Self-generated sequences

Let $k \geq 1$ be an integer. The *Oldenburger-Kolakoski sequence* (A000002), denoted S_k, is the unique sequence over the alphabet $\{1, k\}$, starting with 1, which is identical to its own run-length sequence, e.g. $S_2 = 12211212212211211221211212211...$

Let $\delta(k)$ be the density of $1's$ in S_k. Keane's conjectured that $\delta(2) = \frac{1}{2}$. It is "easy" to compute $\delta(k)$ for k odd, e.g.

$$\begin{aligned}
\delta(3) &= \frac{7}{6} - \frac{1}{6}\sqrt[3]{\frac{43+3\sqrt{177}}{2}} - \frac{1}{6}\sqrt[3]{\frac{43-3\sqrt{177}}{2}}, \\
\delta(5) &= 1 - \frac{1}{4}\sqrt[3]{3+2\sqrt{2}} - \frac{1}{4}\sqrt[3]{3-2\sqrt{2}}, \\
\delta(7) &= \frac{17}{18} - \frac{1}{18}\sqrt[3]{\frac{371+9\sqrt{1497}}{2}} - \frac{1}{18}\sqrt[3]{\frac{371-9\sqrt{1497}}{2}}.
\end{aligned}$$

Iterated matrices

Let $k \in \mathbb{Z}$ be a parameter. Consider the permutation matrices

$$
\begin{aligned}
A_0 &= (1)_{1\times 1}, \\
B_0 &= (1)_{1\times 1}, \\
A_{n+1} &= \begin{pmatrix} 0_{2^n\times 2^n} & A_n \\ B_n & 0_{2^n\times 2^n} \end{pmatrix}_{2^{n+1}\times 2^{n+1}}, \\
B_{n+1} &= \begin{pmatrix} 0_{2^n\times 2^n} & (A_n)^k \\ (B_n)^k & 0_{2^n\times 2^n} \end{pmatrix}_{2^{n+1}\times 2^{n+1}}.
\end{aligned}
$$

Eigenvalues

For all $n \geq 1$,

(i) trivial: if k is odd then any eigenvalue λ of $\frac{1}{2}(A_n + B_n)$ satisfies $\lambda \in \{-1, +1\}$;

(ii) non-trivial[1]: if $k = 0$ then any eigenvalue λ of $\frac{1}{2}(A_n + B_n)$ satisfies $\lambda = \cos(r\pi)$, for some $r \in \mathbb{Q}$;

(iii) open problem: we don't know the algebraic structure of the eigenvalues of $\frac{1}{2}(A_n + B_n)$ for k even and $k \neq 0$.

[1]Proved by the speaker.