Outer billiards outside regular polygons: sets of full measure and an existence of an aperiodic point

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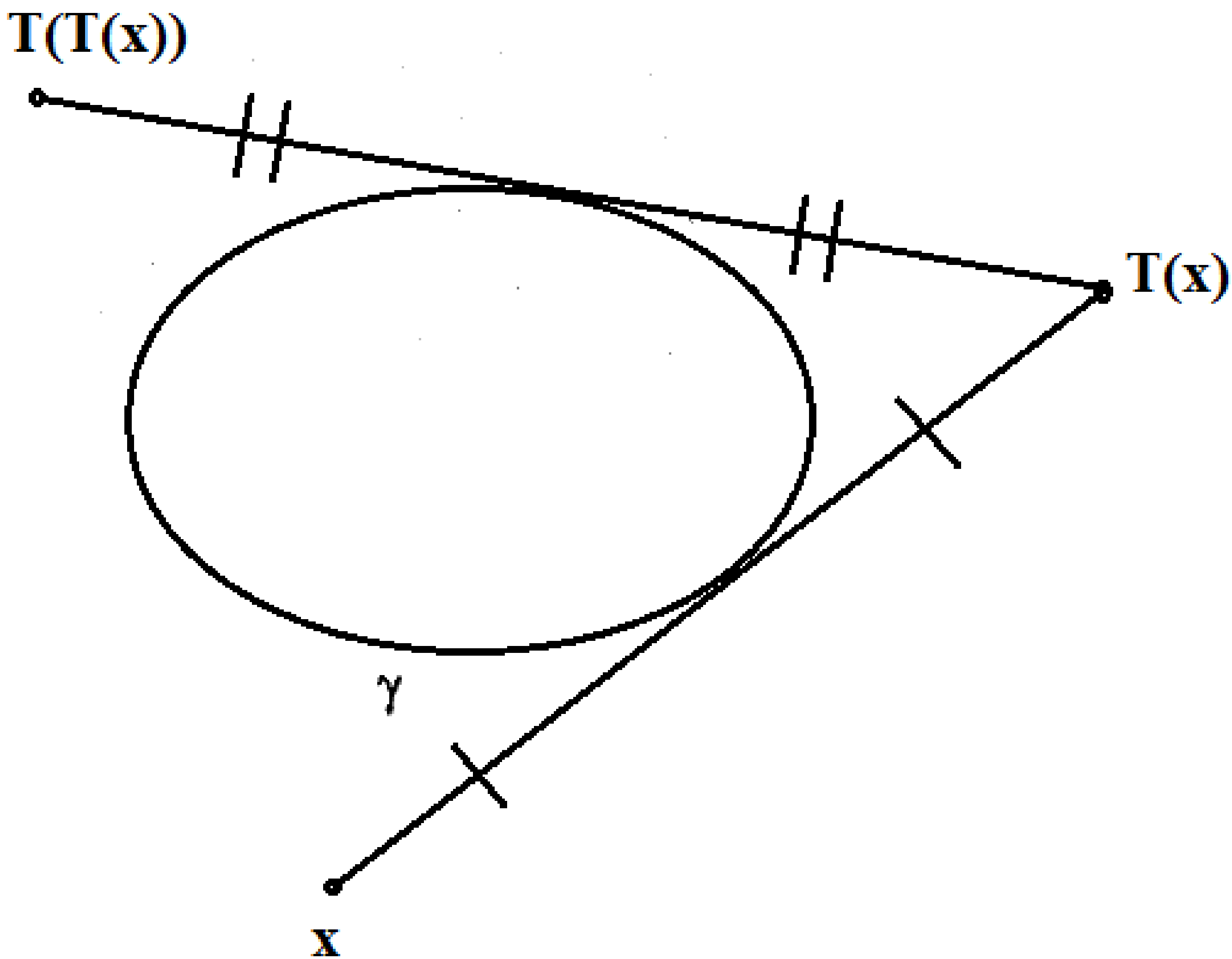
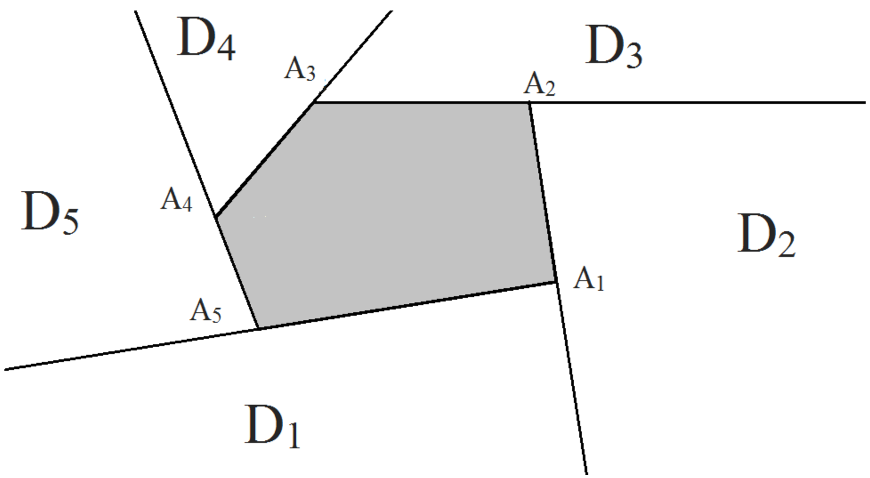
Marseille, 22.11.2017.

Outer billiard map T: definition

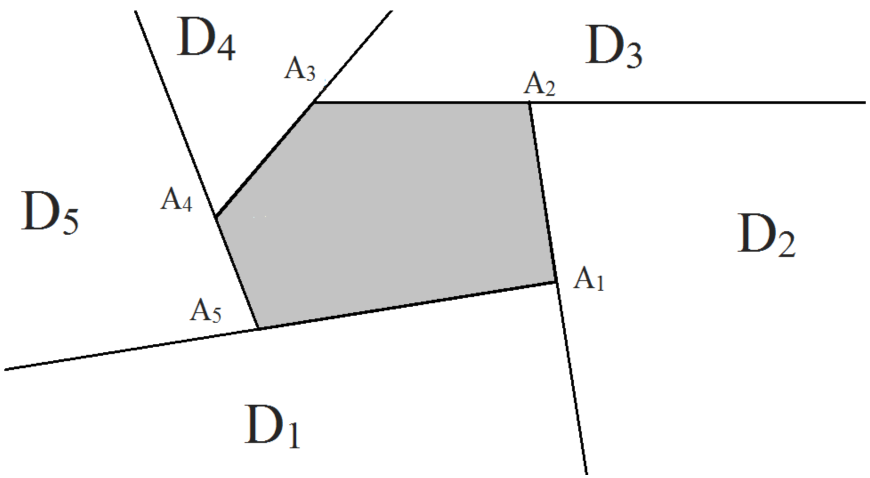


Outer billiards outside convex polygons



- In case of n-gons (n = 5 in the picture), there are set of points for which tangent point is not defined; this set consists of n rays;
- For these points, outer billiard map is not defined;
- The complement of these rays is divided by these rays into n pieces. For each of them, the map is a rotation around the corresponding vertex;
- So, outer billiard map outside convex polygon is a piecewise isometry.

Outer billiards outside convex polygons



- We will say that all points outside polygon are divided into three parts:
  - finite points (point x is finite if $T^n(x)$ is not defined for some integer (may be negative) n);
  - periodic points (if $T^n(x) = x$ for some n);
  - aperiodic points (all other points).

Outer billiards outside regular polygons: problems and current state

- Open problems:
  - for which natural n, there are *aperiodic* points for outer billiard outside regular n-gon?
  - for which natural n, set of periodic points for outer billiard outside regular n-gon is set of full measure?
- In cases: n = 3, 4, 6, there are no aperiodic points; then, sets of periodic points are of full measure. These cases are affine-equivalent to lattice polygons.
- In 1993, S.Tabachnikov proved that aperiodic points exist in case of regular pentagon. Later, he proved that for this case, set of periodic points is of full measure.
- Cases n = 3, 4, 6, 5 are only known cases.

The main results

- Theorem (R., arXiv:1505.06332, 2015): There exists an aperiodic point for outer billiards outside regular dodecagon and octagon.
- Theorem 2: For outer billiard outside regular octagon, set of periodic points is a set of full measure.
- Theorem 3: All possible periods for outer billiard outside regular polygon are union of the following sets:
  - $\{12 * 9^n - 4*(-3)^n,\ 12 * 9^n + 4*(-3)^n,\ 4*9^n \mid n \in \mathbb{Z},\ n \geqslant 0\}$;
  - $\{8,\ 8k,\ 8k*9^n \mid n, k \in \mathbb{Z},\ n \geqslant 0,\ k \geqslant 2\}$;
  - $\{24k*9^n - 4*(-3)^n + 12*9^n \mid n, k \in \mathbb{Z},\ n \geqslant 0,\ k \geqslant 2\}$;
  - $\{24k*9^n + 4*(-3)^n - 4*9^n \mid n, k \in \mathbb{Z},\ n \geqslant 0,\ k \geqslant 2\}$.

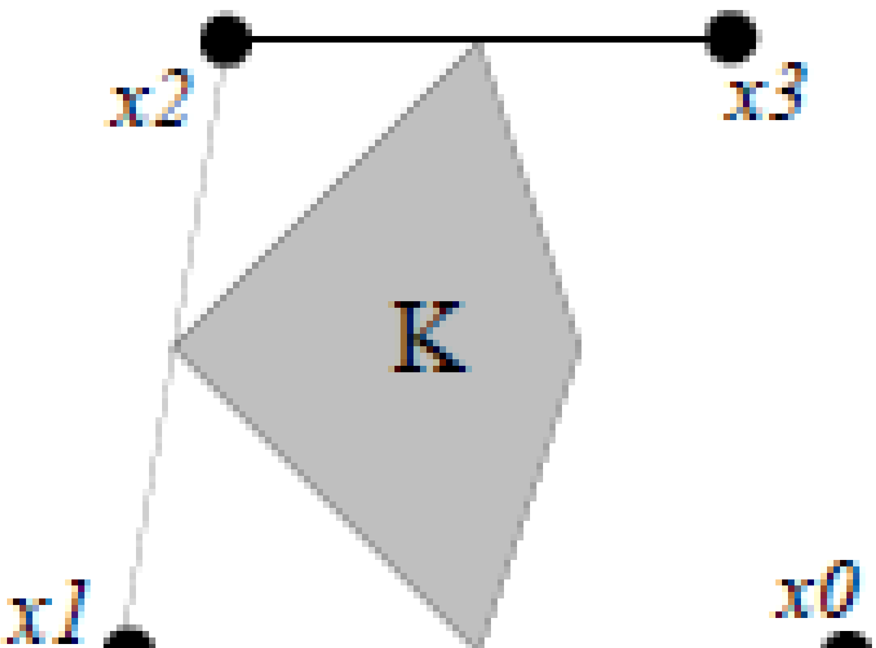
Thank you for attention!
Questions?

History of outer billiard maps: a Moser – Neumann question

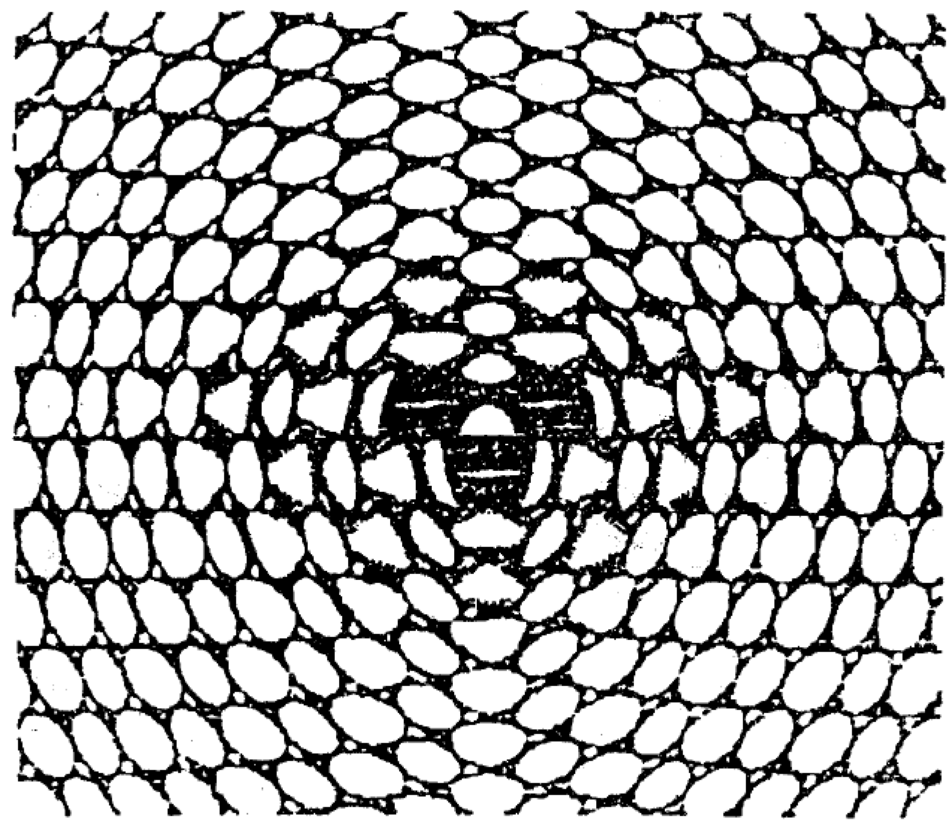
- Outer billiards were introduced in 1950s by B.Neumann;
- J.Moser popularized outer billiards in 1970s, as a toy model of celestial mechanics;
- Also, Moser wrote down a **question** asked earlier by Neumann: is it possible for outer billiard trajectory to be unbounded?

Moser – Neumann question: YES

- R.Schwartz kite (2007):



- Semicircle (D. Dolgopyat , B. Fayad, 2009)



Outer billiards outside convex polygons: problem of aperiodic points

- Theorem (Vivaldi-Shaidenko (1987), Kolodziej (1989), Gutkin-Simanyi (1991)): if table is quasi-rational, then all outer billiard map trajectories are bounded;
- Corollary: all trajectories for lattice and regular polygons are bounded.
- Also, C.Culter in 2004 proved that for outer billiard outside any convex polygon, there exists a periodic point; the proof was published in 2007 by S.Tabachnikov.
- We consider two problems:
  - for which natural n, there are *aperiodic* points for outer billiard outside regular n-gon?
  - for which natural n, set of periodic points for outer billiard outside regular n-gon is set of full measure?