Products of two Cantor sets and separable two-dimensional quasicrystal models

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Cantor sets

Definition
A set $K \subset \mathbb{R}$ is a Cantor set if $K$ is compact, perfect and nowhere dense.

Example (Middle-1/3 Cantor set)

$K_0$:

$K_1$:

$K_2$:

$K_3$:

$K_4$:

$K_5$:

$\vdots$

Figure: $K := \bigcap_{n=0}^{\infty} K_n$ is the middle-1/3 Cantor set.
Sums of Cantor sets

- Denote the thickness of $K$ by $\tau(K)$.

**Example**

- Middle-$\alpha$ Cantor set has thickness $1/\alpha$.

**Lemma (Newhouse Gap Lemma)**

Let $K, L$ be Cantor sets with $\tau(K) \cdot \tau(L) > 1$. Then $K + L$ is a disjoint union of finitely many closed intervals.
Let $K, L$ be Cantor sets, and assume that $\min K, \min L = 0$. Then, $K \cdot L$ is an interval if

$$\tau(L) \geq \frac{2\tau(K) + 1}{\tau(K)^2}, \text{ or } \tau(K) \geq \frac{2\tau(L) + 1}{\tau(L)^2}.$$ 

This estimate is optimal. In particular, if

$$\tau(K) = \tau(L) \geq \frac{1 + \sqrt{5}}{2},$$

$K \cdot L$ is an interval.
Figure: (A): $K \cdot L$ is always a closed interval.
Theorem (T. ’15)

Let $K, L$ be Cantor sets with $0 \in K, L$. Then, if

$$2 (\tau(K) + 1)(\tau(L) + 1) \leq (\tau(K)\tau(L) - 1)^2,$$  \hfill (1)

$K \cdot L$ is an interval. This estimate is optimal. In particular, if

$$\tau(K) = \tau(L) \geq 1 + \sqrt{2},$$

$K \cdot L$ is an interval.
\((1 + \sqrt{2}, 1 + \sqrt{2})\)

**Figure**: (A): \(K \cdot L\) is always a closed interval.
Consider the following substitution:

\[ P = \begin{cases} 
0 & \rightarrow 01 \\
1 & \rightarrow 0 
\end{cases} \]

This substitution defines the *Fibonacci substitution sequence* \( \{\omega_n\} \):

\[ 0 \rightarrow 01 \rightarrow 010 \rightarrow 01001 \rightarrow 01001010 \rightarrow \cdots \]

For \( \lambda > 1 \), define

\[ p_n = \begin{cases} 
\lambda & \text{if } \omega_n = 0 \\
1 & \text{if } \omega_n = 1 
\end{cases} \]
One dimensional quasicrystal model

For \( \lambda > 1 \), define the operator \( H_{\lambda} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \) by

\[
(H_{\lambda} \phi)_n = p_{n+1} \phi_{n+1} + p_n \phi_{n-1}.
\]

It is known that the spectrum \( \Sigma_{\lambda} \) is a (dynamically defined) Cantor set.

Theorem ('11, D. Damanik, A. Gorodetski)

\[
\lim_{\lambda \to 1} \tau(\Sigma_{\lambda}) = \infty.
\]
The spectrum of the Labyrinth model is given by $\Sigma_{\lambda} \cdot \Sigma_{\lambda}$.

**Theorem (’15, T.)**

*For sufficiently small $\lambda > 0$, the spectrum $\Sigma_{\lambda} \cdot \Sigma_{\lambda}$ is an interval.*
Thank you! :)