Invariant measures for actions of congruent monotilable amenable groups

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CIRM, Marseille, November 2017
Framework

Dynamical System: \((X, T, G)\) such that
- \(X\) compact metric space (usually Cantor),
- \(G\) countable group.
- \(T\) action of \(G\) on \(X\) by homeomorphisms.

Example: \(X = A^G\), \(T\) is the \(G\)-shift on \(X\).

Invariant measure: \(\mu\) probability measure on \(X\) such that \(\forall A \in B(X), \mu(T^g(A)) = \mu(A) \forall g \in G\).

\(G\) amenable \(\iff\) the set of invariant measures \(\mathcal{M}(X, T, G)\) is a nonempty convex set. Extreme points: ergodic measures.

\(\Rightarrow\) Amenable group: \(G\) admits a Følner sequence of finite subsets \((F_n)_{n \geq 0}\).

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\lim_{n \to \infty} \frac{|F_n g \setminus F_n|}{|F_n|} = 0 \quad \forall g \in G
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\(\mathcal{M}(X, T, G)\) is a Choquet Simplex: convex set in which any element is written in a unique way in terms of the extreme points.
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- \(\mathcal{M}(X, T, G)\) is a Choquet Simplex: convex set in which any element is written in a unique way in terms of the extreme points.
Questions

Given any Choquet simplex $K$, is there a (minimal) action of $G$ on a Cantor space $X$ such that $M(X, T, G)$ is affine homeomorphic to $K$? If we prescribed $G$ as well? Also...

$M(X, T, G)$ is an invariant for Orbit Equivalence. ($X, T, G$) and ($Y, S, \Gamma$) are Orbit equivalent if there exists an homeomorphism $h: X \rightarrow Y$ such that for all $x \in X \{ Tg(x) : g \in G \} = \{ S\gamma(h(x)) : \gamma \in \Gamma \}$. 

Paulina CECCHI B. (IRIF/USACH)  
Inv. measures and group actions  
CIRM, Nov. 2017 3 / 6
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Also...

- $\mathcal{M}(X, T, G)$ is an invariant for **Orbit Equivalence**.
- $(X, T, G)$ and $(Y, S, \Gamma)$ are **Orbit equivalent** if there exists an homeomorphism $h : X \rightarrow Y$ such that for all $x \in X$

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\{ T^g(x) : g \in G \} = \{ S^\gamma(h(x)) : \gamma \in \Gamma \}
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It is known...

Given any Choquet simplex $K$, there is a Toeplitz subshift $\left( X, T, Z \right)$ on $X = \{0, 1\}$ such that $K \sim M(\left( X, T, Z \right))$.

Given any Choquet simplex $K$ and any residually finite countable amenable group $G$, there exists a Toeplitz $G$-subshift $\left( X, T, G \right)$ on a Cantor space $X$ such that $K \sim M(\left( X, T, G \right))$. 

$\Rightarrow$ Residually finite: $G$ admits a decreasing sequence of finite index normal subgroups with trivial intersection.
It is known...

- (Downarowicz 91) Given any Choquet simplex $K$ there is a Toeplitz subshift $(X, T, \mathbb{Z})$ on $X = \{0, 1\}^\mathbb{Z}$ such that $K \cong \mathcal{M}(X, T, \mathbb{Z})$.
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$\rightsquigarrow$ Amenable residually finite groups are congruent monotileable

(Cortez-Petite 14).
Results

Theorem (C., Cortez 17)
For any Choquet simplex $K$ and any congruent monotileable amenable group $G$, there exists a minimal action $T$ of $G$ on the Cantor set $X$ (a $G$-subshift), such that $K \sim M(X, T, G)$. If $G$ is abelian, the action is free.

Theorem (C., Cortez 17)
Any countable amenable nilpotent group is congruent monotileable.

Corollary
Any Choquet simplex can be seen as the set of invariant measures of a free minimal action of $Q$ on the Cantor space.
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