S-adic sequences
A bridge between dynamics, arithmetic, and geometry

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(joint work with P. Arnoux, V. Berthé, M. Minervino, and W. Steiner)

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REVIEW OF PART 1

Sturmian sequences and rotations

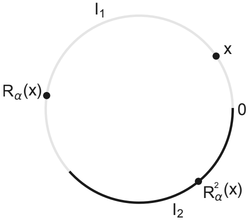
Definition (Sturmian Sequence)

A sequence $w \in \{1,2\}^{\mathbb{N}}$ is called a Sturmian sequence if its complexity function satisfies $p_w(n) = n+1$ for all $n \in \mathbb{N}$.

Definition (Nat'l codings of rotations)

- Rotation by α: $R_\alpha : \mathbb{T} \to \mathbb{T}$ with $x \mapsto x + \alpha \pmod 1$.
- R_α can be regarded as a two interval exchange of the intervals $I_1 = [0, 1-\alpha)$ and $I_2 = [1-\alpha, 1)$.
- $w = w_1 w_2 \ldots \in \{1,2\}^{\mathbb{N}}$ is a natural coding of R_α if there is $x \in \mathbb{T}$ such that $w_k = i$ if and only if $R_\alpha^k(x) \in I_i$ for each $k \in \mathbb{N}$.

Morse and Hedlund



Natural coding: 112...

Figure: Two iterations of the irrational rotation R_α on the circle $\mathbb{T}$.

Theorem (Morse and Hedlund, 1940)

- *A sequence $w \in \{1,2\}^{\mathbb{N}}$ is Sturmian if and only if there exists $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ such that w is a natural coding for R_α.*
- *A Sturmian system $(X_{\boldsymbol{\sigma}}, \Sigma)$ is measurably conjugate to an irrational rotation.*

Strategy of proof

Both are S-adic

$$u = \lim_{n\to\infty} \sigma_{i_0} \circ \sigma_{i_1} \circ \cdots \circ \sigma_{i_n}(2)$$

- Sturmian sequences: Since they are balanced.
- Nat'l codings of rotations: By induction:
  - Consider the rotation R by α on the interval $J = [-1, \alpha)$ with the partition $P_1 = [-1, 0)$ and $P_2 = [0, \alpha)$.
  - natural coding u of the orbit of 0 by R.
  - Let R' be the first return map of R to the interval $J' = \left[\alpha \left\lfloor \frac{1}{\alpha} \right\rfloor - 1, \alpha\right)$.
  - Let v be a natural coding of the orbit of 0 for R'.

The induction

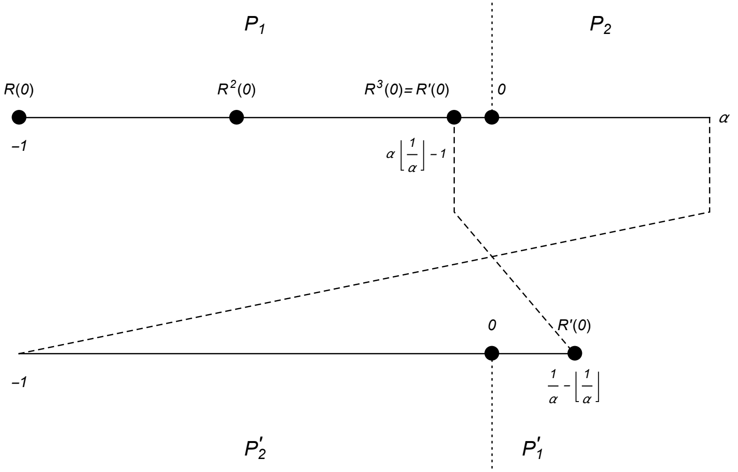
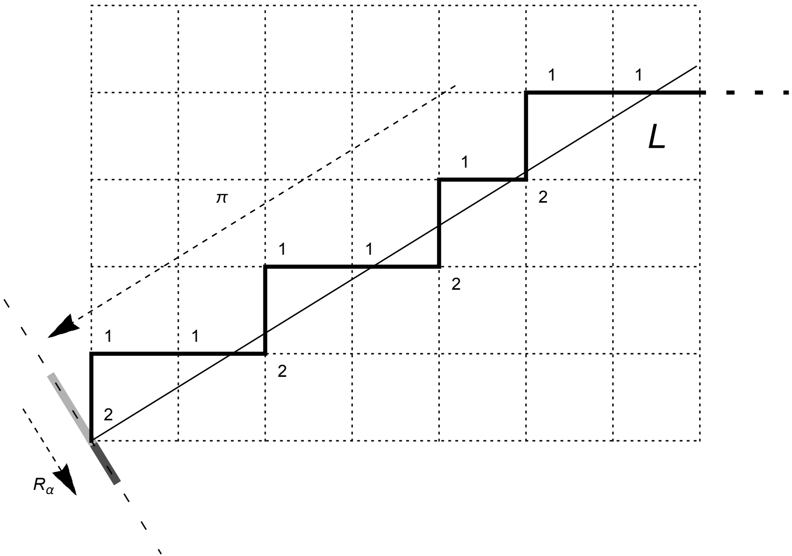
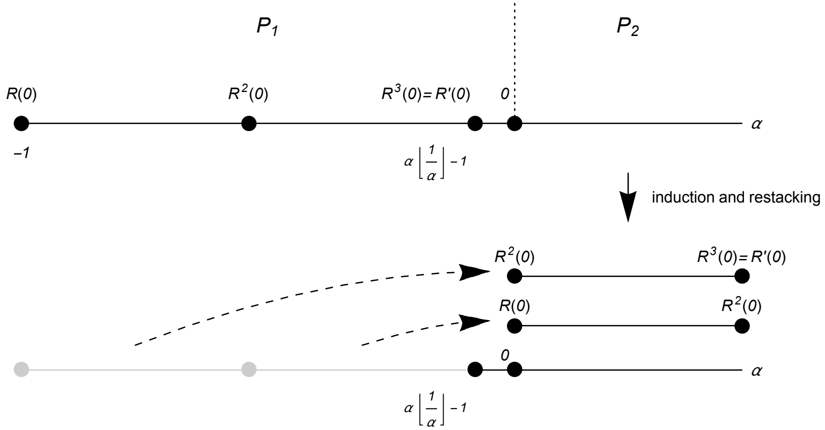


Figure: The rotation R' induced by R.


1
1
1
1
1
1
1
π
L
2
2
2
2
R_α

Inducing with restacking



The intervals $[R(0), R^2(0))$ and $[R^2(0), R^3(0))$ are stacked on one interval of the induced rotation. No information lost!

Restacking and renormalizing boxes

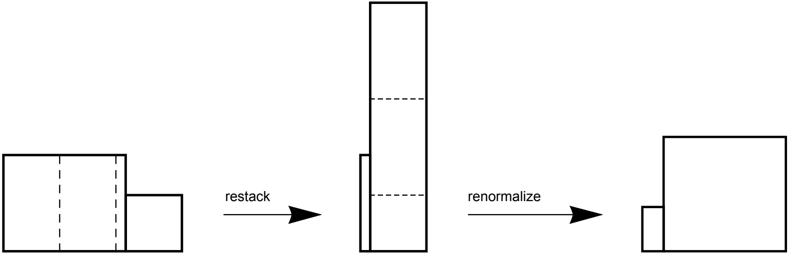


Figure: Step 1: Restack the boxes. Step 2: Renormalize in a way that the larger box has length 1 again.

$$\begin{array}{rclrcl} a & = & \text{length of large } \square, & b & = & \text{length of small } \square, \\ d & = & \text{height of large } \square, & c & = & \text{height of small } \square. \end{array}$$

Mapping in two variables since $\sup\{a, b\} = 1$ and $ad + bc = 1$.

The associated mapping

- Δ_m: Set of pairs of rectangles $(a \times d, b \times c)$ as above such that $a > b$ is equivalent to $d > c$ (the one with larger height has also larger width) with $\sup\{a, b\} = 1$ and $ad + bc = 1$.
- $\Delta_m = \Delta_{m,0} \cup \Delta_{m,1}$, where $a = 1$ in $\Delta_{m,0}$ and $b = 1$ in $\Delta_{m,1}$.

Definition

The map Ψ is defined on $\Delta_{m,1}$ by

$$(\mathrm{a}, \mathrm{d}) \mapsto \left(\left\{\frac{1}{\mathrm{a}}\right\}, \mathrm{a} - \mathrm{d}^2\mathrm{a}\right),$$

and analogously on $\Delta_{m,0}$. This is the natural extension of the Gauss map.

Boxes and Sturmian words

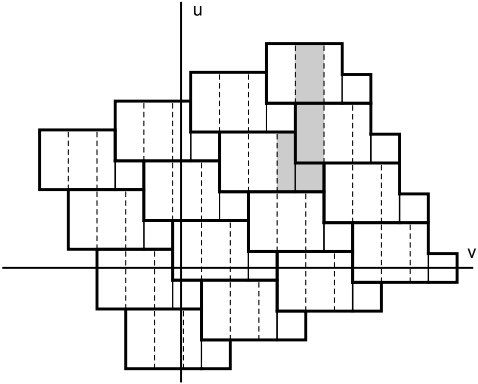


Figure: The vertical line is coded by a Sturmian word u, the horizontal line by a Sturmian word v. The restacking procedure desubstitutes u and substitutes v.

PART 2

S-adic sequences

Contents

The underlying papers

- Cassaigne, J., Ferenczi, S., and Zamboni, L. Q., Imbalances in Arnoux-Rauzy sequences. Ann. Inst. Fourier (Grenoble) 50 (2000), no. 4, 1265–1276.
- Berthé, V. and Delecroix, V., Beyond substitutive dynamical systems: S-adic expansions. Numeration and substitution 2012, 81–123, RIMS Kôkyûroku Bessatsu, B46, Res. Inst. Math. Sci. (RIMS), Kyoto, 2014.
- Berthé, V., Steiner, W., and Thuswaldner, J., Geometry, dynamics, and arithmetic of *S*-adic shifts, preprint, 2016 (available at `https://arxiv.org/abs/1410.0331`).

The first example: Rauzy (1982)

The tribonacci substitution

$$\sigma : \quad \begin{array}{l} 1 \mapsto 12, \\ 2 \mapsto 13, \\ 3 \mapsto 1. \end{array}$$

- This has a fixpoint:
- $X_{(\sigma)} = \overline{\{\Sigma^k w \ : \ k \in \mathbb{N}\}}$ orbit closure.
- $(X_{(\sigma)}, \Sigma)$ associated substitutive dynamical system.

Rauzy (1982) proved that $(X_{(\sigma)}, \Sigma)$ is conjugate to a rotation on the 2-dimensional torus $\mathbb{T}^2$.

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The tribonacci substitution

$$\sigma: \quad \begin{array}{l} 1 \mapsto 12, \\ 2 \mapsto 13, \\ 3 \mapsto 1. \end{array}$$

- This has a fixpoint:

$$\sigma^0(1) = 1$$

- $X_{(\sigma)} = \overline{\{\Sigma^k w \, : \, k \in \mathbb{N}\}}$ orbit closure.
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The first example: Rauzy (1982)

The tribonacci substitution

$$\sigma : \begin{array}{l} 1 \mapsto 12, \\ 2 \mapsto 13, \\ 3 \mapsto 1. \end{array}$$

- This has a fixpoint:

$$\sigma^1(1) = 12$$

- $X_{(\sigma)} = \overline{\{\Sigma^k w \,:\, k \in \mathbb{N}\}}$ orbit closure.
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The first example: Rauzy (1982)

The tribonacci substitution

$$\sigma : \quad \begin{aligned} 1 &\mapsto 12, \\ 2 &\mapsto 13, \\ 3 &\mapsto 1. \end{aligned}$$

- This has a fixpoint:

  $$\sigma^2(1) = 1213$$

- $X_{(\sigma)} = \overline{\{\Sigma^k w \,:\, k \in \mathbb{N}\}}$ orbit closure.
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The first example: Rauzy (1982)

The tribonacci substitution

$$\sigma : \quad \begin{aligned} 1 &\mapsto 12, \\ 2 &\mapsto 13, \\ 3 &\mapsto 1. \end{aligned}$$

- This has a fixpoint:

  $$\sigma^3(1) = 1213121$$

- $X_{(\sigma)} = \overline{\{\Sigma^k w \,:\, k \in \mathbb{N}\}}$ orbit closure.
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The first example: Rauzy (1982)

The tribonacci substitution

$$\sigma: \begin{array}{l} 1 \mapsto 12, \\ 2 \mapsto 13, \\ 3 \mapsto 1. \end{array}$$

- This has a fixpoint:

$$\sigma^4(1) = 1213121121312$$

- $X_{(\sigma)} = \overline{\{\Sigma^k w \ : \ k \in \mathbb{N}\}}$ orbit closure.
- $(X_{(\sigma)}, \Sigma)$ associated substitutive dynamical system.

Rauzy (1982) proved that $(X_{(\sigma)}, \Sigma)$ is conjugate to a rotation on the 2-dimensional torus $\mathbb{T}^2$.

The first example: Rauzy (1982)

The tribonacci substitution

$$\sigma : \quad \begin{aligned} 1 &\mapsto 12, \\ 2 &\mapsto 13, \\ 3 &\mapsto 1. \end{aligned}$$

- This has a fixpoint:

$$\sigma^5(1) = 1213121121312121312112 13$$

- $X_{(\sigma)} = \overline{\{\Sigma^k w \,:\, k \in \mathbb{N}\}}$ orbit closure.
- $(X_{(\sigma)}, \Sigma)$ associated substitutive dynamical system.

Rauzy (1982) proved that $(X_{(\sigma)}, \Sigma)$ is conjugate to a rotation on the 2-dimensional torus $\mathbb{T}^2$.

The first example: Rauzy (1982)

The tribonacci substitution

$$\sigma: \quad \begin{aligned} 1 &\mapsto 12, \\ 2 &\mapsto 13, \\ 3 &\mapsto 1. \end{aligned}$$

- This has a fixpoint:

  $w = \lim_{n\to\infty} \sigma^n(1) = 1213121121312121312112131213\ldots$

- $X_{(\sigma)} = \overline{\{\Sigma^k w \,:\, k \in \mathbb{N}\}}$ orbit closure.
- $(X_{(\sigma)}, \Sigma)$ associated substitutive dynamical system.

Rauzy (1982) proved that $(X_{(\sigma)}, \Sigma)$ is conjugate to a rotation on the 2-dimensional torus $\mathbb{T}^2$.

The Rauzy Fractal

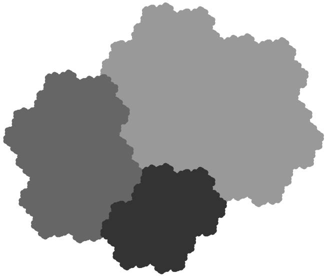


Figure: The classical Rauzy fractal

The main tool in Rauzy's proof is this fractal set on which one can "visualize" the rotation. Fractals instead of intervals !!!

A possible generalization to three letters

In the following definition a right special factor of a sequence $w \in \{1,2,3\}^{\mathbb{N}}$ is a subword v of w for which there are distinct letters $a, b \in \{1,2,3\}$ such that va and vb both occur in w. A left special factor is defined analogously.

Definition (Arnoux-Rauzy sequences, 1991)

A sequence w is called Arnoux-Rauzy sequence if $p_w(n) = 2n + 1$ and if w has only one right special factor and only one left special factor for each given length n.

Hope: Arnoux-Rauzy sequences behave like Sturmian sequences. In particular, they code rotations on $\mathbb{T}^2$.

A substitutive representation

Lemma (Arnoux and Rauzy, 1991)

Let the Arnoux-Rauzy substitutions $\sigma_1, \sigma_2, \sigma_3$ *be defined by*

$$\sigma_1 : \begin{array}{l} 1 \mapsto 1, \\ 2 \mapsto 12, \\ 3 \mapsto 13, \end{array} \qquad \sigma_2 : \begin{array}{l} 1 \mapsto 21, \\ 2 \mapsto 2, \\ 3 \mapsto 23, \end{array} \qquad \sigma_3 : \begin{array}{l} 1 \mapsto 31, \\ 2 \mapsto 32, \\ 3 \mapsto 3. \end{array}$$

Then for each Arnoux-Rauzy sequence w *there exists a sequence* $\boldsymbol{\sigma} = (\sigma_{i_n})$*, where* (i_n) *takes each symbol in* $\{1, 2, 3\}$ *an infinite number of times, such that* w *has the same collection of subwords as*

$$u = \lim_{n \to \infty} \sigma_{i_0} \circ \sigma_{i_1} \circ \cdots \circ \sigma_{i_n}(1).$$

$(X_w, \Sigma) = (X_{\boldsymbol{\sigma}}, \Sigma)$ is the associated S-adic system.

Minimality and unique ergodicity

- Let w be an Arnoux-Rauzy sequence with directive sequence $\boldsymbol{\sigma}$.
- (M_n) sequence of incidence matrices of $\boldsymbol{\sigma} = (\sigma_n)$. For each m there is $n > m$ such that $M_m \cdots M_{n-1}$ is positive.
- This implies that $(X_{\boldsymbol{\sigma}}, \Sigma)$ is minimal.
- Since w has linear complexity function p_w we may invoke a result of Boshernitzan to conclude that $(X_{\boldsymbol{\sigma}}, \Sigma)$ is uniquely ergodic.
- Arnoux and Rauzy proved that w is a coding of an exchange of six intervals.

Unbalanced Arnoux-Rauzy sequences

Definition

Let $C \geq 1$ be an integer. We say that a sequence $w \in \{1, 2, 3\}^{\mathbb{N}}$ is C-balanced if each pair of factors (u, v) of w having the same length satisfies $\big||u|_a - |v|_a\big| \leq C$.

By a clever combinatorial construction one can prove:

Lemma (Cassigne, Ferenczi, and Zamboni, 2000)

There exists an Arnoux-Rauzy sequence which is not C-balanced for any $C \geq 1$.

Consequence: Diameter of Rauzy fractal is not bounded.

The generalization breaks down

Theorem (Cassigne, Ferenczi, and Zamboni, 2000)

There exists an Arnoux-Rauzy sequence w for which (X_w, Σ) is not conjugate to a rotation.

Definition (Bounded remainder set)

For a dynamical system (X, T, μ) a set $A \subset X$ is called a bounded remainder set if there exist real numbers $a, C > 0$ such that for all $N \in \mathbb{N}$ and μ-almost all $x \in X$ we have

$$\big| |\{n < N \ : \ T^n(x) \in A\}| - aN \big| < C.$$

To prove the theorem one has to use a theorem of Rauzy saying that rotations give rise to bounded remainder sets. The unbalanced Arnoux-Rauzy sequence w constructed above doesn't permit a bounded remainder set and, hence, the system (X_w, Σ) cannot be conjugate to a rotation.

Weak Mixing...

- Let (X, T, μ) be a dynamical system with invariant measure μ.
- The transformation T is called weakly mixing for each $A, B \subset X$ of positive measure we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{0\le k<n} |\mu(T^{-k}(A) \cap B) - \mu(A)\mu(B)| = 0.$$

- Weak mixing is equivalent to the fact that 1 is the only measurable eigenvalue of T and the only eigenfunctions are constants (in this case the dynamical system is said to have continuous spectrum).

... and what it really is (after Halmos)

- 90% gin and 10% vermouth in a glass.
- Let V be the original region of the vermouth.
- Let F be a given part of the glass.
- $\mu(T^{-n}F \cap V)/\mu(V)$ is the amount of vermouth in F after n stirrings.

Stirring (which is applying T)

- Ergodic stirring: the amount of vermouth in F is 10% on average.
- Mixing stirring: amount of vermouth in F is close to 10% after a while.
- Weakly mixing stirring: amount of vermouth in F is close to 10% after a while apart from few exceptions.

Weakly mixing Arnoux-Rauzy sequences

- Given an Arnoux-Rauzy sequence

$$u = \lim_{n\to\infty} \sigma_{i_1}^{k_1} \circ \sigma_{i_2}^{k_2} \circ \cdots \circ \sigma_{i_n}^{k_n}(1)$$

  with $i_n \neq i_{n+1}$.
- (n_ℓ) the sequence of indices for which $i_n \neq i_{n+2}$.
- u is uniquely defined by the sequences (k_n) and (n_ℓ)

Theorem (Cassaigne-Ferenczi-Messaoudi, 2008)

For an Arnoux-Rauzy word w with directive sequence $\boldsymbol{\sigma}$ and associated squences (k_n) and (n_ℓ) the system $(X_{\boldsymbol{\sigma}}, \Sigma, \mu)$ (with μ being the unique invariant measure) is weakly mixing if

$$k_{n_i+2} \text{ is unbounded}, \quad \sum_{\ell\geq 1} \frac{1}{k_{n_\ell+1}} < \infty, \quad \text{and} \quad \sum_{\ell \geq 1} \frac{1}{k_{n_\ell}} < \infty.$$

Rauzy's program

Generalize the Sturmian setting to dimension $d \geq 3$; $|\mathcal{A}| = d$.

- Sequences generated by substitutions over the alphabet $\mathcal{A} = \{1, \ldots, d\}$.
- Generalized continued fraction algorithms.
- Rauzy fractals.
- Rotations on $\mathbb{T}^{d-1}$.
- Flows on $SL_d(\mathbb{Z}) \setminus SL_d(\mathbb{R})$ (Weyl chamber flow).

Problems we have to deal with

A lot of new difficulties pop up in the general case.

- No unconditional generalization is possible in view of the counterexamples in the last section.
- The theory of generalized continued fractions is less complete.
- The structure of lattices in $\mathbb{R}^d$ is more complicated than in $\mathbb{R}^2$.
- The projections of the “broken” line is a fractal, not an interval.
- The Weyl chamber flow is an $\mathbb{R}^{d-1}$-action (hence, not a “flow” in the strict sense).

Start from the beginning: substitutions

Definition (Substitution)

Let $\mathcal{A} = \{1, \ldots, d\}$ be an alphabet.
A substitution is a (nonerasing) endomorphism on $\mathcal{A}^*$.

It is sufficient to define a substitution σ on $\mathcal{A}$

Example

- Fibonacci substitution $\sigma(1) = 12$, $\sigma(2) = 1$.
- Sturmian substitutions.
- Tribonacci Substitution $\sigma(1) = 12$, $\sigma(2) = 13$, $\sigma(3) = 1$.
- Arnoux-Rauzy Substitutions.

On $\mathcal{A}^{\mathbb{N}}$ a substitution σ is defined by concatenation setting

$$\sigma(w_0 w_1 \ldots) = \sigma(w_0)\sigma(w_1)\ldots$$

The mapping σ is continuous on $\mathcal{A}^{\mathbb{N}}$ (w.r.t. usual topology).

Abelianization and incidence matrix

- Given a substitution σ on $\mathcal{A}$.
- The incidence matrix: the $|\mathcal{A}| \times |\mathcal{A}|$ matrix M_σ whose columns are the abelianized images of $\sigma(a)$ for $i \in \mathcal{A}$, *i.e.*, $M_\sigma = (m_{ij}) = (|\sigma(j)|_i)$.
- The abelianization: $\mathbf{l} : \mathcal{A}^* \to \mathbb{N}$, $\mathbf{l}(w) = (|w|_1, \ldots, |w|_d)^t$.

We have the commutative diagram

$$\begin{array}{ccc} \mathcal{A}^* & \xrightarrow{\sigma} & \mathcal{A}^* \\ \downarrow \mathbf{l} & & \downarrow \mathbf{l} \\ \mathbb{N} & \xrightarrow{M_\sigma} & \mathbb{N} \end{array}$$

which says that $\mathbf{l}\sigma(w) = M_\sigma \mathbf{l} w$ holds for each $w \in \mathcal{A}^*$.

Properties and notations

Definition

Let $\sigma : \mathcal{A}^* \to \mathcal{A}^*$ be a substitution.

- σ is called unimodular if $|det\, M_\sigma| = 1$.
- σ is called Pisot if the characteristic polynomial of M_σ is the minimal polynomial of a Pisot number.

- $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$ is a sequence of substitutions.
- $\mathbf{M} = (M_{\sigma_n})_{n\in\mathbb{N}} = (M_n)_{n\in\mathbb{N}}$ is the associated sequence of incidence matrices.
- $\sigma_{[m,n)} = \sigma_m \circ \cdots \circ \sigma_{n-1}$ is a block of substitutions.
- $M_{[m,n)} = M_m \cdots M_{n-1}$ is a block of matrices.

.

S-adic sequence

Definition (S-adic sequence)

$\boldsymbol{\sigma} = (\sigma_n)$ is a sequence of substitutions over $\mathcal{A}$.

$S := \{\sigma_n \,:\, n \in \mathbb{N}\}$ (we assume this is finite).

$w \in \mathcal{A}^{\mathbb{N}}$ is an S-adic sequence (or a limit sequence) for $\boldsymbol{\sigma}$ if there exists $(w^{(n)})_{n\in\mathbb{N}}$ with $w^{(n)} \in \mathcal{A}^{\mathbb{N}}$ s.t.

$$w^{(0)} = w, \quad w^{(n)} = \sigma_n(w^{(n+1)}) \quad \text{(for all } n \in \mathbb{N}).$$

In this case we call $\boldsymbol{\sigma}$ the directive sequence for w.

Often there is $a \in \mathcal{A}$ such such that

$$w = \lim_{n\to\infty} \sigma_{[0,n)}(a)$$

(this is related to primitivity).

S-adic shift

Shift on $\mathcal{A}^{\mathbb{N}}$: $\Sigma : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}, \quad \Sigma(w_0 w_1 \ldots) = w_1 w_2 \ldots$

Definition (*S*-adic shift)

For an *S*-adic sequence w Let

$$X_w = \overline{\{\Sigma^k w \ : \ k \in \mathbb{N}\}}.$$

(X_w, Σ) is the *S*-adic shift (or *S*-adic system) generated by w.

Language of a sequence:

$$L(w) = \{u \in \mathcal{A}^* \ : \ u \text{ is a subword of } w\}.$$

Alternative: X_w can also be defined by

$$X_w = \{v \in \mathcal{A}^{\mathbb{N}} \ : \ L(v) \subseteq L(w)\}.$$

Often X_w only depends on σ: $X_\sigma := X_w$.

The substitutive case (well-studied)

Let $\boldsymbol{\sigma} = (\sigma)_{n\in\mathbb{N}}$ is the constant sequence.

Then $(X_{\boldsymbol{\sigma}}, \Sigma) = (X_{(\sigma)}, \Sigma)$ is called substitutive.

- Rauzy (1982): $(X_{(\sigma)}, \Sigma)$ is a rotation on $\mathbb{T}^2$ for σ being the tribonacci substitution.
- Arnoux-Ito (2001) and Ito-Rao (2006): $(X_{(\sigma)}, \Sigma)$ is a rotation on $\mathbb{T}^{d-1}$ for σ unimodular Pisot if some combinatorial conditions are in force.
- Minervino-T. (2014): Generalizations to nonunimodular Pisot substitutions under combinatorial conditions.
- Barge (2016): $(X_{(\sigma)}, \Sigma)$ is a rotation on $\mathbb{T}^{d-1}$ for σ unimodular Pisot under very general conditions.
- Pisot substitution conjecture: $(X_{(\sigma)}, \Sigma)$ is a rotation on $\mathbb{T}^2$ for σ unimodular Pisot.

Some Rauzy fractals

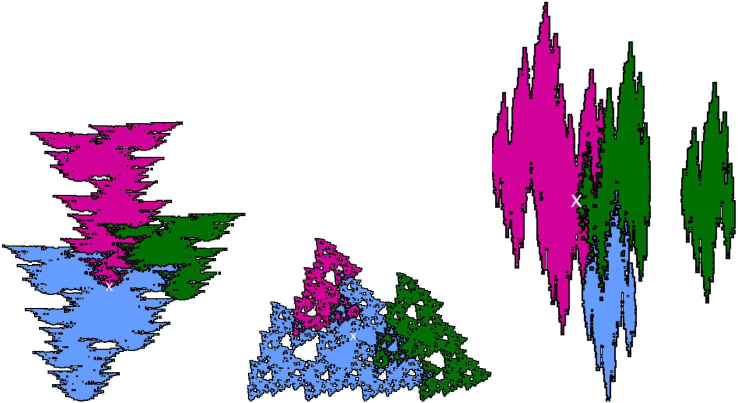


Figure: Rauzy fractals may have holes and can even be disconnected

Generalized continued fraction algorithms

Definition (Generalized continued fraction algorithm)

Let X be a closed subset of the projective space $\mathbb{P}^d$ and let $\{X_i\}_{i \in I}$ be a partition of X (up to a set of measure 0) indexed by a countable set I. Let $\mathcal{M} = \{M_i \,:\, i \in I\}$ be a set of unimodular integer matrices $M_i^{-1} X_i \subset X$ and let

$$M : X \to \mathcal{M}, \quad \mathbf{x} \mapsto M_i \text{ whenever } \mathbf{x} \in X_i.$$

The generalized continued fraction algorithm associated with this data is given by the mapping

$$F : X \to X; \quad \mathbf{x} \mapsto M(\mathbf{x})^{-1}\mathbf{x}.$$

If I is a finite set, the algorithm given by F is called additive, otherwise it is called multiplicative.

The linear Brun continued fraction algorithm

The linear Brun algorithm is defined on

$$B = \{[w_1 : w_2 : w_3] \in \mathbb{P}^2 : 0 \le w_1 < w_2 < w_3\} \subset \mathbb{P}^2$$

For $\mathbf{x} = [w_1 : w_2 : w_3] \in B$ replace w_3 by $w_3 - w_2$ and sort the elements $w_1, w_2, w_3 - w_2$ in increasing order. More precisely, Brun's algorithm is given by

$$F_{\mathrm{B}} : \; B \to B, \quad \mathbf{x} \mapsto [\mathrm{sort}(w_1, w_2, w_3 - w_2)].$$

The linear Brun algorithm subtracts the second largest element of a sorted vector $[w_1, w_2, w_3] \in \mathbb{P}^2$ from the largest one.

The Brun Matrices

Set $\mathcal{M} = \{M_1, M_2, M_3\}$ with

$$M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

The sets $B_i = M_i B \subset B$ partition B up to a set of measure 0.

The Brun continued fraction algorithm can be written as

$$F_{\mathrm{B}} : \mathbf{x} \mapsto M_i^{-1}\mathbf{x}, \quad \text{for } \mathbf{x} \in B_i,$$

This continued fraction algorithm is additive, since it is defined by a finite family $\mathcal{M} = \{M_1, M_2, M_3\}$ of unimodular matrices.

The partition of B

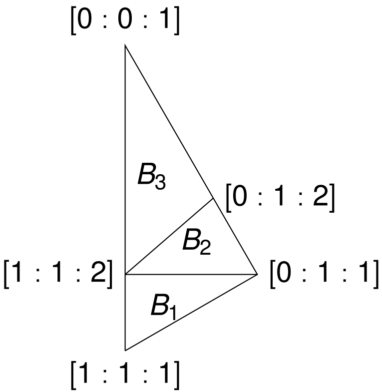


Figure: The partition of B in the three regions B_1, B_2, B_3.

The original Brun continued fraction algorithm

There also exists a projective version of the Brun algorithm.

Definition (Brun algorithm)

The projective additive form of the Brun algorithm is given on $\Delta_2 := \{(x_1, x_2) \in \mathbb{R}^2 \,:\, 0 < x_1 < x_2 < 1\}$ by

$$f_{\mathrm{B}} : (x_1, x_2) \mapsto \begin{cases} \left(\frac{x_1}{1-x_2}, \frac{x_2}{1-x_2}\right), & \text{for } x_2 \leq \frac{1}{2}, \\ \left(\frac{x_1}{x_2}, \frac{1-x_2}{x_2}\right), & \text{for } \frac{1}{2} \leq x_2 \leq 1 - x_1, \\ \left(\frac{1-x_2}{x_2}, \frac{x_1}{x_2}\right), & \text{for } 1 - x_1 \leq x_2. \end{cases}$$

If the linear version of the algorithm performs the mapping $(w_1, w_2, w_3) \mapsto (w_1', w_2', w_3')$ then

$$f_{\mathrm{B}}(w_1/w_3, w_2/w_3) = (w_1'/w_3', w_2'/w_3').$$

Brun Substitutions

Let $S = \{\beta_1, \beta_2, \beta_3\}$ be the set of Brun substitutions

$$\beta_1 : \begin{cases} 1 \mapsto 3, \\ 2 \mapsto 1, \\ 3 \mapsto 23, \end{cases} \qquad \beta_2 : \begin{cases} 1 \mapsto 1, \\ 2 \mapsto 3, \\ 3 \mapsto 23, \end{cases} \qquad \beta_3 : \begin{cases} 1 \mapsto 1, \\ 2 \mapsto 23, \\ 3 \mapsto 3, \end{cases}$$

whose incidence matrices are the Brun matrices M_1, M_2, M_3.
Note that this choice is not canonical.

Using these substitutions we can produce *S*-adic sequences whose abelianizations perform the Brun algorithm.

Primitivity and minimality

Definition (Primitivity)

A sequence **M** of nonnegative matrices from $GL_d(\mathbb{Z})$ is primitive if for each $m \in \mathbb{N}$ there is $n > m$ such that $M_{[m,n)}$ is a positive matrix. A sequence σ of substituitons is primitive if its associated sequence of incidence matrices is primitive.

Definition (Minimality)

Let (X, T) be a topological dynamical system. (X, T) is called minimal if the orbit of each point is dense in X, *i.e.,* if

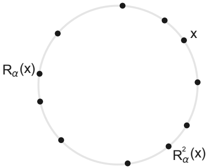
$$\overline{\{T^k x \ : \ k \in \mathbb{N}\}} = X$$

holds for each $x \in X$.

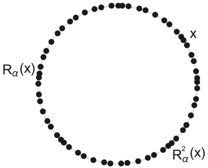
Example: irrational rotation

Example

An irrational rotation R_α on $\mathbb{T}^1$ is minimal.



(10 iterates)



(60 iterates)

More general: If $\boldsymbol{\alpha} \in \mathbb{T}^d$ has irrational and rationally independent coordinates then the rotation by $\boldsymbol{\alpha}$ on $\mathbb{T}^d$ is minimal (Kronecker rotation).

Example

The full shift $(\{1,2\}^{\mathbb{N}}, \Sigma)$ is not minimal: for instance, $1111\ldots$ doesn't have a dense orbit.

Consequences of primitivity

Lemma

If σ is a primitive sequence of unimodular substitutions, the following properties hold.

(i) *There exists at least one and at most $|\mathcal{A}|$ limit sequences for σ.*

(ii) *Let w, w' be two S-adic sequences with directive sequence σ. Then $(X_w, \Sigma) = (X_{w'}, \Sigma)$.*

(iii) *The S-adic shift (X_w, Σ) is minimal.*

If σ is a primitive sequence of substitutions, assertion (ii) of this lemma allows us to define $(X_\sigma, \Sigma) = (X_w, \Sigma)$ for w being an arbitraty S-adic sequence with directive sequence σ.

Recurrence

Definition (Recurrence)

A sequence $\mathbf{M} = (M_n)$ of matrices is called recurrent if for each $m \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that $M_{[0,m)} = M_{[n,n+m)}$. A sequence $\boldsymbol{\sigma} = (\sigma_n)$ of substitutions is called recurrent if for each $m \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that $\sigma_{[0,m)} = \sigma_{[n,n+m)}$.

Lemma (Furstenberg, 1960)

Let $\mathbf{M} = (M_n)$ be a primitive and recurrent sequence of nonnegative matrices from $GL_d(\mathbb{Z})$. Then there is a vector $\mathbf{u} \in \mathbb{R}^d_+$ satisfying

$$\bigcap_{n \geq 0} M_{[0,n)} \mathbb{R}^d_+ = \mathbb{R}_+ \mathbf{u}.$$

Proof of the lemma: Hilbert metric

- $C = \{\mathbb{R}_+\mathbf{w} \,:\, \mathbf{w} \in \mathbb{R}_+^d\}$: space of nonnegative rays through the origin.
- Hilbert metric:

$$d_C(\mathbb{R}_+\mathbf{v}, \mathbb{R}_+\mathbf{w}) = \max_{1\le i,j\le d} \log \frac{v_i w_j}{v_j w_i},$$

- For M nonnegative:

$$d_C(M\mathbb{R}_+\mathbf{v}, M\mathbb{R}_+\mathbf{w}) \le d_C(\mathbb{R}_+\mathbf{v}, \mathbb{R}_+\mathbf{w})$$

- For M positive:

$$d_C(M\mathbb{R}_+\mathbf{v}, M\mathbb{R}_+\mathbf{w}) \le d_C(\mathbb{R}_+\mathbf{v}, \mathbb{R}_+\mathbf{w})$$

- Since (M_n) has infinitely many occurrences of a given positive block, the lemma follows.

Weak convergence & generalized right eigenvector

Definition (Weak convergence & generalized right eigenvector)

If a sequence of nonnegative matrices **M** with elements in $GL_d(\mathbb{Z})$ satisfies

$$\bigcap_{n \geq 0} M_{[0,n)} \mathbb{R}_+^d = \mathbb{R}_+ \mathbf{u}$$

we say that **M** is weakly convergent to **u**. In this case we call **u** a generalized right eigenvector of **M**.

Substitutive case

If $\boldsymbol{\sigma} = (\sigma)$, with σ a Pisot substitution then one can prove that $\mathbf{M} = (M_\sigma)$ contains a positive block. Hence, it contains infinitely many positive blocks. In this case the generalized right eigenvector **u** is the dominant right eigenvector of M_σ.

Unique ergodicity

Definition (Unique ergodicity)

A topological dynamical system (X, T) on a compact space X is said to be uniquely ergodic if there is a unique T-invariant Borel probability measure on X.

- Kyrlov and Bogoliubov: If X is compact then there is at least one invariant measure μ, *i.e.*, $\mu(A) = \mu(T^{-1}A)$, $\forall A$.
- If there is a unique invariant measure μ, it has to be ergodic, otherwise $\nu(B) = \frac{\mu(B \cap E)}{\mu(E)}$ is another invariant measure if E is an invariant set E of μ with $0 < \mu(E) < 1$.
- Unique ergodicity implies that each point is generic, *i.e.*,

$$\frac{1}{N} \sum_{0 \leq n < N} f(T^n x) \longrightarrow \int f d\mu$$

holds for each $x \in X$ and each $f \in C(X)$.

Uniform frequenies of letters and words

Definition

- $w = w_0 w_1 \ldots \in \mathcal{A}^{\mathbb{N}}$.
- $|w_k \ldots w_{\ell-1}|_v$ be the number of occurrences of v in $w_k \ldots w_{\ell-1} \in \mathcal{A}^* \quad (k < \ell$ and $v \in \mathcal{A}^*)$.
- w has uniform frequencies for words if for each $v \in \mathcal{A}^*$

$$\lim_{\ell \to \infty} \frac{|w_k \ldots w_{\ell-1}|_v}{\ell - k} = f_v(w)$$

holds uniformly in k. It has uniform letter frequencies if this is true for each $v \in \mathcal{A}$.

Example

The fixpoint $w = \lim \sigma^n(1)$ of the Fibonacci substitution has uniform letter frequencies $(f_1(w), f_2(w)) = (\varphi^{-1}, \varphi^{-2})$.

Uniform word frequencies and unique ergodicity

Lemma

Let $w \in \mathcal{A}^{\mathbb{N}}$ be a sequence with uniform word frequencies and let $X_w = \overline{\{\Sigma^n w \,:\, n \in \mathbb{N}\}}$ be the shift orbit closure of w. Then (X_w, Σ) is uniquely ergodic.

This criterion can be applied to the S-adic setting:

Lemma

Let $\boldsymbol{\sigma}$ be a sequence of unimodular substitutions with sequence of incidence matrices $\mathbf{M}$. If $\mathbf{M}$ is primitive and recurrent then each sequence $w \in X_{\boldsymbol{\sigma}}$ has uniform word frequencies.

Proof.

- Generalized eigenvector $\mathbf{u}$ describes letter frequencies.
- Dumont-Thomas expansion is used.

The main result

Summing up we get the following result:

Theorem

Let σ be a sequence of unimodular substitutions with associated sequence of incidence matrices $\mathbf{M}$. If $\mathbf{M}$ is primitive and recurrent, (X_σ, Σ) is minimal and uniquely ergodic.

An Example: Brun substitutions

Lemma

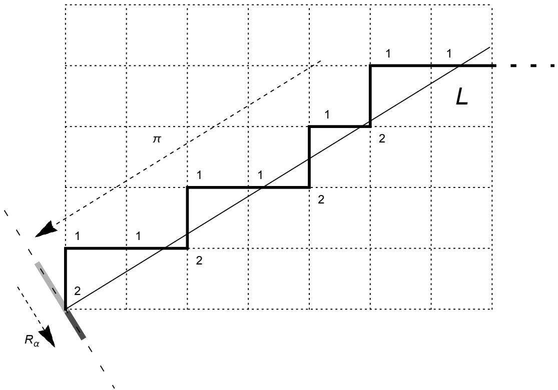
Let $S = \{\sigma_1, \sigma_2, \sigma_3\}$ be the set of Brun substitutions and $\boldsymbol{\sigma} \in S^{\mathbb{N}}$. If $\boldsymbol{\sigma}$ is recurrent and contains the block $(\sigma_3, \sigma_2, \sigma_3, \sigma_2)$ then the associated S-adic system $(X_{\boldsymbol{\sigma}}, \Sigma)$ is minimal and uniquely ergodic.

Proof.

It is immediate that $M_3M_2M_3M_2$ is a strictly positive matrix. Since $\boldsymbol{\sigma}$ is recurrent, it contains the block $(\sigma_3, \sigma_2, \sigma_3, \sigma_2)$ infinitely often. Thus $\boldsymbol{\sigma}$ is primitive and the result follows from the theorem. □

Being recurrent is a generic property.

Looking back to the Sturmian case



- We “see” the rotation on the Rauzy fractal if it has “good” properties.
- It is our aim to establish these properties.

Preparations for the definition

An S-adic Rauzy fractal will be defined in terms of a projection to a hyperplane.

- $\mathbf{w} \in \mathbb{R}^d_{\geq 0} \setminus \{\mathbf{0}\}$.
- $\mathbf{w}^\perp = \{\mathbf{x} \,:\, \mathbf{x} \cdot \mathbf{w} = 0\}$ orthogonal hyperplane
- $\mathbf{w}^\perp$ is equiped with the Lebesgue measure $\lambda_{\mathbf{w}}$.
- The vector $\mathbf{1} = (1, \ldots, 1)^t$ will be of special interest
- $\mathbf{u}, \mathbf{w} \in \mathbb{R}^d_{\geq 0} \setminus \{\mathbf{0}\}$ noncollinear. Then we denote the projection along $\mathbf{u}$ to $\mathbf{w}^\perp$ by $\pi_{\mathbf{u},\mathbf{w}}$.

S-adic Rauzy fractal

Definition (S-adic Rauzy fractals and subtiles)

Let $\boldsymbol{\sigma}$ be a sequence of unimodular substitutions over the alphabet $\mathcal{A}$ with generalized eigenvector $\mathbf{u} \in \mathbb{R}^d_{>0}$.

Let $(X_{\boldsymbol{\sigma}}, \Sigma)$ be the associated S-adic system.

The S-adic Rauzy fractal (in $\mathbf{w}^\perp$, $\mathbf{w} \in \mathbb{R}^d_{\geq 0}$) associated with $\boldsymbol{\sigma}$ is the set

$$\mathcal{R}_{\mathbf{w}} := \overline{\{\pi_{\mathbf{u},\mathbf{w}}\mathbf{l}(p) \ : \ p \text{ is a prefix of a limit sequence of } \boldsymbol{\sigma}\}}.$$

The set $\mathcal{R}_{\mathbf{w}}$ can be naturally covered by the subtiles ($i \in \mathcal{A}$)

$$\mathcal{R}_{\mathbf{w}}(i) := \overline{\{\pi_{\mathbf{u},\mathbf{w}}\mathbf{l}(p) \ : \ pi \text{ is a prefix of a limit sequence of } \boldsymbol{\sigma}\}}.$$

For convenience we set $\mathcal{R}_{\mathbf{1}}(i) = \mathcal{R}(i)$ and $\mathcal{R}_{\mathbf{1}} = \mathcal{R}$.

Illustration of the definition

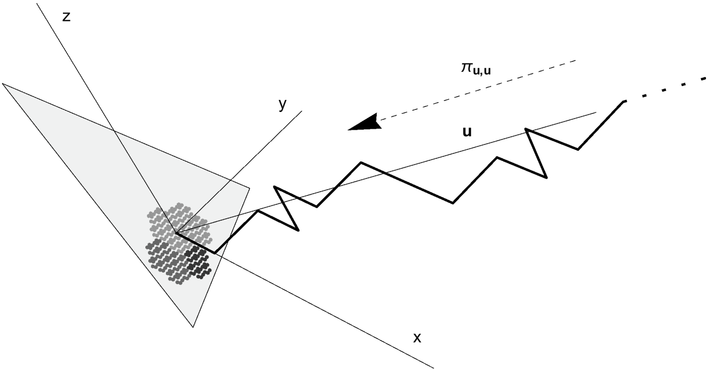


Figure: Definition of $\mathcal{R}_{\mathbf{u}}$ and its subtiles

What we need

We want to “see” the rotation on the Rauzy fractal.

- $\mathcal{R}$ should be bounded.
- $\mathcal{R}$ should be the closure of its interior.
- The boundary $\partial\mathcal{R}$ should have λ_1-measure zero.
- The subtiles $\mathcal{R}(i)$, $i \in \mathcal{A}$, should not overlap on a set of positive measure.
- $\mathcal{R}$ should be the fundamental domain of a lattice, *i.e.*, it can be used as a tile for a lattice tiling.

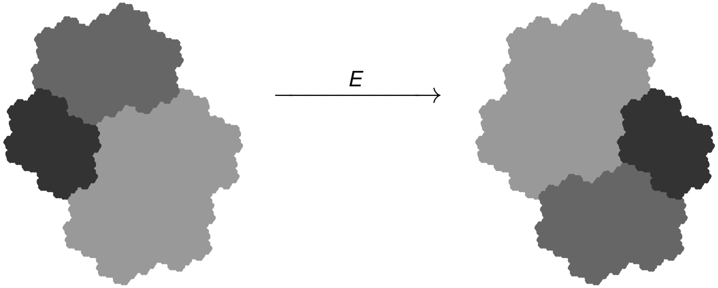
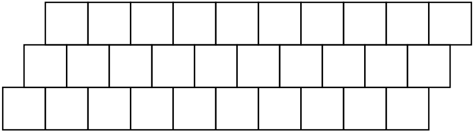


Figure: The domain exchange

Multiple tiling and tiling

Definition (Multiple tiling and tiling)

- Let $\mathcal{K}$ be a collection of subsets of an Euclidean space $\mathcal{E}$.
- Assume that each element of $\mathcal{K}$ is compact and equal to the closure of its interior.
- $\mathcal{K}$ is a multiple tiling if there is $m \in \mathbb{N}$ such that a. e. point (w.r.t. Lebesgue measure) of $\mathcal{E}$ is contained in exactly m elements of $\mathcal{K}$.
- $\mathcal{K}$ is a multiple tiling if $m = 1$.



Discrete hyperplane

- A discrete hyperplane can be viewed as an approximation of a hyperplane by translates of unit hypercubes.
- Pick $\mathbf{w} \in \mathbb{R}^d_{\geq 0} \setminus \{\mathbf{0}\}$ and denote by $\langle \cdot, \cdot \rangle$ the dot product.
- The discrete hyperplanes is defined by

$$\Gamma(\mathbf{w}) = \{[\mathbf{x}, i] \in \mathbb{Z}^d \times \mathcal{A} \ : \ 0 \leq \langle \mathbf{x}, \mathbf{w} \rangle < \langle \mathbf{e}_i, \mathbf{w} \rangle\}$$

  (here $\mathbf{e}_i$ is the standard basis vector).
- Interpret the symbol $[\mathbf{x}, i] \in \mathbb{Z}^d \times \mathcal{A}$ as the hypercube or "face"

$$[\mathbf{x}, i] = \left\{ \mathbf{x} + \sum_{j \in \mathcal{A} \setminus \{i\}} \lambda_j \mathbf{e}_j \ : \ \lambda_j \in [0, 1] \right\}.$$

  Then the set $\Gamma(\mathbf{w})$ turns into a stepped hyperplane that approximates $\mathbf{w}^\perp$ by hypercubes.

Examples of stepped surfaces

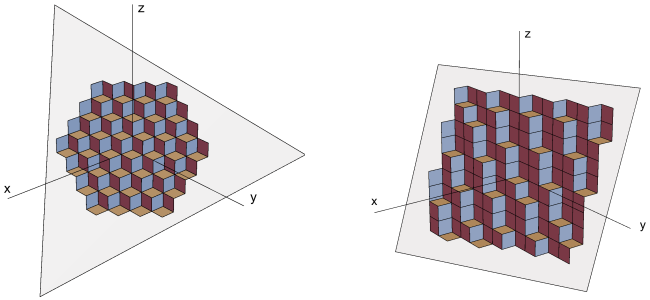


Figure: A subset of a periodic and an aperiodic stepped surface

A finite subset of a discrete hyperplane will be called a patch.

Collections of Rauzy fractals

- Using the concept of discrete hyperplane we define the following collections of Rauzy fractals.
- Let $\boldsymbol{\sigma}$ be a sequence of substitutions with generalized eigenvector $\mathbf{u}$ and choose $\mathbf{w} \in \mathbb{R}^d_{\geq 0} \setminus \{\mathbf{0}\}$.

Definition (Collections of Rauzy fractals)

Set

$$\mathcal{C}_{\mathbf{w}} = \{\pi_{\mathbf{u},\mathbf{w}}\mathbf{x} + \mathcal{R}_{\mathbf{w}}(i) \ : \ [\mathbf{x}, i] \in \Gamma(\mathbf{w})\}.$$

- We will see that these collections often form a tiling of the space $\mathbf{w}^\perp$.
- A special role will be played by the collection $\mathcal{C}_{\mathbf{1}}$ which will give rise to a periodic tiling of $\mathbf{1}^\perp$ by lattice translates of the Rauzy fractal $\mathcal{R}$.