Invariant measures for actions of congruent monotilable amenable groups

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- $\mathcal{M}(X, T, G)$ is a *Choquet Simplex*: convex set in which any element is written in a unique way in terms of the extreme points.

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Also...

- $\mathcal{M}(X, T, G)$ is an invariant for **Orbit Equivalence**.
- (X, T, G) and (Y, S, Γ) are *Orbit equivalent* if there exists an homeomorphism $h : X \to Y$ such that for all $x \in X$

$$\{T^g(x) : g \in G\} = \{S^\gamma(h(x)) : \gamma \in \Gamma\}$$

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  - $\rightsquigarrow$ Residually finite: G admits a decreasing sequence of finite index normal subgroups with trivial interesection.

Congruent Monotileable Amenable Groups

If G is amenable, we may assume

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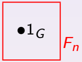

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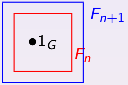

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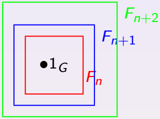

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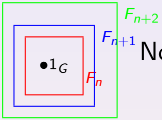

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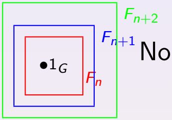

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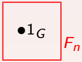

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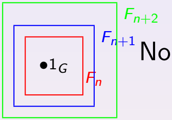

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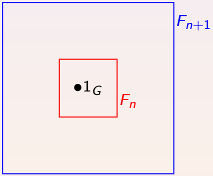

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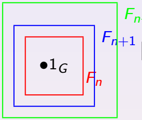

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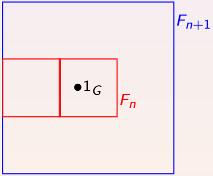

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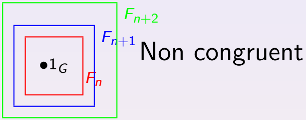

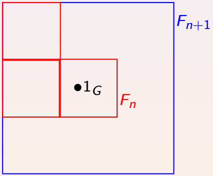

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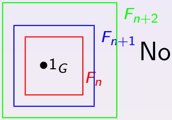

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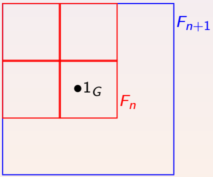

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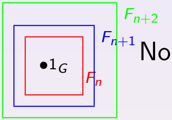

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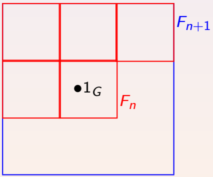

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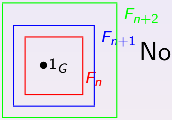

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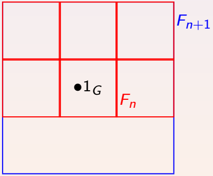

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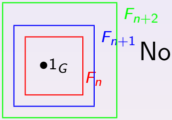

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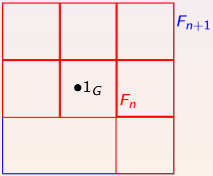

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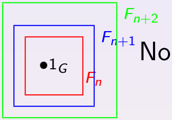

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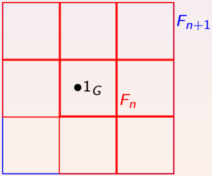

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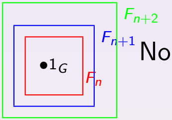

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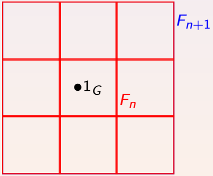

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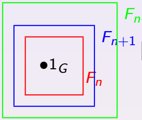

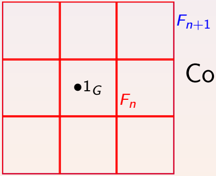

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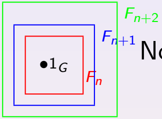

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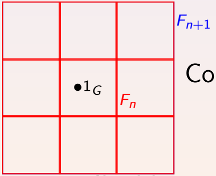

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⇝ Amenable residually finite groups are **congruent monotileable** (Cortez-Petite 14).

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Theorem (C., Cortez 17)

For any Choquet simplex K and any congruent monotileable amenable group G, there exists a minimal action T of G on the Cantor set X (a G-subshift), such that $K \cong \mathcal{M}(X, T, G)$. If G is abelian, the action is free.

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Corollary

Any Choquet simplex can be seen as the set of invariant measures of a free minimal action of $\mathbb{Q}$ on the Cantor space.