Products of two Cantor sets and separable two-dimensional quasicrystal models

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Cantor sets

Definition

A set $K \subset \mathbb{R}$ is a *Cantor set* if K is compact, perfect and nowhere dense.

Example (Middle-1/3 Cantor set)

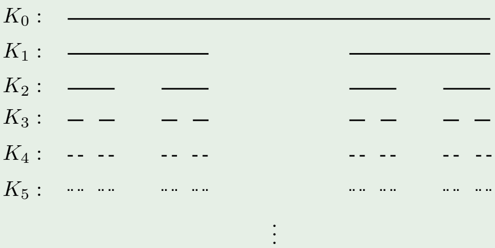


Figure : $K := \bigcap_{n=0}^{\infty} K_n$ is the *middle*-1/3 *Cantor set*.

Sums of Cantor sets

- Denote the *thickness* of K by $\tau(K)$.

Example

- Middle-α Cantor set has thickness $1/\alpha$.

Lemma (Newhouse Gap Lemma)

Let K, L be Cantor sets with $\tau(K) \cdot \tau(L) > 1$. Then $K + L$ is a disjoint union of finitely many closed intervals.

Products of Cantor sets: Main result 1

Theorem (T. '15)

Let K, L be Cantor sets, and assume that $\min K, \min L = 0$. Then, $K \cdot L$ is an interval if

$$\tau(L) \geq \frac{2\tau(K)+1}{\tau(K)^2}, \text{ or } \tau(K) \geq \frac{2\tau(L)+1}{\tau(L)^2}.$$

This estimate is optimal. In particular, if

$$\tau(K) = \tau(L) \geq \frac{1+\sqrt{5}}{2},$$

$K \cdot L$ is an interval.

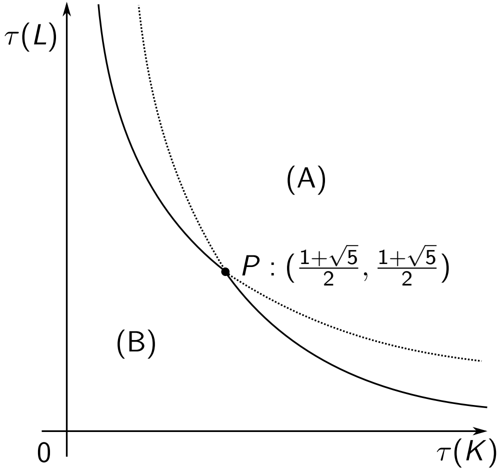


Figure : (A): $K \cdot L$ is always a closed interval.

Products of Cantor sets: Main result 2

Theorem (T. '15)

Let K, L be Cantor sets with $0 \in K, L$. Then, if

$$2\left(\tau(K)+1\right)\left(\tau(L)+1\right) \leq \left(\tau(K)\tau(L)-1\right)^2, \tag{1}$$

$K \cdot L$ is an interval. This estimate is optimal. In particular, if

$$\tau(K) = \tau(L) \geq 1 + \sqrt{2},$$

$K \cdot L$ is an interval.

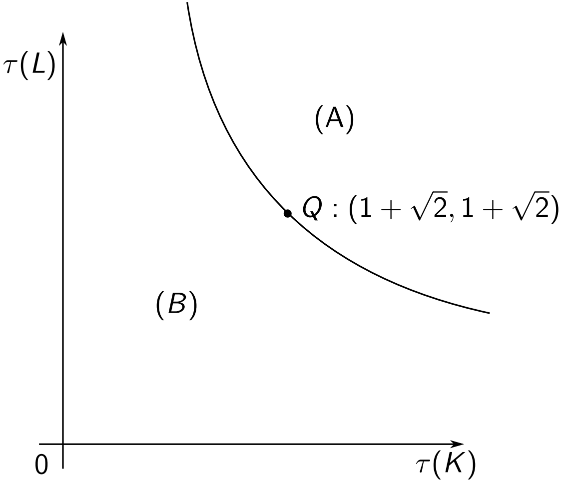


Figure : (A): $K \cdot L$ is always a closed interval.

Fibonacci substitution sequence

- Consider the following substitution:

$$\mathcal{P} = \begin{cases} 0 \longrightarrow 01 \\ 1 \longrightarrow 0 \end{cases}$$

This substitution defines the *Fibonacci substitution sequence* $\{\omega_n\}$:

$$0 \longrightarrow 01 \longrightarrow 010 \longrightarrow 01001 \longrightarrow 01001010 \longrightarrow \cdots .$$

- For $\lambda > 1$, define

$$p_n = \begin{cases} \lambda \ \text{ if } \ \omega_n = 0 \\ 1 \ \text{ if } \ \omega_n = 1. \end{cases}$$

One dimensional quasicrystal model

For $\lambda > 1$, define the operator $H_\lambda : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ by

$$(H_\lambda \phi)_n = p_{n+1}\phi_{n+1} + p_n \phi_{n-1}.$$

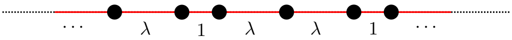


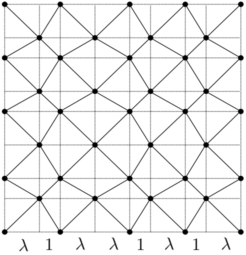
Figure : one dimensional quasicrystal model

- It is known that the *spectrum* Σ_λ is a (dynamically defined) Cantor set.

Theorem ('11, D. Damanik, A. Gorodetski)

$$\lim_{\lambda \to 1} \tau(\Sigma_\lambda) = \infty.$$

Labyrinth model



- The spectrum of the Labyrinth model is given by $\Sigma_\lambda \cdot \Sigma_\lambda$.

Theorem ('15, T.)

For sufficiently small $\lambda > 0$, the spectrum $\Sigma_\lambda \cdot \Sigma_\lambda$ is an interval.

Thank you! :)