The Airy point process in the two-periodic Aztec diamond

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(Joint work with Sunil Chhita and Vincent Beffara)

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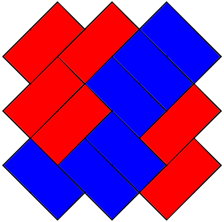
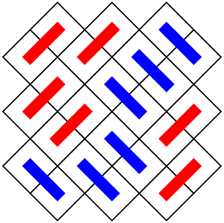
Aztec diamond

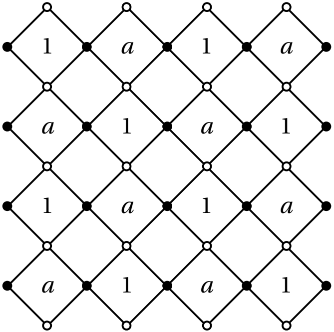
A **domino tiling** of an Aztec diamond shape corresponds to a **dimer configuration** on the Aztec graph



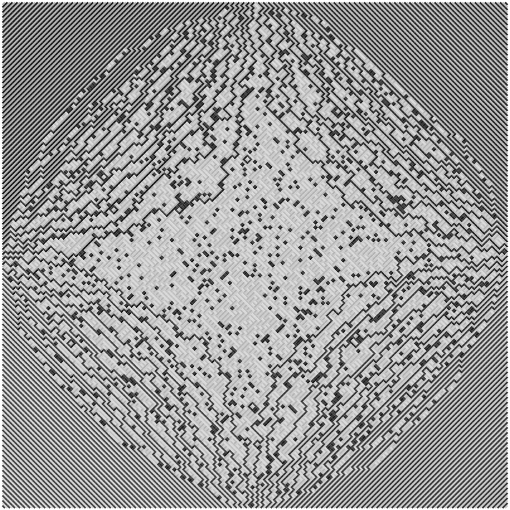


Two Periodic Weighting

The **two-periodic weighting** of the Aztec diamond is defined in the following way. For a two-colouring of the faces, the edge weights around a particular coloured face alternate between a and 1. E.g. for a size 4 Aztec diamond



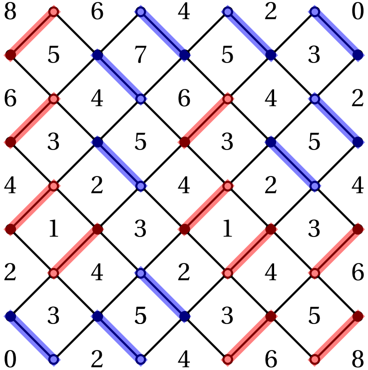
Random tiling of a two-periodic Aztec diamond



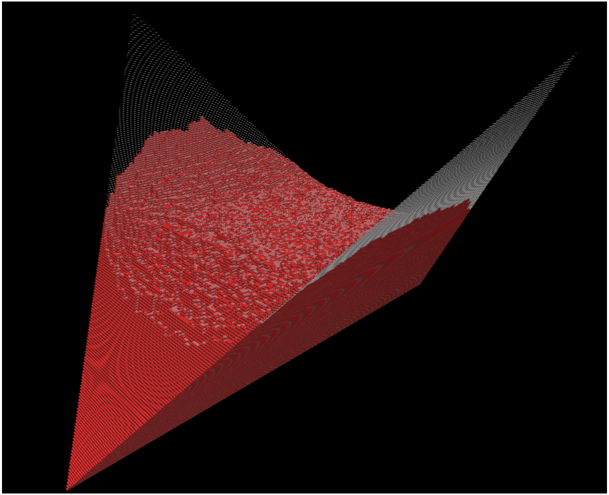
Aztec diamond height function

To each tiling of an Aztec diamond we can associate a **height function**. The heights sit on the faces of the Aztec graph. The height differences between two faces are given by

- $+3$ (-3) if we cross a dimer with a white vertex to the right (left)
- $+1$ (-1) if we do not cross a dimer and have a white vertex to the left (right)

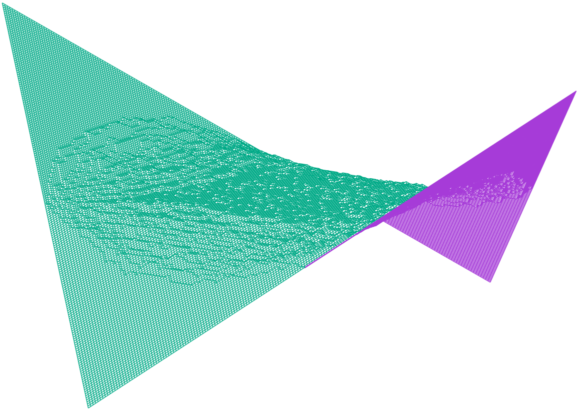


Two-periodic Aztec diamond height function



Picture by B. Young

Two-periodic Aztec diamond height function



Picture by V. Beffara

Kasteleyn Matrix

Let $\nu(e)$ be the weights. We choose a **Kasteleyn sign**, $s(e)$, $|s(e)| = 1$, for each edge with certain properties, and then define the **Kasteleyn matrix** $\mathbb{K}$ with elements

$$\mathbb{K}(b_i, w_j) = s(b_i w_j)\nu(b_i w_j).$$

For the Aztec diamond graph we can take

$$\mathbb{K}(b, w) = \begin{cases} \nu(bw) & \text{if } e = bw \text{ is horizontal} \\ \mathrm{i}\nu(bw) & \text{if } e = bw \text{ is vertical} \\ 0 & \text{otherwise (i.e. no edge between } b \text{ and } w) \end{cases}$$

Kasteleyn's theorem

Let $\mathbb{K}$ be a Kasteleyn matrix

Theorem (Kasteleyn)

$$\det(\mathbb{K}) = SZ,$$

where Z is the partition function, and $|S| = 1$.

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It follows from Kasteleyn's theorem that

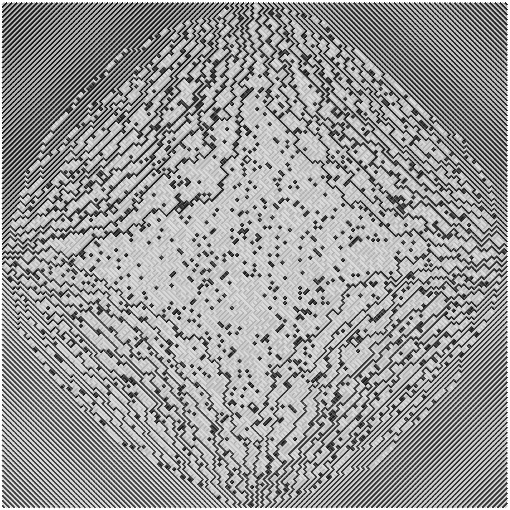
Theorem (Montroll-Potts-Ward, Kenyon)

If $e_i = (b_i, w_i)$, then the probability that $e_1, \ldots, e_m$ belong to a dimer cover is

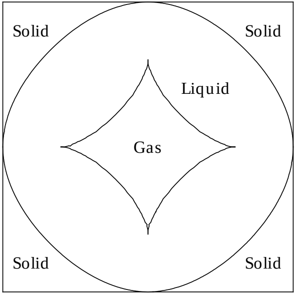
$$\mathbb{P}(e_1, \ldots, e_m) = \det\left(\mathbb{K}(b_i, w_i)\mathbb{K}^{-1}(w_i, b_j)\right)_{1 \le i,j \le m}$$

This means that the dimers form a **determinantal point process** with correlation kernel $K(e_i, e_j) = \mathbb{K}(b_i, w_i)\mathbb{K}^{-1}(w_i, b_j)$, $e_i = b_i w_i$.

Phases



Phases



The curve in the picture is a degree 8 curve with two real components. We get three regions which are called **solid, liquid and gas**.

Phases

Kenyon, Okounkov and Sheffield have characterized the different **limiting translation invariant Gibbs measures** that are possible for bipartite dimer models on the plane.

There are three classes of Gibbs measures, **solid**, **liquid** and **gas**, defined via the infinite, limiting inverse Kasteleyn matrices $\mathbb{K}^{-1}_{\text{solid}}$, $\mathbb{K}^{-1}_{\text{liquid}}$ and $\mathbb{K}^{-1}_{\text{gas}}$.

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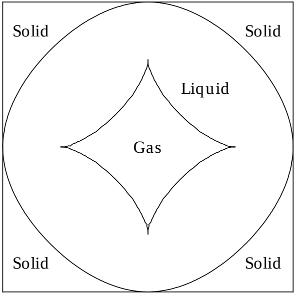
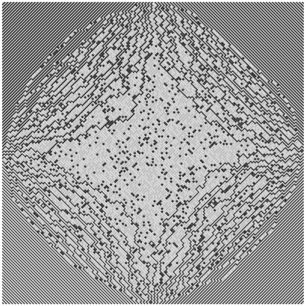
Correlations between dominos decay polynomially with distance in the liquid region, and exponentially in the gas region.

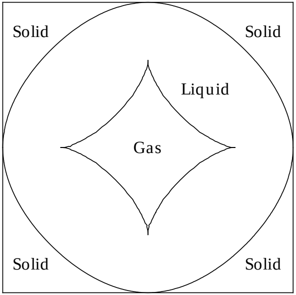
Liquid-gas boundary

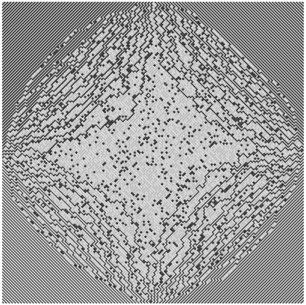
We now have two types of boundaries, the liquid-solid boundary and the liquid-gas boundary.





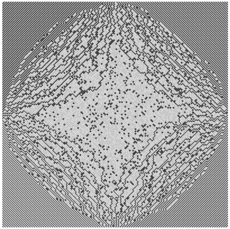
Liquid-gas boundary

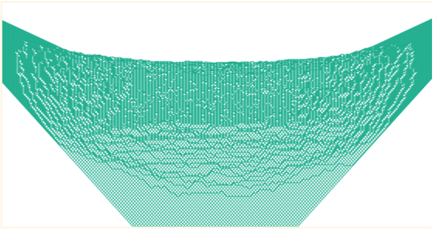




What can we say about the edge fluctuations at the liquid-gas boundary?

Liquid-gas boundary



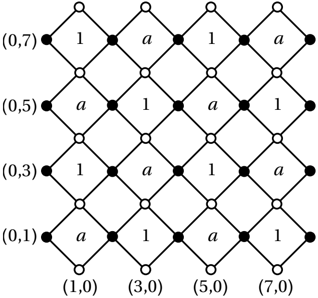


What can we say about the edge fluctuations at the liquid-gas boundary?

At the liquid-solid boundary we would expect to see the Airy process.

At the liquid-gas boundary the situation is less clear. How should we even define this boundary microscopically?

Formula for the inverse Kasteleyn matrix in the two-periodic case



The coordinate system that we use is indicated in the figure.

The inverse Kasteleyn Matrix

Theorem (Chhita-J. based on Chhita-Young)

Consider an Aztec diamond of size $n = 4m$ with the two-periodic weighting and let $\mathbb{K}_m$ be its Kasteleyn matrix. Then,

$$\mathbb{K}_m^{-1}((x_1,x_2),(y_1,y_2)) = \mathbb{K}_{gas}^{-1}((x_1,x_2),(y_1,y_2)) - \sum_{i=1}^{4} B_i((x_1,x_2),(y_1,y_2)),$$

where $\mathbb{K}_{gas}^{-1}((x_1,x_2),(y_1,y_2))$ is the inverse Kasteleyn matrix for a whole plane gas phase, and $B_1,\dots,B_4$ are related by a symmetry with B_1, which has the form

$$B_1(x,y) = \frac{1}{(2\pi \mathrm{i})^2}\int_{|\omega_1|=r}\frac{d\omega_1}{\omega_1}\int_{|\omega_2|=1/r} d\omega_2 \frac{Y_{\epsilon_1,\epsilon_2}(\omega_1,\omega_2)}{\omega_2-\omega_1}\frac{H_{x_1+1,x_2}(\omega_1)}{H_{y_1,y_2+1}(\omega_2)}.$$

Here $Y_{\epsilon_1,\epsilon_2}(\omega_1,\omega_2)$ is a complicated non-asymptotic factor,

$$H_{x_1,x_2}(\omega) = \frac{\omega^{2m}G(\omega)^{2m-x_1/2}}{G(1/\omega)^{2m-x_2/2}}, \quad G(\omega) = \frac{1}{\sqrt{2c}}(\omega - \sqrt{\omega^2+2c}),$$

and $c = a/(1+a^2)$ with $0 < c < 1/2$.

Airy kernel point process

The **extended Airy point process** is a determinantal point process on parallel lines $\{\beta_q\} \times \mathbb{R}$, $1 \leq q \leq L_2$ in $\mathbb{R}^2$. We can think of it as a random measure μ_{Ai} defined via a Laplace transform. Let A_p, $1 \leq p \leq L_1$, be disjoint intervals in $\mathbb{R}$, $w_{p,q} \in \mathbb{C}$,

$$\mathbb{E}\left[\exp\left(\sum_{p=1}^{L_2} \sum_{q=1}^{L_1} w_{p,q} \mu_{\mathrm{Ai}}(\{\beta_q\} \times A_p) \right) \right]$$
$$= \det\left(I + (e^{\Psi} - 1) K_{\mathrm{extAi}} \right)_{L^2(\{\beta_1, \ldots, \beta_q\} \times \mathbb{R})},$$

where

$$\Psi(x) = \sum_{q=1}^{L_1} \sum_{p=1}^{L_2} w_{p,q} \mathbb{I}_{\{\beta_q\} \times A_p}(x).$$

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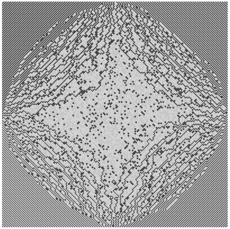
Recall that the **extended Airy kernel** is given by

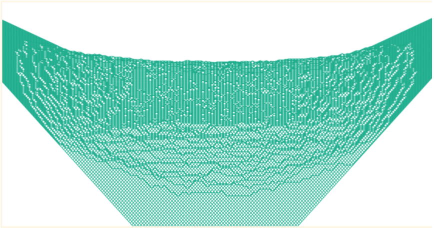
$$K_{\mathrm{extAi}}(\tau_q, \xi_1; \tau_2, \xi_2) = -\phi_{\tau_1,\tau_2}(\xi_1, \xi_2) + \tilde{K}_{\mathrm{extAi}}(\tau_1, \xi_1; \tau_2, \xi_2),$$

where

$$\tilde{K}_{\mathrm{extAi}}(\tau_1, \xi_1; \tau_2, \xi_2) = \int_0^{\infty} e^{-\lambda(\tau_1 - \tau_2)}\mathrm{Ai}\,(\xi_1 + \lambda)\mathrm{Ai}\,(\xi_2 + \lambda)\,d\lambda.$$

Random measure





Height differences between two points in a vicinity of the liquid gas boundary are due to two effects:

- Small and basically independent height fluctuations due to the "surrounding gas phase".
- Long distance correlated effects due to the large scale structures that we see in the figure.
- By taking suitable averages of height differences we could hope to eliminate the small scale gas effects. This is the idea behind the definition of a certain random measure.

Random measure

Consider a two-periodic Aztec diamond of size $n = 4m$.

We want to imbed the intervals A_p as discrete intervals in the Aztec diamond.z
Consider only one interval, $A = [a^l, a^r]$. We want to imbed $M = [(\log m)^4]$ copies of it a certain distance apart.

Random measure

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Define for $s \in \mathbb{Z}$, $1 \le k \le M = [(\log m)^4]$

$$z_k(s) = \big(4[m(1+\xi)] - 2[\beta^2\lambda_1(2m)^{1/3}] + s + \frac{1}{2}\big)(1,1) \\ - 2[\beta\lambda_2(2m)^{2/3} + k\lambda_2(\log m)^2](-1,1).$$

These are points sitting at the midpoints of edges. We then have the **discrete lines**, $1 \le k \le M$,

$$\mathcal{L}_m(k) = \{z_k(s)\,;\, s \in \mathbb{Z}\},$$

and $\mathcal{L}_m$ is the union of all of them.

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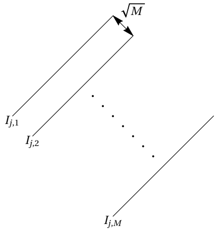
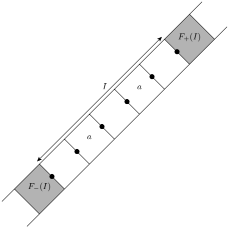
Imbed A in $\mathcal{L}_m(k)$ as a **discrete interval**,

$$I_k = \{z_k(s)\,;\, 2[a^l\lambda_1(2m)^{1/3}] - 1 \le s < 2[a^r\lambda_1(2m)^{1/3}] + 1\}.$$



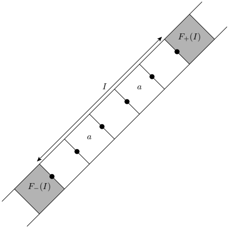
Random measure



The **height change** along a discrete interval

$$\Delta h(I) = h(F_+(I)) - h(F_-(I)).$$

Random measure



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Random measure:

$$\mu_m(\{\beta\} \times A) = \frac{1}{4M} \sum_{k=1}^{M} \Delta h(I_k).$$

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Theorem

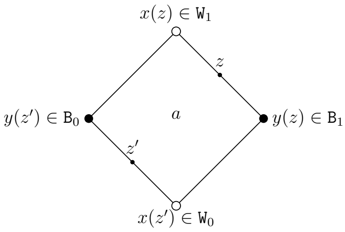
The random measure μ_m converges weakly to μ_{Ai}

For one interval we need to prove

$$\lim_{m \to \infty} \mathbb{E}\left[e^{w\mu_m(A)}\right] = \mathbb{E}\left[e^{w\mu_{\mathrm{Ai}}(A)}\right],$$

for $w \in \mathbb{C}$, $|w| < R$.

Particle process

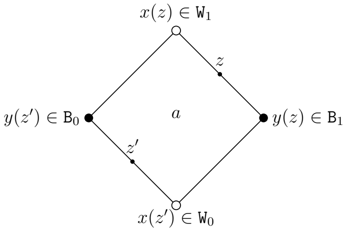


In the figure we see an a-face.

$z, z' \in \mathcal{L}_m$

There is a **parity**: $\epsilon(z') = 0$, crossing the dimer $(x(z'), y(z'))$ gives a positive height change; $\epsilon(z) = 1$, crossing the dimer $(x(z), y(z))$ gives a negative height change.

Particle process



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There is a **parity**: $\epsilon(z') = 0$, crossing the dimer $(x(z'), y(z'))$ gives a positive height change; $\epsilon(z) = 1$, crossing the dimer $(x(z), y(z))$ gives a negative height change.
Particle at z if and only if dimer covering $(x(z), y(z))$

Since the dimers form a determinantal process so do the particles. Kernel,

$$K_m(z, z') = ai\mathbb{K}_m^{-1}(x(z'), y(z)), \quad z, z' \in \mathcal{L}_m.$$

Height differences in terms of particle process

Indicator function for a discrete interval

$$\mathbb{I}_k(z) = \begin{cases} 1 & \text{if } z \in I_k \\ 0 & \text{if } z \notin I_k. \end{cases}$$

The height difference along I_k can be written

$$\frac{1}{4}\Delta h(I_k) = \sum_i (-1)^{\epsilon(z_i)} \mathbb{I}_k(z_i),$$

where the sum is over all particles. Note that we are counting particles with a sign factor.

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By the definition of the random measure we get

$$\mathbb{E}\left[e^{w\mu_m(A)}\right] = \mathbb{E}\left[e^{\frac{1}{M}\sum_i \psi(z_i)}\right] = \det\left(I + (e^{\frac{1}{M}\psi} - 1)K_m\right)_{\ell^2(\mathcal{L})},$$

where

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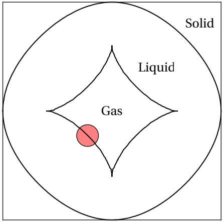
Use the cumulant expansion,

$$\log\det\left(I + (e^{\frac{1}{M}\psi} - 1)K_m\right)_{\ell^2(\mathcal{L})}$$
$$= \sum_{s=1}^{\infty} \frac{1}{M^s} \sum_{r=1}^{s} \frac{(-1)^s}{r} \sum_{\substack{\ell_1+\cdots+\ell_r=s \\ \ell_i\geq 1}} \frac{\operatorname{tr}\left(\psi^{\ell_1}K_m \ldots \psi^{\ell_r}K_m\right)}{\ell_1! \ldots \ell_r!}$$

Asymptotics for the inverse Kasteleyn matrix at the liquid-gas boundary

Let $x = (x_1, x_2)$ be a white vertex and $y = (y_1, y_2)$ a black vertex. Scaling around a point at the liquid-gas boundary:

$$(x_1, x_2) = (4[\rho m] + 2[c_1 \xi_1 m^{1/3}](1,1) - 2[c_2 \tau_1 m^{2/3}](-1,1),$$
$$(y_1, y_2) = (4[\rho m] + 2[c_1 \xi_2 m^{1/3}](1,1) - 2[c_2 \tau_2 m^{2/3}](-1,1).$$



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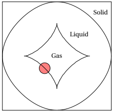

Asymptotics

$$\mathbb{K}_m^{-1}(x,y) = \mathbb{K}_{\text{gas}}^{-1}(x,y) - (\text{pre-factor})\tilde{K}_{\text{extAi}}(\tau_1, \xi_1 + \tau_1^2; \tau_2, \xi_2 + \tau_2^2) m^{-1/3}(1 + o(1))$$

as $m \to \infty$.

Thank you for listening!