Delone sets and tilings

School on Tiling Dynamical Systems, CIRM, Luminy

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Plan of the talk

1. Classes of Delone sets: finitely generated, finite type, Meyer
2. Inflation symmetries
3. Substitution Delone sets and tilings
4. Associated dynamical systems and their spectral properties (if time permits)

Bibliography

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Delone sets

Definition. A set $X \subset \mathbb{R}^d$ is *Delone* if it is

(a) *Uniformly discrete:* $\exists\, r > 0$ such that $\sharp(B(y,r) \cap X) \le 1\ \forall\, y \in \mathbb{R}^d$, and

(b) *Relatively dense:* $\exists\, R > 0$ such that $B(y,R) \cap X \neq \emptyset$ for all $y \in \mathbb{R}^d$.

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Definition. A Delone set X in $\mathbb{R}^d$ is

- **finitely generated** if $[X]$ (equivalently $[X - X]$) is finitely generated.

$$[X] = \left\{ \sum_{i=1}^{k} n_i x_i : \; n_i \in \mathbb{Z}, \; x_i \in X, \; k \in \mathbb{N} \right\}.$$

- of **finite type** if $X - X$ is a discrete closed set, that is, $(X - X) \cap B(0, N)$ is finite for all $N > 0$.

Properties of Delone sets

finitely generated $\overset{\not\Rightarrow}{\Leftarrow}$ finite type

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finite type $\Longleftrightarrow$ finite local complexity (FLC)

Address map

Definition. Let X be a finitely generated Delone set in $\mathbb{R}^d$. Note that $[X]$ is a free Abelian group, hence it has a *basis* (set of free generators): $[X] = \mathbb{Z}[v_1, \ldots, v_s]$ for some $s \geq d$. Then

$$\phi\Big(\sum_{i=1}^{s} n_i v_i\Big) = (n_1, \ldots, n_s).$$

is the address map. It depends on the choice of basis; defined up to left-multiplication by an element of $GL(s, \mathbb{Z})$.

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- $s = d$ iff $[X]$ is a lattice in $\mathbb{R}^d$
- the address map describes X using s "internal dimensions"

Characterization of finite type

Theorem 1 (Lagarias 1999) *Let $X \subset \mathbb{R}^d$ be a Delone set. Then X is of finite type if and only if X is finitely generated and any address map is* **globally Lipschitz on X**: *there exists $C > 0$ such that*

$$\|\phi(x) - \phi(x')\| \le C\|x - x'\| \quad \text{for all} \quad x, x' \in X.$$

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$$\|\phi(x) - \phi(x')\| \le C\|x - x'\| \quad \text{for all} \quad x, x' \in X.$$

Proof.

Finite type $\Longleftarrow$ Address map is Lipschitz on X

If $\|x - x'\| \le N$, then

$$\|\phi(x - x')\| = \|\phi(x) - \phi(x')\| \le CN.$$

The map ϕ is 1-to-1 and into $\mathbb{Z}^s$, hence $(X - X) \cap B(0, N)$ is finite. □

Characterization of finite type (cont.)

Note: the address map is usually NOT continuous on $[X]$, which is generically dense in $\mathbb{R}^d$ (unless $[X]$ is a lattice).

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Finite type $\Longrightarrow$ Fin. gen. & the address map is Lipschitz on X

Lemma 1. *Let X be a Delone set with parameters (R, r). Then there exist $k, C_1 > 0$ such that for any two points $x, x' \in X$ there is a chain $x = x_0, x_1, \dots, x_{m-1}, x_m = x'$ in X, with*

(a) $\|x_i - x_{i-1}\| \le kR$ *for all* i;

(b) $m \le C_1 \|x - x'\|$.

Easy: one can take, e.g., $k = 4$ and $C_1 = (2R)^{-1}$.

Characterization of finite type (cont.)

Now let $\phi : [X] \to \mathbb{Z}^s$ be an address map, and define

$$C_2 := \max\{\|\phi(y)\| : \ y \in (X - X) \cap B(0, kR)\}.$$

Using $\mathbb{Z}$-linearity of ϕ on $[X]$ we have for all $x, x' \in X$, by the Lemma,

$$\begin{aligned} \|\phi(x) - \phi(x')\| &\le \sum_{i=1}^{m} \|\phi(x_i) - \phi(x_{i-1})\| \\ &= \sum_{i=1}^{m} \|\phi(x_i - x_{i-1})\| \\ &\le C_2 m \le C_2 C_1 \|x - x'\|. \end{aligned}$$

□

Meyer sets

Definition. A Delone set X in $\mathbb{R}^d$ is *Meyer* if $X - X$ is uniformly discrete (equivalently, Delone).

Theorem 2 (Meyer 1970, 1972; Lagarias 1999) *For a Delone set X in $\mathbb{R}^d$ the following are equivalent:*

(i) *X is Meyer, that is, $X - X$ is Delone.*

(ii) *there is a finite F such that $X - X \subset X + F$*
(this was the original definition of Y. Meyer).

(iii) *X is fin. generated and the address map $\phi : [X] \to \mathbb{Z}^s$ is* almost linear*:*

$$\exists \text{ linear } L : \mathbb{R}^d \to \mathbb{R}^s,\ C_2 > 0 : \quad \|\phi(x) - Lx\| \le C_2 \ \text{ for all } x \in X.$$

(iv) *X is a subset of a non-degenerate cut-and-project set.*

Cut-and-project sets

Definition. Let Λ be a full rank lattice in $\mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^m$. Let $\pi^{\|}$ and $\pi^{\perp}$ be the orthogonal projections onto $\mathbb{R}^d$ and $\mathbb{R}^m$. A *window* Ω is a bounded open subset of $\mathbb{R}^m$. The *cut-and-project set* $X(\Lambda, \Omega)$ associated with the data (Λ, Ω) is

$$X(\Lambda, \Omega) = \pi^{\|}\big(\{w \in \Lambda : \ \pi^{\perp}(w) \in \Omega\}\big).$$

Sometimes we say that $\mathbb{R}^d$ is the "physical space" and $\mathbb{R}^m$ is the "internal space". The cut-and-project set is *non-degenerate* if $\pi^{\|} : \mathbb{R}^n \to \mathbb{R}^d$ is one-to-one. It is *irreducible* if $\pi^{\perp}(\Lambda)$ is dense in $\mathbb{R}^m$. Cut-and-project set (sometimes with different requirements for the window) are also called *model sets*.

Example of a cut-and-project set

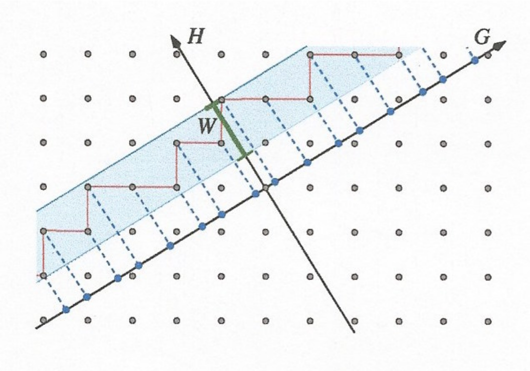

Figure: "Fibonacci set". From the paper *"A short guide to pure point diffraction in cut-and-project sets"* by C. Richard and N. Strungaru, Journal of Physics A: Math. and Theor., Vol. 50, No. 15, 2017

Characterization of Meyer sets (about the proof)

- (i) $\Rightarrow$ (ii), that is, ($X - X$ Delone) $\Rightarrow$ ($X - X \subset X + F$ for F finite) was proved by Lagarias (1996)
- (ii) $\Rightarrow$ (iii), that is

$X - X \subset X + F \Rightarrow$ the address map is almost linear.

It is clear that X is of finite type. Construct $L : \mathbb{R}^d \to \mathbb{R}^s$ as an "ideal address map": for each $y \in \mathbb{R}^d$ define

$$L(y) = \lim_{k \to \infty} \frac{\phi(x_k)}{2^k}, \quad \text{where} \quad x_k \in X \quad \text{satisfies} \quad \|x_k - 2^k y\| \le R.$$

Need to prove that the limit exists and is unique (independent of x_k). Then show that it is linear and within a constant from ϕ on X.

- (iii) $\Rightarrow$ (i), that is, address map is almost linear implies $X - X$ is uniformly discrete.

It is enough to show that there is a lower bound on the norm of

$$z \in (X - X) - (X - X),$$

whenever $z \neq 0$. Suppose $\|z\| \le R$. We have $\|Lz - \phi(z)\| \le 4C_2$, since ϕ is $\mathbb{Z}$-linear on $[X]$, L is linear, and $\|\phi(x) - Lx\| \le C_2, \quad x \in X$. Therefore,

$$|\phi(z)\| \le 4C_2 + \|L\|R.$$

Since ϕ is 1-to-1 on $[X]$ and $\phi(z) \in \mathbb{Z}^s$, there are only finitely many possibilities for z. □

Classes of algebraic integers

A (complex) number η is an *algebraic integer* if $p(\eta) = 0$ for some monic polynomial $p \in \mathbb{Z}[x]$, that is, $p(x) = x^n + \sum_{j=0}^{n-1} a_j x^j$, $a_j \in \mathbb{Z}$. The Galois conjugates of η are the other roots of the minimal polynomial for η.

Definition. Let η be a real algebraic integer greater than one.

(a) η is a *Pisot number* or *Pisot-Vijayaraghavan (PV)*-number if all Galois conjugates satisfy $|\eta'| < 1$.

(b) η is a *Salem number* if for all conjugates $|\eta'| \le 1$ and at least one satisfies $|\eta'| = 1$.

(c) η is a *Perron number* if for all conjugates $|\eta'| < \eta$.

(d) η is a *Lind number* if for all conjugates $|\eta'| \le \eta$ and at least one satisfies $|\eta'| = \eta$.

Inflation symmetries

A Delone set X has an *inflation symmetry* by a real $\eta > 1$ if $\eta X \subset X$.

Theorem 3 (Lagarias 1999 + "folklore") *Let X be a Delone set in $\mathbb{R}^d$ such that $\eta X \subseteq X$ for a real number $\eta > 1$.*

(i) *If X is finitely generated, then X is an algebraic integer.*

(ii) *If X is a Delone set of finite type, then η is a Perron number or a Lind number.*

(ii') *If X is* repetitive *Delone set of finite type, then η is a Perron number.*

(iii) *If X is a Meyer set, then η is a Pisot number or a Salem number.*

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(ii') *If X is* repetitive *Delone set of finite type, then η is a Perron number.*

(iii) *If X is a Meyer set, then η is a Pisot number or a Salem number.*

Definition. A Delone set X is *repetitive* if every X-cluster occurs relatively dense in $\mathbb{R}^d$.

Example: β-integers

Fix $\beta > 1$, with $\beta \notin \mathbb{N}$. Let $X_\beta = X_\beta^+ \bigcup (-X_\beta^+)$, where

$$X_\beta^+ = \left\{ \sum_{j=0}^{N} a_j \beta^j,\ a_j \in \{0, 1, \ldots, \lfloor \beta \rfloor\},\ \text{greedy expansion} \right\}$$

Then X_β is relatively dense in $\mathbb{R}$ and $\beta X \subset X$.

- X_β is Delone iff the orbit of 1 under $T_\beta(x) = \lfloor \beta x \rfloor$ does not accumulate to 0.

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- Delone X_β is finitely generated iff β is an algebraic integer.

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- X_β is Delone iff the orbit of 1 under $T_\beta(x) = \lfloor \beta x \rfloor$ does not accumulate to 0.
- Delone X_β is finitely generated iff β is an algebraic integer.
- Delone X_β is of finite type iff β is a Parry β-number, i.e., the orbit $\{T_\beta^n(1)\}_{n\geq 0}$ is finite.

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- Delone X_β is of finite type iff β is a Parry β-number, i.e., the orbit $\{T_\beta^n(1)\}_{n\geq 0}$ is finite.
- If β is Pisot, then X_β is Meyer.

Example: Delone sets from self-similar tilings

Let $\mathcal{T}$ be a self-similar tiling in $\mathbb{R}^d$ with inflation symmetry by $\eta > 1$ (definition will be given later.) Then the set of *control points* of the tiles is a Delone set with inflation symmetry by $\eta > 1$.

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- Consider a primitive substitution on a finite alphabet and make it into a tiling. On $\mathbb{R}$: interval tiles with lengths corresponding to the Perron-Frobenius eigenvector; the control points are the endpoints. Obtain a finite type Delone set $X(\mathcal{T})$.

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- For every Perron number $\eta > 1$ there is a finite type Delone set $X = X(\mathcal{T}) \subset \mathbb{R}$, such that $\eta X \subset X$ [D. Lind (1984)].

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- Such an X is Meyer if and only if η is Pisot.

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- For every Perron number $\eta > 1$ there is a finite type Delone set $X = X(\mathcal{T}) \subset \mathbb{R}$, such that $\eta X \subset X$ [D. Lind (1984)].
- Such an X is Meyer if and only if η is Pisot.
- For every Salem number η there exists X Meyer such that $\eta X \subset X$ (not from substitution) [Y. Meyer (1972)].

Algebraicity of inflations

Lemma 2. *Let X be a finitely generated Delone set in $\mathbb{R}^d$ such that $QX \subset X$ for some expanding linear map $Q : \mathbb{R}^d \to \mathbb{R}^d$. Then all eigenvalues of Q are algebraic integers.*

Proof.

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- $\{v_j\}_{j \le s}$ spans $\mathbb{R}^d$, hence $\mathrm{rank}(V) = d$.
- Let $\mathbf{e}$ be a left eigenvector of Q for an eigenvalue λ. Then $\lambda \mathbf{e} V = \mathbf{e} QV = \mathbf{e} VM \ \Rightarrow \ \mathbf{e} V$ is a left eigenvector for M.
- $\mathbf{e} V \neq 0$, because the rows of V are linearly independent.
- $M \in \mathbb{Z}^{s \times s}$, so all its eigenvalues are algebraic integers. □

Inflations for finite type Delone sets are Lind

Proof. Assume $0 \in X$, $\eta X \subset X$ for $\eta > 1$, and $X \subset \mathbb{R}^d$ is a Delone set of finite type. Continue the argument in the Lemma.

- Let γ be a Galois conjugate of η. Since η is an eigenvalue of M, so is γ. We want to prove $|\gamma| \le \eta$.
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- $\eta V = QV = VM \;\Rightarrow\; \eta x = \eta V\phi(x) = VM\phi(x) \;\Rightarrow\; \phi(\eta x) = M\phi(x)$.
- $\phi(\eta^n x) = M^n \phi(x)$.
- $\|M^n\phi(x)\| = \|\phi(\eta^n x)\| = \|\phi(\eta^n x) - \phi(0)\| \le C\|\eta^n x\| = C\eta^n\|x\|$

  by Theorem 1. It follows that $|\gamma| \le \eta$. □

Example revisited: the set of β-integers X_β

- Fix $\beta > 1$ algebraic integer, such that $\{T_\beta^n 1\}_{n\ge 0}$ does not accumulate to zero. Then $X_\beta \subset \mathbb{R}$ is Delone and $[X_\beta] = \mathbb{Z}[\beta]$.
- Free generators can be chosen $v_j = \beta^{j-1}$, $j \le s$, where s is the degree of β. Let $c_0 + c_1 x + \cdots + c_{s-1}x^{s-1} + x^s$ be the minimal (with integer coefficients) polynomial for β.
- We have $Qx = \beta x$ on $\mathbb{R}$, and $QX_\beta \subset X_\beta$. Then $QV = VM$, where $V = [v_1, \dots, v_s]$ (a $1 \times s$ matrix), and

$$M = \begin{pmatrix} 0 & \dots & \dots & 0 & -c_0 \\ 1 & 0 & \dots & 0 & -c_1 \\ 0 & 1 & \dots & 0 & -c_2 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 0 & -c_{s-2} \\ 0 & 0 & \dots & 1 & -c_{s-1} \end{pmatrix}$$

The set of β-integers (cont.)

- Let ϕ be the associated address map, $\phi : [X_\beta] = \mathbb{Z}[\beta] \to \mathbb{R}^s$. We have

$$\phi(\beta^n) = M^n \phi(1) = M^n \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

- Now suppose that β is Pisot. Then we have

$$\phi(\beta^n) = \beta^n e_\beta + O(\varrho^n), \tag{1}$$

where e_β is the eigenvector of M corresponding to β and $\varrho \in (0,1)$ is the maximal absolute value of the Galois conjugates of β.

The set of β-integers (cont.)

- Define $L : \mathbb{R} \to \mathbb{R}^s$ by $L(x) = xe_\beta$, a linear map.
- We want to show that $\|\phi(x) - Lx\| \le C$ on X_β, whence X_β is a Meyer set.
- In view of (1) we have for $x = \sum_{j=0}^N a_j\beta^j \in X_\beta^+$:

$$\begin{aligned}\|\phi(x) - Lx\| &= \left\|\phi\Big(\sum_{j=0}^N a_j\beta^j\Big) - L\Big(\sum_{j=0}^N a_j\beta^j\Big)\right\| \\ &= O\Big(\max_j |a_j| \cdot \sum_{j=0}^N \varrho^j\Big) = O(1).\end{aligned}$$

$\square$

The set of β-integers (cont.)

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□

- The same proof works e.g. for the set of endpoints of a self-similar tiling on $\mathbb{R}$ with a Pisot inflation factor.

Delone m-sets

Consider Delone sets where each point has a "color" or "type" from a finite list. Formally:

Definition. An m-multiset in $\mathbb{R}^d$ is

- $\overline{\Lambda} = \Lambda_1 \times \cdots \times \Lambda_m \subset \mathbb{R}^d \times \cdots \times \mathbb{R}^d$ (m copies).
- We also write $\overline{\Lambda} = (\Lambda_1, \ldots, \Lambda_m) = (\Lambda_i)_{i \le m}$.
- $\overline{\Lambda} = (\Lambda_i)_{i \le m}$ is a *Delone m-set* in $\mathbb{R}^d$ if each Λ_i is Delone and

$$\mathrm{supp}(\overline{\Lambda}) := \bigcup_{i=1}^{m} \Lambda_i \subset \mathbb{R}^d \text{ is Delone.}$$

Substitution Delone sets

Definition. Let $Q : \mathbb{R}^d \to \mathbb{R}^d$ be a linear expanding map (i.e. all eigenvalues are greater than one in absolute value).

- $\overline{\Lambda} = (\Lambda_i)_{i \le m}$ is a *substitution Delone m-set with expanding map Q* if there are finite sets $\mathcal{D}_{ij}$ for $i, j \le m$ (possibly empty) such that

$$\Lambda_i = \biguplus_{j=1}^{m} (Q\Lambda_j + \mathcal{D}_{ij}), \quad i \le m. \tag{2}$$

- The *substitution matrix* S is $\mathsf{S}_{ij} = \sharp(\mathcal{D}_{ij})$.

Algebraicity of expansion map

Lemma 3. *Suppose that $\overline{\Lambda}$ is a substitution Delone m-set with expansion map Q. If $\mathrm{supp}(\overline{\Lambda})$ is finitely generated, then all eigenvalues of Q are algebraic integers.*

Proof sketch. Consider the set of "inter-atomic vectors"

$$\Xi(\overline{\Lambda}) := \bigcup_{i=1}^{m} (\Lambda_i - \Lambda_i).$$

We have $[\Xi(\overline{\Lambda})]$ finitely generated, and $Q(\Xi(\overline{\Lambda})) = \Xi(\overline{\Lambda})$. Then the proof proceeds as in Lemma 2 above. □

Theorem 4 (Lagarias and Wang 2003). *If $\overline{\Lambda}$ is a primitive substitution Delone m-set with expansion map Q, then the Perron-Frobenius (PF) eigenvalue of the substitution matrix* S *equals* $|\det(Q)|$.

Perron-Frobenius condition

Theorem 4 (Lagarias and Wang 2003). *If $\overline{\Lambda}$ is a primitive substitution Delone m-set with expansion map Q, then the Perron-Frobenius (PF) eigenvalue of the substitution matrix* S *equals* $|\det(Q)|$.

Denote $\lambda(\mathsf{S})$ = PF eigenvalue of S. In fact,

$$\begin{array}{ccc} \overline{\Lambda} \quad \text{is relatively dense} & \Rightarrow & \lambda(\mathsf{S}) \geq |\det(Q)| \\ \overline{\Lambda} \quad \text{is uniformly discrete} & \Rightarrow & \lambda(\mathsf{S}) \leq |\det(Q)| \\ \mathrm{supp}(\overline{\Lambda}) \quad \text{is Delone} & \Rightarrow & \lambda(\mathsf{S}) = |\det(Q)|. \end{array}$$

Adjoint system of equations

Given

$$\Lambda_i = \biguplus_{j=1}^{m} (Q\Lambda_j + \mathcal{D}_{ij}), \quad i \le m,$$

set up the *adjoint system of equations*

(3) $$QA_j = \bigcup_{i=1}^{m} (\mathcal{D}_{ij} + A_i), \quad j \le m.$$

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Theory of (graph-directed) iterated function systems: (3) has a unique solution $(A_1, \ldots, A_m)$ where $\emptyset \ne A_i \subset \mathbb{R}^d$ are compact.

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Theory of (graph-directed) iterated function systems: (3) has a unique solution $(A_1, \ldots, A_m)$ where $\emptyset \ne A_i \subset \mathbb{R}^d$ are compact.

Theorem 4 (Lagarias and Wang 2003). *If $(\Lambda_i)_{i \le m}$ is a primitive substitution Delone m-set, then all $A_i = \mathrm{clos}(A_i^\circ)$ (closure of the interior), and interiors in the RHS of (3) are disjoint.*

Definition.

- $\mathcal{A} = \{T_1, \ldots, T_m\}$ is a set of *tiles* in $\mathbb{R}^d$; these will be our *prototiles*.
- Each tile is the closure of its interior and has a "type" (or "color").
- $\mathcal{P}_{\mathcal{A}}$ is the set of patches made up of *translated* prototiles.
- $\omega : \mathcal{A} \to \mathcal{P}_{\mathcal{A}}$ is a *tile-substitution* with expanding map Q if there exist finite sets $\mathcal{D}_{ij} \subset \mathbb{R}^d$ for $i, j \le m$, such that

$$\omega(T_j) = \bigcup_{i=1}^{m} (T_i + \mathcal{D}_{ij}), \tag{4}$$

and $\mathrm{supp}(\omega(T_j)) = QA_j$ for all $j \le m$.

Self-affine and self-similar tilings

- The substitution (4) is extended by $\omega(x + T_j) = Qx + \omega(T_j)$, and to patches and tilings by $\omega(P) = \bigcup\{\omega(T) : \ T \in P\}$.
- ω can be iterated, producing larger and larger patches $\omega^k(T_j)$.
- Substitution matrix: $\mathsf{S}_{ij} := \sharp(\mathcal{D}_{ij})$. The substitution ω is primitive if S is primitive.
- If $\omega(\mathcal{T}) = \mathcal{T}$ for a primitive ω, we say that $\mathcal{T}$ is *self-affine*. Usually FLC is also assumed.
- If $Q = \eta\mathcal{O}$ for some $\eta > 1$ and an orthogonal linear transformation $\mathcal{O}$, then $\mathcal{T}$ is *self-similar*.
- For a self-similar tiling in $\mathbb{R}^2 \cong \mathbb{C}$ consider the *complex expansion factor* $\lambda \in \mathbb{C}$, $|\lambda| > 1$, by identifying Q with $z \mapsto \lambda z$.

From a substitution tiling to a substitution Delone set

If $\omega(\mathcal{T}) = \mathcal{T}$, then $\mathcal{T} = \bigcup_{j=1}^m (T_j + \Lambda_j)$ for some Delone Λ_j, hence

$$\begin{aligned}
\biguplus_{i=1}^m (T_i + \Lambda_i) = \mathcal{T} = \omega(\mathcal{T}) &= \biguplus_{j=1}^m \big(\omega(T_j) + Q\Lambda_j\big) \\
&= \biguplus_{j=1}^m \Big(\biguplus_{i=1}^m (T_i + \mathcal{D}_{ij}) + Q\Lambda_j\Big) \\
&= \biguplus_{i=1}^m \Big(T_i + \biguplus_{j=1}^m (Q\Lambda_j + \mathcal{D}_{ij})\Big).
\end{aligned}$$

Thus $\Lambda_i = \biguplus_{j=1}^m (Q\Lambda_j + \mathcal{D}_{ij}), \quad i \le m$. In general, Λ_i need not be disjoint, but can be made so by translating T_j's. Then $(\Lambda_i)_{i \le m}$ is a substitution Delone m-set.

From a substitution Delone set to a substitution tiling

Let $\overline{\Lambda} = (\Lambda_i)_{i \le m}$ be a substitution Delone set with expansion Q. Consider the solution of the adjoint equation $(A_1, \ldots, A_m)$ and define the prototiles to be $T_i = (A_i, i)$.

Question. Is $\biguplus_{j=1}^{m}(T_j + \Lambda_j)$ necessarily a tiling of $\mathbb{R}^d$?

From a substitution Delone set to a substitution tiling

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Question. Is $\biguplus_{j=1}^{m} (T_j + \Lambda_j)$ necessarily a tiling of $\mathbb{R}^d$?

[Lagarias and Wang (2003)] showed that, in general, "no". However, there are verifiably sufficient conditions for "yes". Then we say that $\overline{\Lambda}$ is *representable*.

This was extended in [Lee, Moody and Solomyak (2003)].

Definition. For a substitution Delone m-set $\overline{\Lambda} = (\Lambda_i)_{i \le m}$ satisfying (2), define a matrix $\Phi = (\Phi_{ij})_{i,j=1}^m$ whose entries are finite (possibly empty) families of linear affine transformations on $\mathbb{R}^d$ given by

$$\Phi_{ij} = \{f : x \mapsto Qx + a : \ a \in \mathcal{D}_{ij}\}.$$

We define $\Phi_{ij}(\mathcal{X}) := \bigcup_{f \in \Phi_{ij}} f(\mathcal{X})$ for a set $\mathcal{X} \subset \mathbb{R}^d$. For an m-set $\overline{\mathcal{X}} = (\mathcal{X}_i)_{i \le m}$ let

$$\Phi(\overline{\mathcal{X}}) = \left(\bigcup_{j=1}^m \Phi_{ij}(\mathcal{X}_j) \right)_{i \le m}.$$

Thus $\Phi(\overline{\Lambda}) = \overline{\Lambda}$ by definition. We say that Φ is an *m-set substitution*.

Representable substitution Delone sets (cont.)

Let $\overline{\Lambda}$ be a substitution Delone m-set and Φ the associated m-set substitution.

- $\overline{\Lambda}$-cluster $\overline{P} = (P_i)_{i \le m}$ is *legal* if it is a translate of a subcluster of $\Phi^k(\{x_j\})$ for some $x_j \in \Lambda_j$ and $k \in \mathbb{N}$. (Here $\{x_j\}$ is an m-set which is empty in all coordinates other than j, for which it is a singleton.)
- $\overline{\Lambda}$ is representable if and only if every $\overline{\Lambda}$-cluster is legal [LMS (2003)]
- $\overline{\Lambda}$-cluster $\overline{P}$ is *generating* if $\overline{P} \subset \Phi(\overline{P})$ and $\overline{\Lambda} = \lim_{n\to\infty} \Phi^n(\overline{P})$.
- $\overline{\Lambda}$ is representable if there exists a legal generating cluster.

Application: pseudo-self-affine tilings

Definition. A repetitive FLC tiling $\mathcal{T}$ of $\mathbb{R}^d$ is *pseudo-self-affine with expansion* Q if $\mathcal{T}$ is *locally derivable* from $Q\mathcal{T}$ (see N. P. Frank's Lecture Notes for precise definition).

- E. A. Robinson, Jr. conjectured that every pseudo-self-affine tiling is *mutually locally derivable* with a self-affine tiling. This was settled for $d = 2$ in [Frank and Solomyak (2001)] and in higher dimensions in [Solomyak (2005)]
- Caveat: in both papers we had to pass from Q to a higher power Q^k, that is, a pseudo-self-affine tiling with expansion Q was proved to be MLD with a self-affine tiling with expansion Q^k for some $k \in \mathbb{N}$.

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- Caveat: it is harder to control topological properties.

Characterization of expansion maps

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Theorem (Thurston). *If $\mathcal{T}$ is a self-similar tiling of $\mathbb{R}^2 \cong \mathbb{C}$, then its expansion constant $\lambda \in \mathbb{C}$ is* **complex Perron.**

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- (My personal view: more likely to work for some power λ^k.)
- What about self-affine? Higher dimensions?

Characterization of expansion maps (cont.)

Theorem (Kenyon 1990, Kenyon and Solomyak 2010). *Let Q be a diagonalizable (over $\mathbb{C}$) expansion map on $\mathbb{R}^d$, and let $\mathcal{T}$ be a self-affine tiling of $\mathbb{R}^d$ with expansion Q. Then every eigenvalue of Q is an algebraic integer, and if λ is an eigenvalue of Q and γ is a Galois conjugate of λ, then either $|\gamma| < |\lambda|$, or γ is also an eigenvalue of Q of greater or equal multiplicity.*

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- It is conjectured that the "generalized Perron condition" is also sufficient, at least in the weak sense: given a generalized Perron Q, there exists a self-affine tiling with expansion Q^k for some k.

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- It is conjectured that the "generalized Perron condition" is also sufficient, at least in the weak sense: given a generalized Perron Q, there exists a self-affine tiling with expansion Q^k for some k.
- Possible strategy: construct a substitution Delone m-set.

Dynamical systems from Delone sets and tilings (in brief)

- For a Delone m-set Λ in $\mathbb{R}^d$ let X_Λ be the collection of all m-sets each of whose clusters is a translate of a Λ-cluster.
- X_Λ is the "hull" of Λ. Natural $\mathbb{R}^d$ action by translations.
- Introduce the usual "big ball" metric ρ, so that $(X_\Lambda, \mathbb{R}^d)$ is compact and $(X_\Lambda, \mathbb{R}^d)$ is a topological dynamical system ($\mathbb{R}^d$-action).

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- Introduce the usual "big ball" metric ρ, so that $(X_\Lambda, \mathbb{R}^d)$ is compact and $(X_\Lambda, \mathbb{R}^d)$ is a topological dynamical system ($\mathbb{R}^d$-action).
- Invariant (Borel probability) measures.
- Assuming FLC, the system $(X_\Lambda, \mathbb{R}^d)$ is uniquely ergodic iff Λ has Uniform Cluster Frequencies (UCF), i.e. for any cluster P, the limit

$$\mathrm{freq}(P, \Lambda) = \lim_{N \to \infty} \frac{\sharp\{x \in \mathbb{R}^d : \ x + P \subset B(0, N) \cap \Lambda\}}{Vol(B(0, N))} \geq 0$$

exists uniformly in $x \in \mathbb{R}^d$.

- For a measure-preserving system $(X_\Lambda, \mathbb{R}^d, \mu)$ consider the associated group of unitary operators $\{U_g\}_{g \in \mathbb{R}^d}$ on $L^2(X_\Lambda, \mu)$:

$$U_g f(\Gamma) = f(-g + \Gamma), \quad \Gamma \in X_\Lambda, \ g \in \mathbb{R}^d.$$

- A vector $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ is an *eigenvalue* for the $\mathbb{R}^d$-action if there exists an eigenfunction $0 \not\equiv f \in L^2(X_\Lambda, \mu)$:

$$U_g f = e^{2\pi i \langle g, \alpha \rangle} f, \quad g \in \mathbb{R}^d.$$

Here $\langle g, \alpha \rangle$ is the scalar product in $\mathbb{R}^d$.

Spectrum of systems from substitution Delone sets

- A primitive FLC substitution Delone *m*-set has UCF $\Rightarrow$ uniquely ergodic system $(X_\Lambda, \mathbb{R}^d, \mu)$.

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Spectrum of systems from substitution Delone sets

- A primitive FLC substitution Delone *m*-set has UCF $\Rightarrow$ uniquely ergodic system $(X_\Lambda, \mathbb{R}^d, \mu)$.
- Parallel theory for self-affine tilings; in fact, many of the proofs use the link with tilings.

Theorem 5 (Lee-Solomyak 2008). *Let Λ be a representable primitive FLC substitution Delone m-set. The set of eigenvalues for the $\mathbb{R}^d$-action is relatively dense in $\mathbb{R}^d$ if and only if $\mathrm{supp}(\Lambda)$ is a Meyer set.*

Corollary. *A representable primitive FLC substitution Delone m-set is "pure point diffractive" if and only if its support is Meyer.*

This answered a question of Lagarias.

About the proof of Theorem 5

- The new part was

relatively dense eigenvalues $\Rightarrow$ Meyer set

- The proof proceeds via Pisot families. A set $\mathcal{P}$ of algebraic integers is a *Pisot family* if for every $\lambda \in \mathcal{P}$ and every Galois conjugate λ' of λ, if $\lambda' \notin \mathcal{P}$, then $|\lambda'| < 1$.
- If λ is a Pisot number, then $\{\lambda\}$ is a Pisot family. If λ is a complex Pisot number, then $\{\lambda, \overline{\lambda}\}$ is a Pisot family. Let $\|x\| := \mathrm{dist}(x, \mathbb{Z})$.

Theorem (Körnei 1987, Mauduit 1989) *Let $\lambda_1, \ldots, \lambda_r$ be distinct algebraic numbers $|\lambda_i| \geq 1$, $i \leq r$, and let $P_1, \ldots, P_r$ be nonzero polynomials with complex coefficients. If $\sum_{i=1}^r P_i(n)\lambda_i^n \in \mathbb{R}$ for all n and*

$$\lim_{n\to\infty} \left\| \sum_{i=1}^{r} P_i(n)\lambda_i^n \right\| = 0 \implies \{\lambda_1, \ldots, \lambda_r\} \text{ is a Pisot family}$$

About the proof of Theorem 5 (cont.)

The link from eigenvalues to number theory proceeds via the following (restated in different terms here):

Theorem (Solomyak 1997) *Let $\Lambda = (\Lambda_i)_{i \le m}$ be a representable primitive FLC substitution Delone m-set. Let $\Xi(\Lambda) = \bigcup_{i=1}^{m} (\Lambda_i - \Lambda_i)$ be the set of "inter-atomic" vectors. If $\alpha \in \mathbb{R}^d$ is an eigenvalue for $(X_\Lambda, \mathbb{R}^d, \mu)$, then*

$$\lim_{n \to \infty} \| \langle Q^n x, \alpha \rangle \| = 0 \quad \textit{for all } x \in \Xi(\Lambda).$$

Partial converse

Theorem (Lee-Solomyak 2012) *Let $\Lambda = (\Lambda_i)_{i \leq m}$ be a representable primitive FLC substitution Delone m-set. Suppose that the expansion map Q has irreducible over $\mathbb{C}$ characteristic polynomial. Then the following are equivalent:*

(i) *The spectrum of Q is a Pisot family;*

(ii) *the set of eigenvalues of $(X_\Lambda, \mathbb{R}^d, \mu)$ is relatively dense in $\mathbb{R}^d$;*

(iii) *$(X_\Lambda, \mathbb{R}^d, \mu)$ is not weakly mixing (i.e. has a non-zero eigenvalue);*

(iv) *$\mathrm{supp}(\Lambda)$ is a Meyer set.*