Tiling Dynamical Systems (school)
CIRM, November 20-24, 2017

Multiscale Substitution Tilings

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November 21st, 2017

A generalization of standard substitution tilings

- $\mathcal{F} = \{T_1, \ldots, T_n\}$ - a finite set of prototiles in $\mathbb{R}^d$.
- $\zeta_1, \ldots, \zeta_m \in (0,1)$ - *contraction factors* (more than one!).
- $\mathcal{F}^*(\zeta_1, \ldots, \zeta_m)$ - patches by tiles of the form $\zeta_j T_i$.
- A *multiscale subdivision rule* is a mapping $H : \mathcal{F} \to \mathcal{F}^*(\zeta_1, \ldots, \zeta_m)$, with $supp(H(T_i)) = T_i$.

Example: $\#\mathcal{F} = 2, \quad (\zeta_1, \zeta_2) = (\frac{1}{3}, \frac{2}{3})$

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The inflation process

- We define an action of $\mathbb{R}_{\geq 0}$ on patches, S_t(patch) (t is a continuous *time* parameter).
- For $t = 0$, $S_0(T_i) = H(T_i)$.
- As time progresses, inflate at constant speed, where every tile that reaches its original size is dissected via H.
- A *multiscale substitution tiling* is a tiling of $\mathbb{R}^d$ that is a limit, in the right topology, of a sequence of elements of the form $S_t(T_i)$ (i may varies), and their translations.

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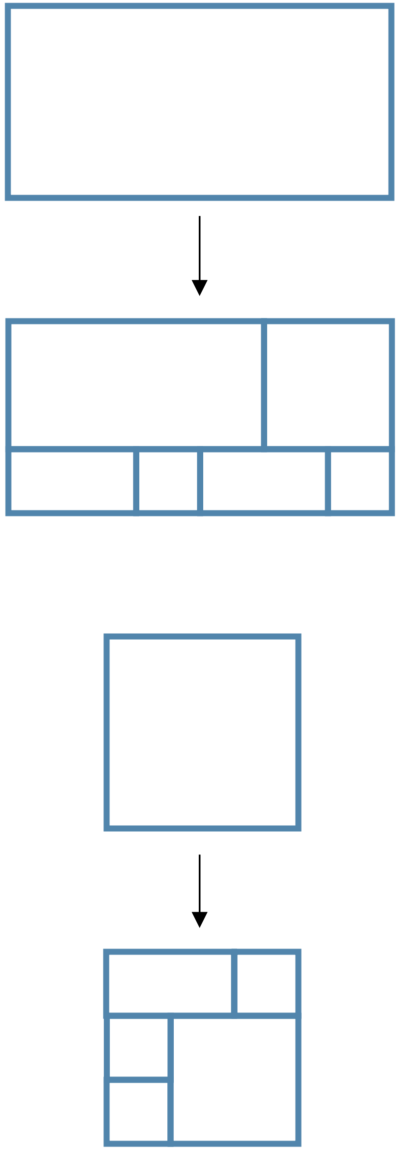
Illustration

Substitution rule:



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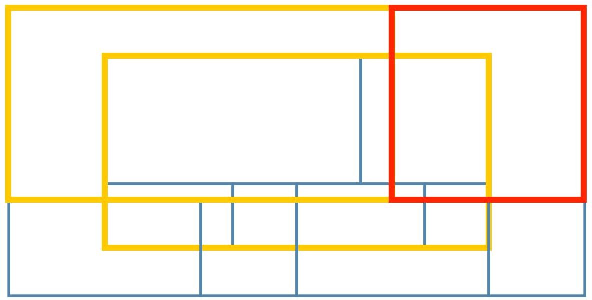

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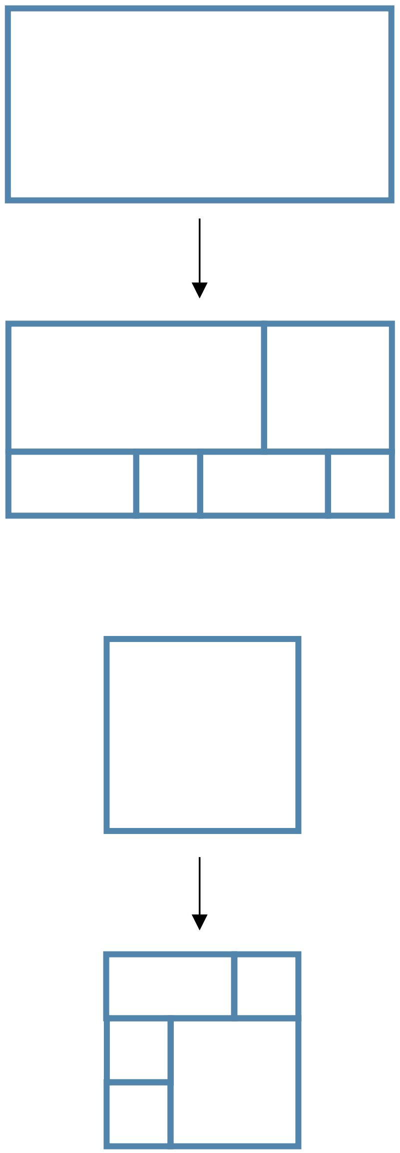

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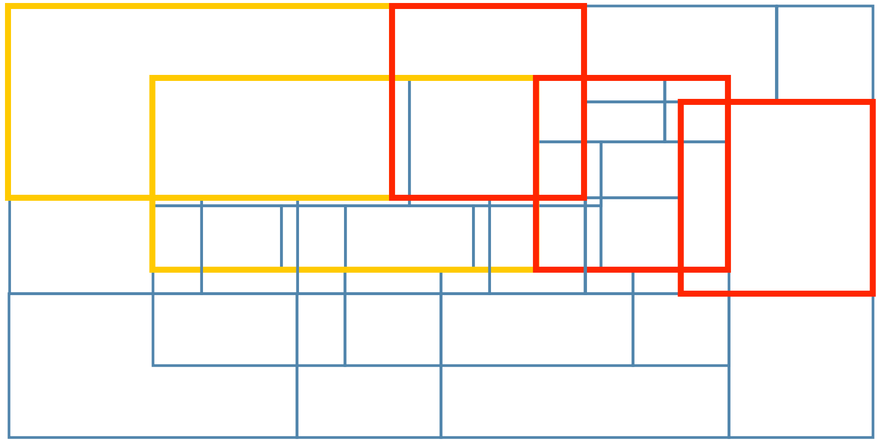

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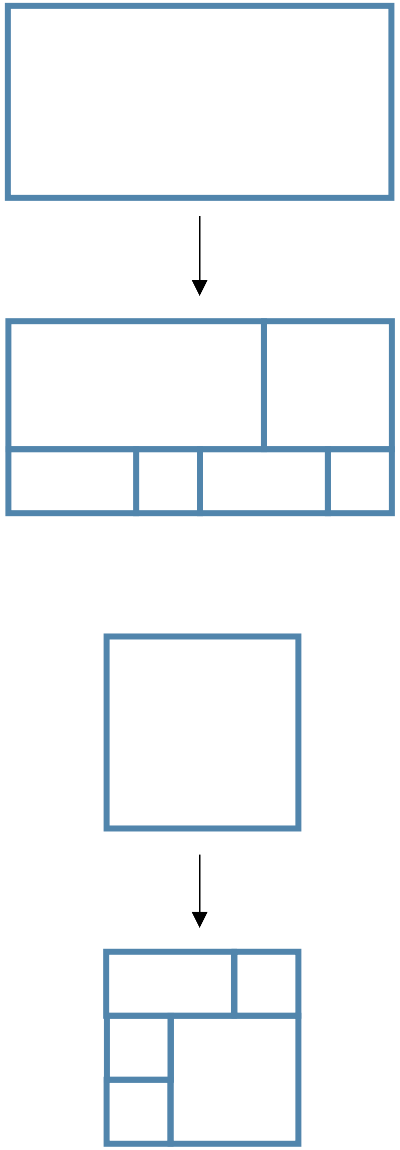

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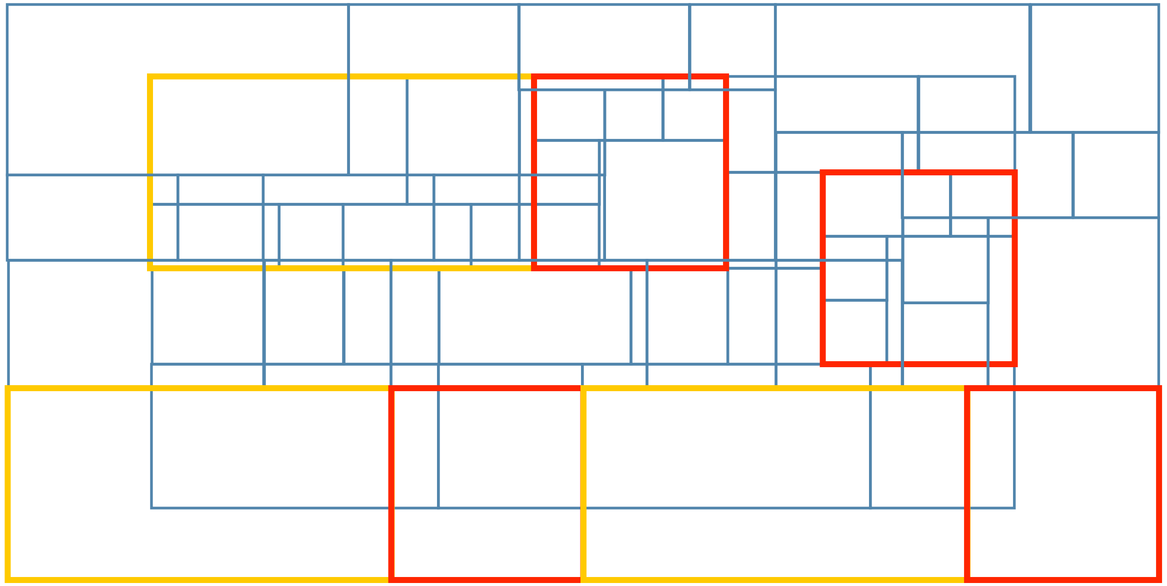

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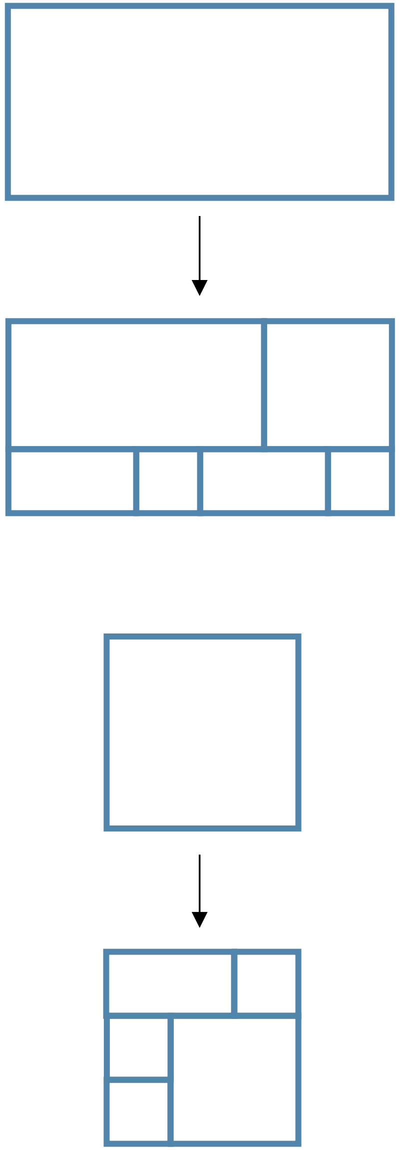

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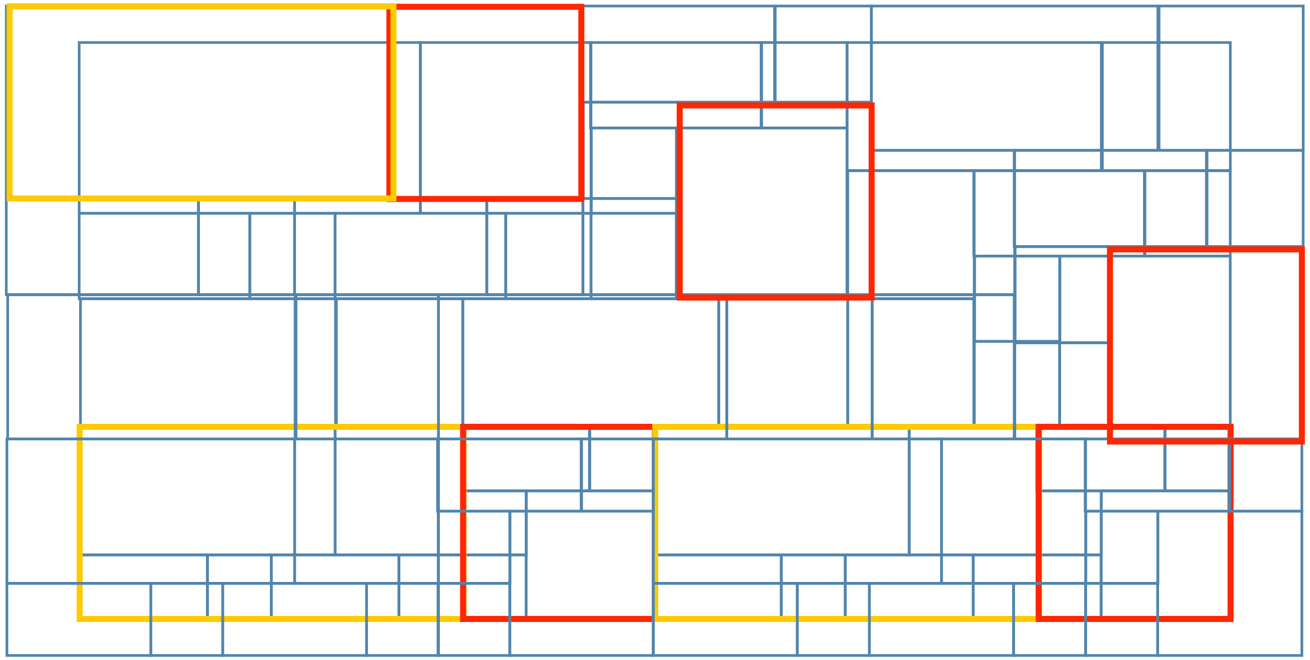

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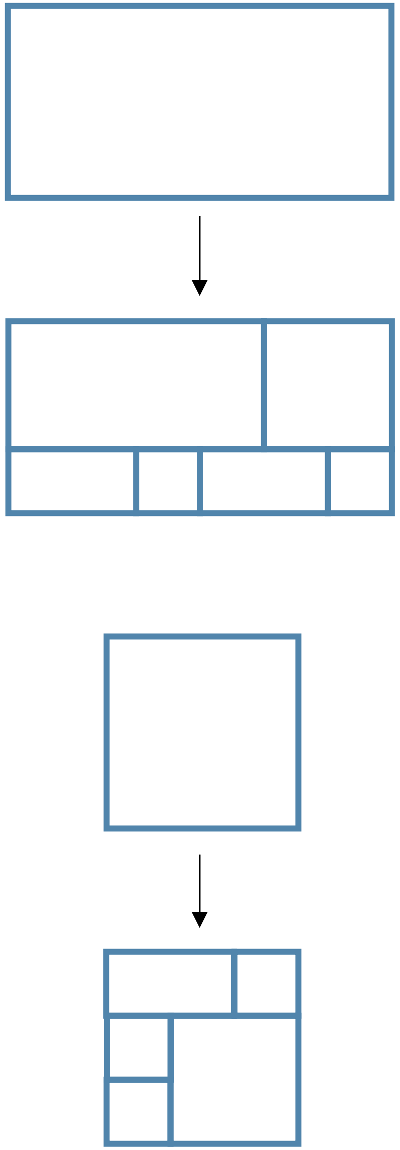

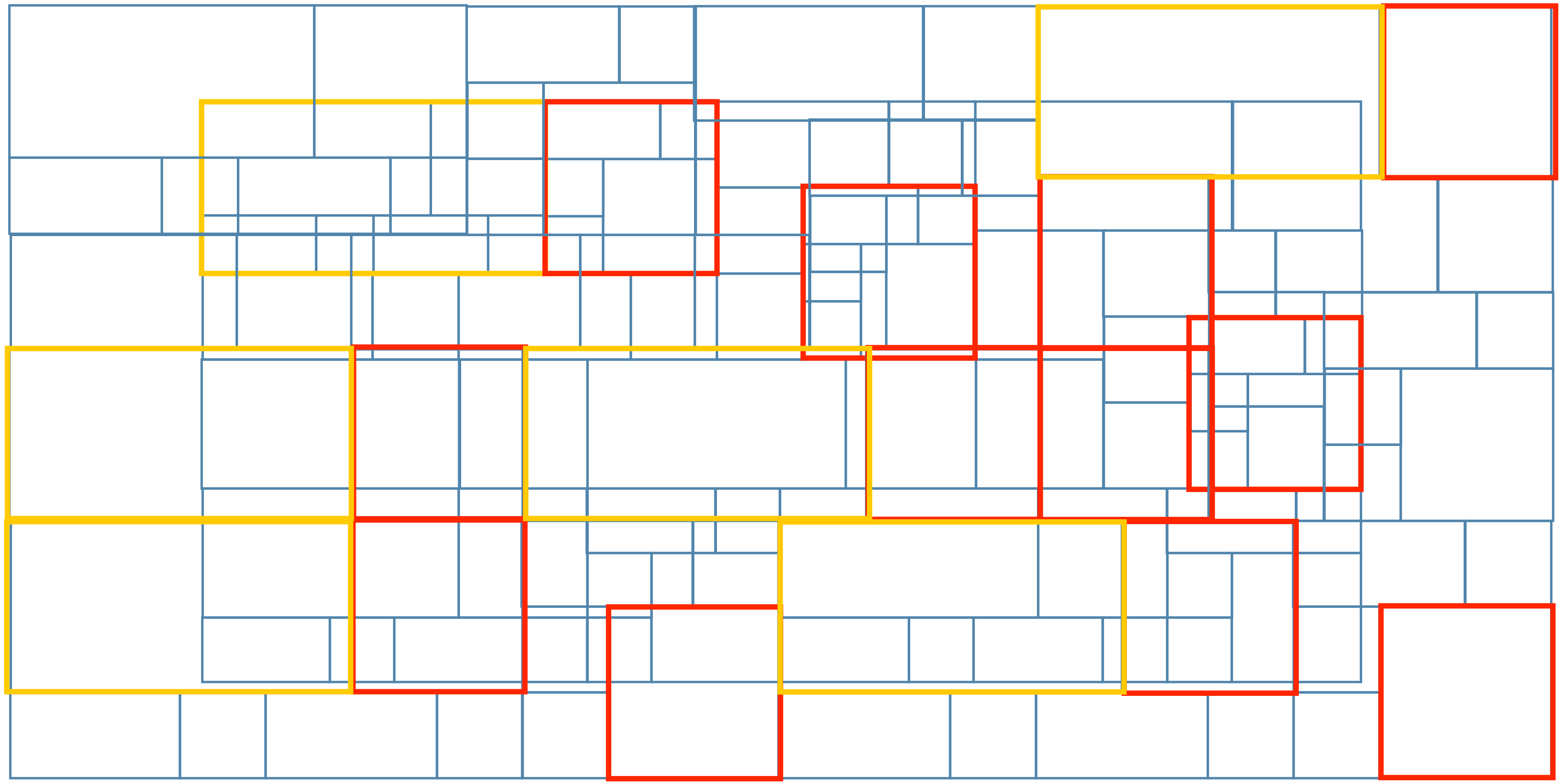

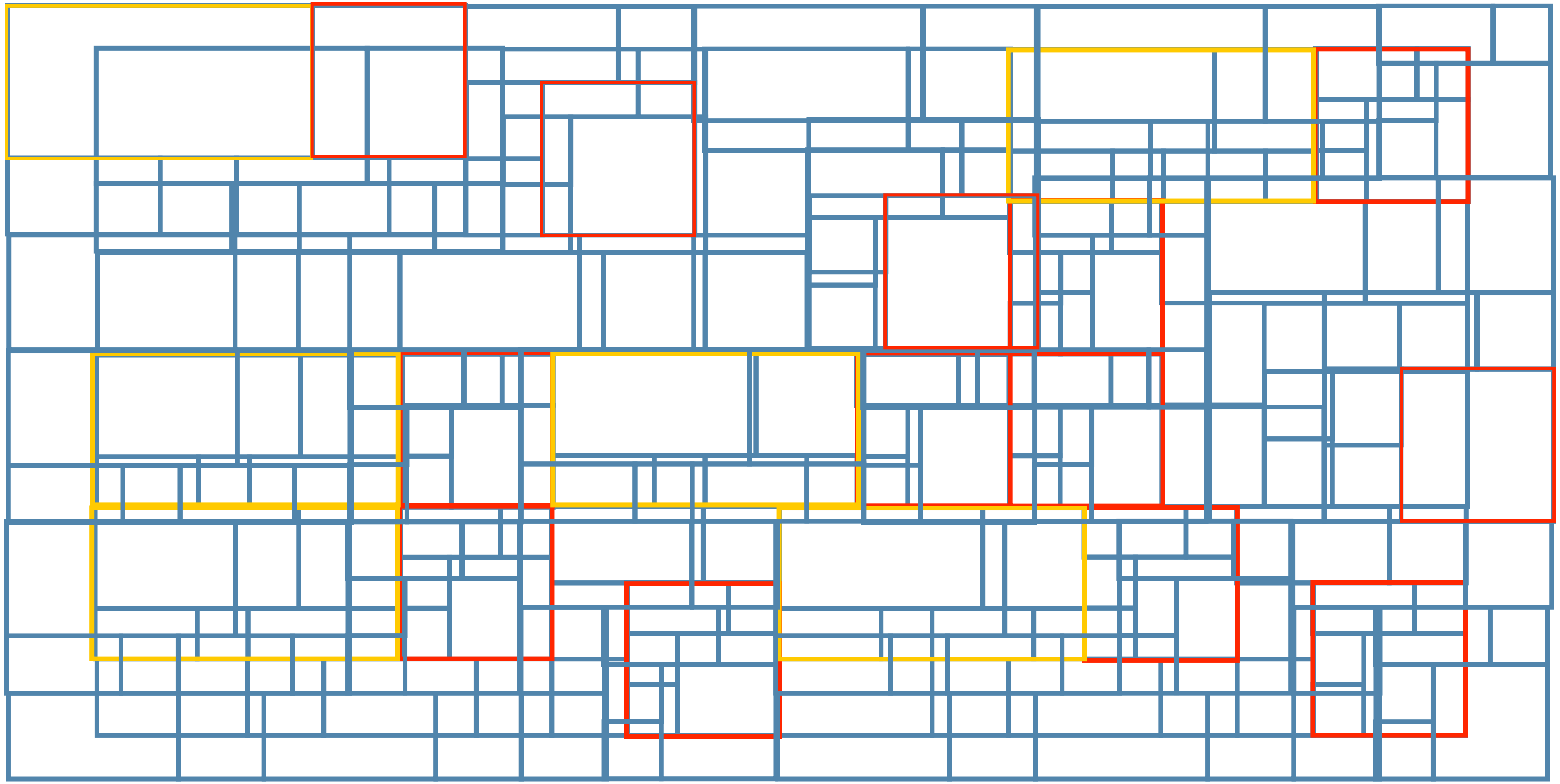

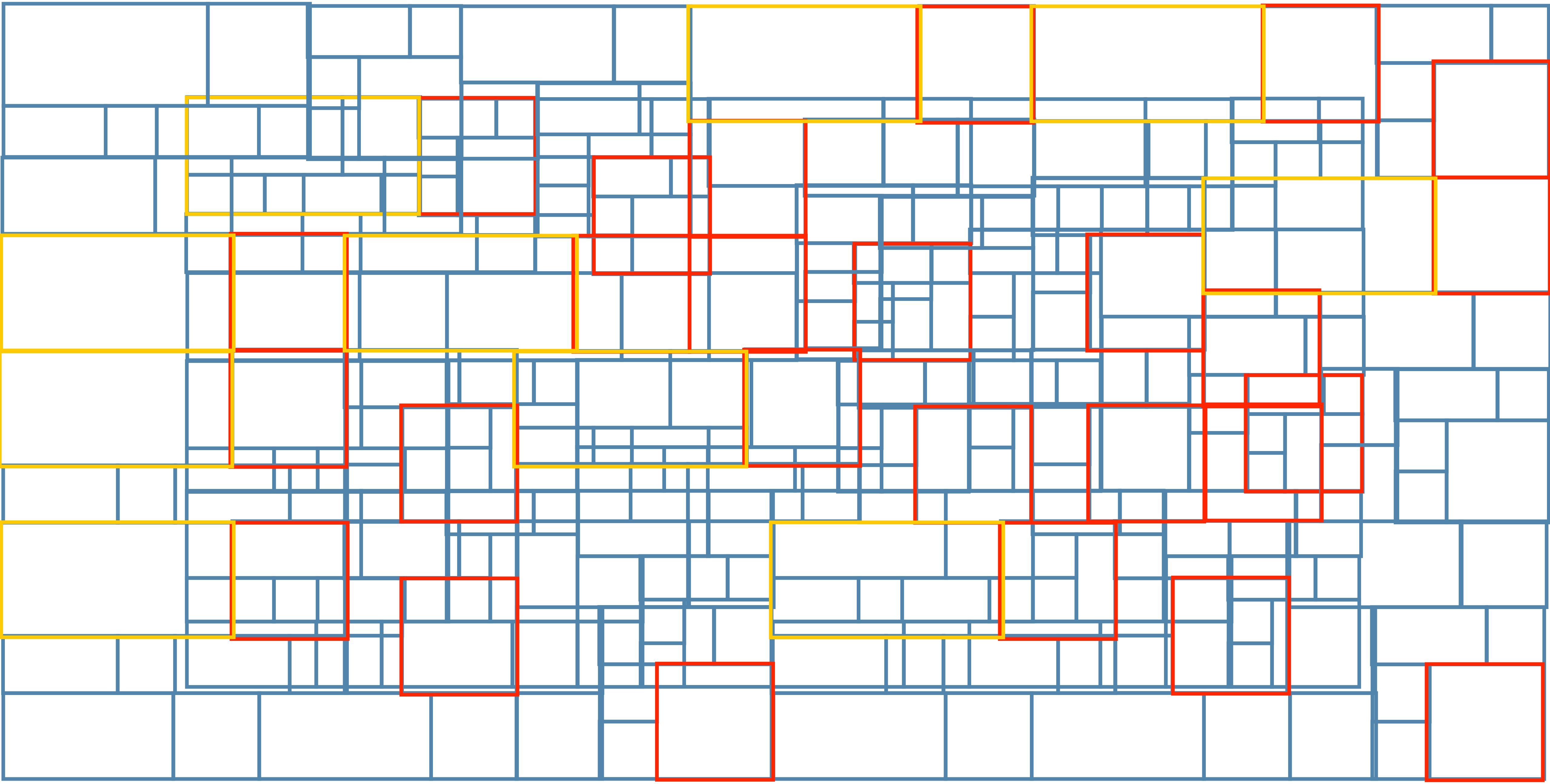

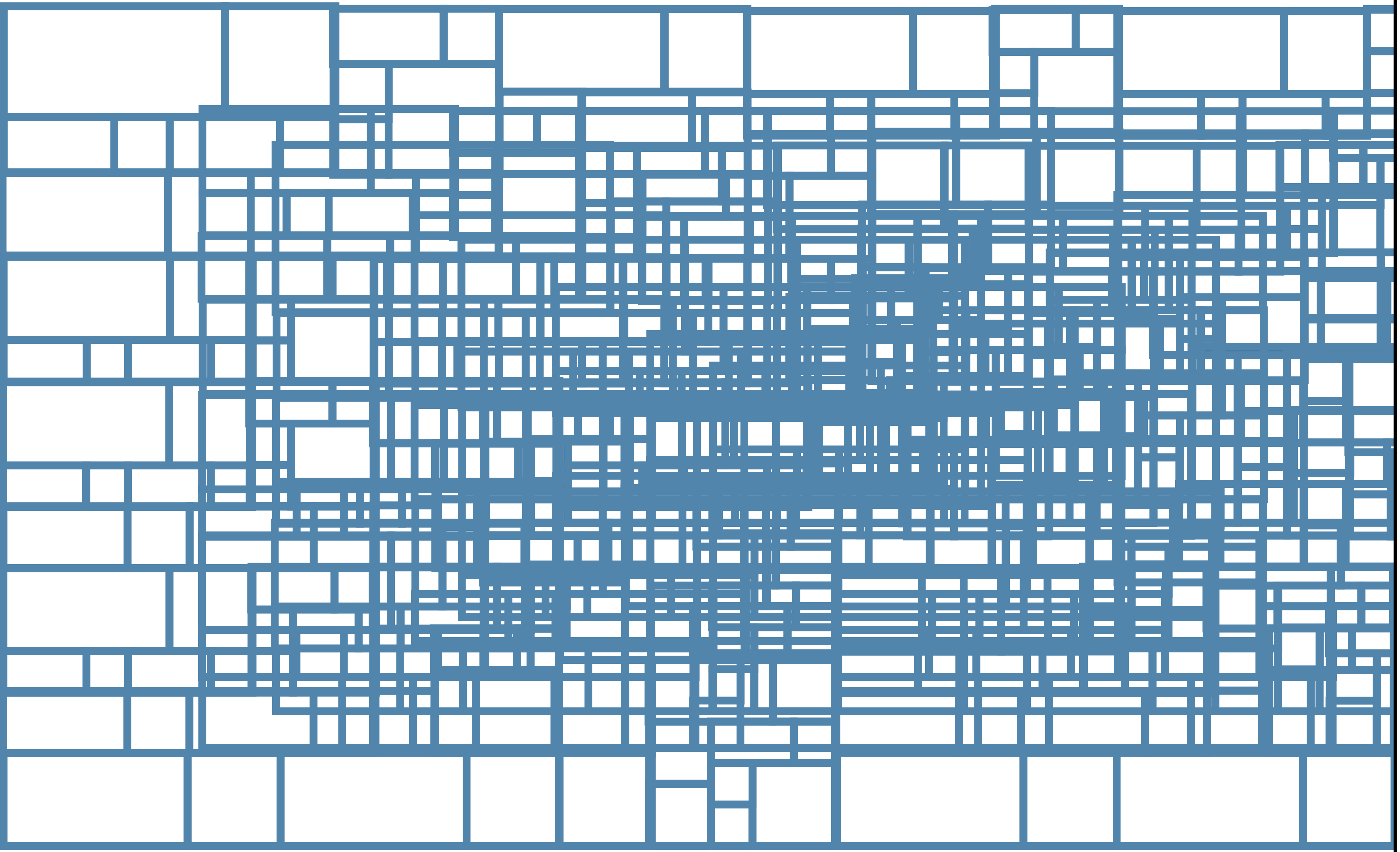

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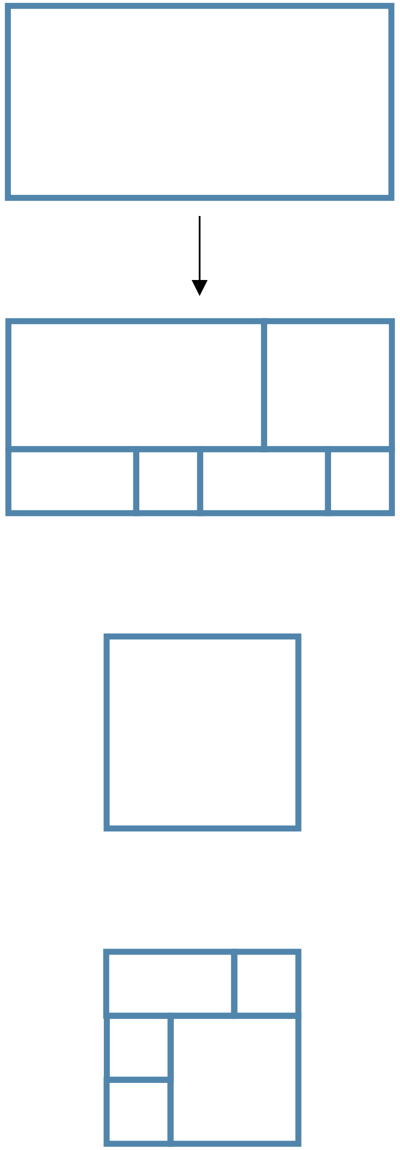

The Chabauty-Fell Topology (the "rubber" topology)

- Let $\mathcal{X}$ be the space of closed subsets of $\mathbb{R}^d$. For $F_1, F_2 \in \mathcal{X}$ define:
$$D(F_1,F_2) \overset{\text{def}}{=} \inf\left(\left\{\varepsilon > 0 : \begin{array}{l} F_1 \cap B(0,1/\varepsilon) \subseteq U_\varepsilon(F_2) \\ F_2 \cap B(0,1/\varepsilon) \subseteq U_\varepsilon(F_1) \end{array}\right\} \cup \{1\}\right)$$
- $(\mathcal{X}, D)$ is a compact metric space!
- $\mathcal{X}_H \neq \varnothing$: If $t_k \to \infty$ the $supp[S_{t_k}(T_i)]$ exhaust $\mathbb{R}^d$, and hence every partial limit of $(S_{t_k}(T_i))_{k=1}^{\infty}$ is a tiling of $\mathbb{R}^d$.
- Every tile in every tiling in $\mathcal{X}_H$ is similar to a prototile.
- Questions: In what ways this construction is similar/different from the standard substitution framework?

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