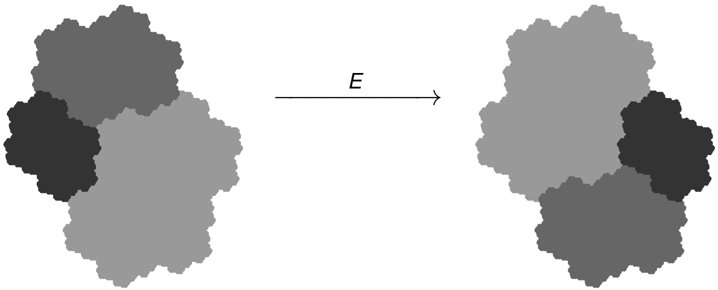


Figure: The domain exchange and rotation

Lyapunov exponents

- Shift $(S^{\mathbb{N}}, \Sigma, \nu)$ with ν an ergodic probability measure.
- With each $\gamma = (\gamma_n)_{n\in\mathbb{N}} \in S^{\mathbb{N}}$, associate the linear cocycle operator $A(\gamma) = {}^tM_0$
- Then the Lyapunov exponents $\theta_1, \theta_2, \ldots, \theta_d$ of $(S^{\mathbb{N}}, \Sigma, \nu)$ are recursively defined by

$$\begin{aligned}
\theta_1 + \theta_2 + \cdots + \theta_k &= \\
&= \lim_{n\to\infty} \frac{1}{n} \int_{E_G} \log \| \wedge^k \left(A(\Sigma^{n-1}(x)) \cdots A(\Sigma(x))A(x)\right) \| \, d\nu(x) \\
&= \lim_{n\to\infty} \frac{1}{n} \int_{E_G} \log \| \wedge^k ({}^tM_{[0,n)}) \| \, d\nu \\
&= \lim_{n\to\infty} \frac{1}{n} \int_{E_G} \log \| \wedge^k M_{[0,n)} \| \, d\nu
\end{aligned}$$

for $1 \leq k \leq d$, where $\wedge^k$ denotes the k-fold wedge product.

Pisot condition

Definition (Pisot condition)

We say that $(S^{\mathbb{N}}, \Sigma, \nu)$ satisfies the Pisot condition if

$$\theta_1 > 0 > \theta_2 \geq \theta_3 \geq \cdots \geq \theta_d.$$

This means that a typical sequence of substitution from $S^{\mathbb{N}}$ has a sequence of incidence matrices that on the long run is expanding in one direction and contracting in on a hyperplane. It is thus exponentially convergent to one direction.

Metric theory

Theorem

Let S be a finite set of unimodular substitutions and consider the shift $(S^{\mathbb{N}}, \Sigma, \nu)$. Assume that this shift is ergodic and that it satisfies the Pisot condition. Assume further that ν assigns positive measure to each (non-empty) cylinder, and that there exists a cylinder corresponding to a substitution with positive incidence matrix. Then for a.a. sequences $\boldsymbol{\sigma} \in S^{\mathbb{N}}$ the assertions of the above theorem hold provided that $\mathcal{C}_{\mathbf{1}}$ forms a tiling of $\mathbf{1}^{\perp}$. In particular, the S-adic system $(X_{\boldsymbol{\sigma}}, \Sigma)$ is measurably conjugate to a rotation on $\mathbb{T}^{d-1}$.

Arnoux-Rauzy sequences are a.a. rotations

Theorem

Let S be the set of Arnoux-Rauzy substitutions over three letters and consider the shift $(S^{\mathbb{N}}, \Sigma, \nu)$ for some shift invariant ergodic probability measure ν which assigns positive measure to each cylinder. Then $(S^{\mathbb{N}}, \Sigma, \nu)$ satisfies the Pisot condition. Moreover, for ν-almost all sequences $\boldsymbol{\sigma} \in S^{\mathbb{N}}$ the collection $\mathcal{C}_{\mathbf{1}}$ forms a tiling, the S-adic shift $(X_{\boldsymbol{\sigma}}, \Sigma)$ is measurably conjugate to a translation on the torus $\mathbb{T}^2$, and the words in $X_{\boldsymbol{\sigma}}$ form natural codings of this translation.

This shows that the counterexample of Cassaigne, Ferenczi, and Zamboni is exceptional.

Concrete rotational Arnoux-Rauzy systems

Definition

A directive sequence $\boldsymbol{\sigma} = (\sigma_n) \in S^{\mathbb{N}}$ that contains each α_i ($i = 1, 2, 3$) infinitely often is said to have bounded weak partial quotients if there is $h \in \mathbb{N}$ such that $\sigma_n = \sigma_{n+1} = \cdots = \sigma_{n+h}$ does not hold for any $n \in \mathbb{N}$.

Theorem

Let $S = \{\alpha_1, \alpha_2, \alpha_3\}$ be the set of Arnoux-Rauzy substitutions over three letters. If $\boldsymbol{\sigma} \in S^{\mathbb{N}}$ is recurrent, contains each α_i ($i = 1, 2, 3$) infinitely often and has bounded weak partial quotients, then the collection $\mathcal{C}_1$ forms a tiling, the S-adic shift $(X_{\boldsymbol{\sigma}}, \Sigma)$ is measurably conjugate to a translation on the torus $\mathbb{T}^2$, and the words in $X_{\boldsymbol{\sigma}}$ form natural codings of this translation.

Bun sequences are a.a. rotations

Theorem

Let $S = \{\beta_1, \beta_2, \beta_3\}$ be the set of Brun substitutions, and consider the shift $(S^{\mathbb{N}}, \Sigma, \nu)$ for some shift invariant ergodic probability measure ν that assigns positive measure to each cylinder. Then $(S^{\mathbb{N}}, \Sigma, \nu)$ satisfies the Pisot condition. Moreover, for ν-almost all sequences $\boldsymbol{\sigma} \in S^{\mathbb{N}}$ the collection $\mathcal{C}_{\mathbf{1}}$ forms a tiling, the S-adic shift $(X_{\boldsymbol{\sigma}}, \Sigma)$ is measurably conjugate to a translation on the torus $\mathbb{T}^2$, and the words in $X_{\boldsymbol{\sigma}}$ form natural codings of this translation.

By the weak convergence of Brun's algorithm for almost all $(x_1, x_2) \in \Delta_2 = \{(x, y) : 0 < x < y < 1\}$ (w.r.t. to the two-dimensional Lebesgue measure), there is a bijection Φ defined for almost all $(x_1, x_2) \in \Delta_2$ that makes the diagram

$$\begin{array}{ccc} \Delta_2 & \xrightarrow{T_{\mathrm{Brun}}} & \Delta_2 \\ \downarrow \Phi & & \downarrow \Phi \\ S^{\mathbb{N}} & \xrightarrow{\Sigma} & S^{\mathbb{N}} \end{array}$$

commutative and that provides a measurable conjugacy between $(\Delta_2, F_{\mathrm{B}}, \lambda_2)$ and $(S^{\mathbb{N}}, \Sigma, \nu)$.

Natural codings for a.a. $\mathbb{T}^2$-rotations

Theorem

For almost all $(x_1, x_2) \in \Delta_2$, the S-adic shift $(X_{\boldsymbol{\sigma}}, \Sigma)$ with $\boldsymbol{\sigma} = \Phi(x_1, x_2)$ is measurably conjugate to the translation by $\left(\frac{x_1}{1+x_1+x_2}, \frac{x_2}{1+x_1+x_2}\right)$ on $\mathbb{T}^2$; then each $\omega \in X_{\boldsymbol{\sigma}}$ is a natural coding for this translation, $\mathcal{L}_{\boldsymbol{\sigma}}$ is balanced, and the subpieces of the Rauzy fractal provide bounded remainder sets for this translation.

This result has the following consequence.

Corollary

For almost all $\mathbf{t} \in \mathbb{T}^2$, there is $(x_1, x_2) \in \Delta_2$ such that the S-adic shift $(X_{\boldsymbol{\sigma}}, \Sigma)$ with $\boldsymbol{\sigma} = \Phi(x_1, x_2)$ is measurably conjugate to the translation by $\mathbf{t}$ on $\mathbb{T}^2$. Moreover, the words in $X_{\boldsymbol{\sigma}}$ form natural codings of the translation by $\mathbf{t}$.

Linear natural codings?

- We believe that the codings mentioned on the previous slide have linear factor complexity.
- S. Labbé and J. Leroy are currently working on a proof of the fact that S-adic words with $S = \{\beta_1, \beta_2, \beta_3\}$ have linear factor complexity.
- We thus get bounded remainder sets whose characteristic infinite words have linear factor complexity, contrarily to the examples provided e.g. by Chevallier or Grepstad and Lev.