The Kontsevich matrix integral and Painlevé hierarchy

Rigorous asymptotics and universality at the soft and hard edges of the spectrum in random matrix theory

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Abstract

The Kontsevich integral is a matrix integral (aka "Matrix Airy function") whose logarithm, in the appropriate formal limit, generates the intersection numbers on $\mathcal{M}_{g,n}$. In the same formal limit it is also a particular tau function of the KdV hierarchy; truncation of the times yields thus tau functions of the first Painlevé hierarchy. This, however is a purely formal manipulation that pays no attention to issues of convergence.

The talk will try to address two issues: Issue 1: how to make an analytic sense of the convergence of the Kontsevich integral to a tau function for a member of the Painlevé I hierarchy? Which particular solution(s) does it converge to? Where (for which range of the parameters)?

Issue 2: it is known that (in fact for any β) the correlation functions of K points in the GUE_β ensemble of size N are dual to the correlation functions of N points in the $GUE_{4/\beta}$ of size K. For $\beta = 2$ they are self-dual.

Consider $\beta = 2$: this duality is lost if the matrix model is not Gaussian; however we show that the duality resurfaces in the scaling limit near the edge (soft and hard) of the spectrum.

In particular we want to show that the correlation functions of K points near the edge of the spectrum converge to the Kontsevich integral of size K as $N \to \infty$.

This line of reasoning was used by Okounkov in the GUE_2 for his "edge of the spectrum model". This is based on joint work with Mattia Cafasso (Angers).

Time permitting I will discuss a work-in-progress with G. Ruzza (SISSA) extending these results to the generating function of open intersection numbers.

Based on

1. M. B. M. Cafasso, *The Kontsevich matrix integral: convergence to the Painlevé hierarchy and Stokes' phenomenon* , arXiv:1603.06420
2. M. B. M. Cafasso, *Universality of the matrix Airy and Bessel functions at spectral edges of unitary ensembles*, arXiv:1610.06108
3. M.B., B. Dubrovin, D. Yang, *Correlation functions of the KdV hierarchy and applications to intersection numbers over* $\overline{\mathcal{M}}_{g,n}$, Physica D, 2016.
4. M. B. G. Ruzza, "something something... *Penner model* ... something something", in progress.

The main actor: Kontsevich (Penner) matrix integral

$$Z_n^P(x;Y) := \frac{\int_{H_n} \mathrm{d}M \det(M+iY)^K \mathrm{e}^{\mathrm{Tr}\left(i\frac{M^3}{3} - YM^2 + ixM\right)}}{\int_{H_n} \mathrm{d}M \mathrm{e}^{-\mathrm{Tr}\,(YM^2)}} =$$

$$= \left\langle \det(M+iY)^K \mathrm{e}^{\mathrm{Tr}\left(i\frac{M^3}{3} + ixM\right)} \right\rangle_{GUE(Y)}$$

$$Y = \mathrm{diag}(y_1, \ldots, y_n); \quad \Re(y_j) > 0.$$

The main goal

Understand what happens as $n \to \infty$ in analytic way.

- Introduced by Kontsevich ('92) to prove Witten's conjecture ('90).
- With the addition of blue term it is currently proposed as generalized tool to study moduli space of open Riemann surfaces.
- It is called **Matrix Airy** function; a sort of unitary model with external source.

$$\overline{\mathcal{M}}_{g,n} := \left\{ \text{equiv. classes of R. S of genus } g \text{ with } n \text{ marked points} \right\}$$

[Deligne–Mumford]

- A point in $\overline{\mathcal{M}}_{g,m}$ is a Riemann surface (algebraic curve) with n points, up to biholomorphic equivalence.
- $\dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,m} = 3g - 3 + m$ (an "orbifold"). Nonetheless differential geometry and integration on $\mathcal{M}_{g,m}$ is possible.
- There are "tautological" line bundles $\mathcal{L}_j$, whose fiber at the point $[\mathcal{C}]$ is $T^\star_{p_j}\mathcal{C}$. As with any line bundle, one can associate a "curvature form", the Chern class of $\mathcal{L}_j$ which we denote ψ_j. They are **two–forms**; to integrate over $\overline{\mathcal{M}}_{g,m}$ we need $3g - 3 + m$ ($\dim_{\mathbb{R}} \overline{M}_{g,m} = 6g - 6 + 2m$).

Intersection numbers

$$\left\langle \tau_0^{k_0} \tau_1^{k_1} \dots \right\rangle := \int_{\overline{\mathcal{M}}_{g,m}} \psi_1^{\ell_1} \wedge \cdots \wedge \psi_m^{\ell_m}$$

$k_j = \sharp$ of times j appears as exponent

Conjecture (Witten, '91)

$$F^{\text{Witten}}(T_0, T_1, \ldots,) := \sum_{k_1, k_2, \ldots} \left\langle \tau_0^{k_0} \tau_1^{k_1} \ldots \tau_\ell^{k_\ell} \ldots \right\rangle_{\substack{\overline{\mathcal{M}}_{g,m} \\ m=\sum k_j, \\ 3g-3=\sum (j-1)k_j}} \prod_{j=0}^{\infty} \frac{T_j^{k_j}}{k_j!}$$

as a formal Taylor series, defines a τ–function of the KdV hierarchy;

$$\tau(\vec{T}) = \mathrm{e}^{F^{\text{Witten}}(\vec{T})}, \qquad U(T_0, T_1, \ldots) := \frac{\partial^2 F(\vec{T})}{\partial T_0^2} \quad \text{solves}$$

$$\begin{cases} \dfrac{\partial U}{\partial T_1} = U \dfrac{\partial U}{\partial T_0} + \dfrac{1}{12} \dfrac{\partial^3 U}{\partial T_0^3} & \text{(Korteweg-deVries equation)} \\ U(T_0, 0, \ldots,) = T_0. \end{cases}$$

$$\sum_{k \geqslant 0} T_{k+1} \frac{\partial}{\partial T_k} \ln \tau(\vec{T}) + \frac{T_0^2}{2} \equiv 0 \qquad \text{(String equation)}$$

Remark

The above statement is completely formal; the series for F does not converge. It is useful to recursively compute the coefficients and hence the intersection numbers.

Introduce the *Miwa* variables[1]

$$T_j = T_j(Y) := -(2j-1)!! \sum_{\ell=1}^{n} \frac{1}{y_\ell^{2j+1}} = -(2j-1)!! \,\mathrm{Tr}\, \frac{1}{Y^{2j+1}}$$

Theorem (Kontsevich'92)

$$F_n^{Kont}(Y) = \sum_{k_1,k_2,\dots\geqslant 0} C_n(\vec{k}) \prod_{j=0}^{\infty} \frac{T_j^{k_j}}{k_j!} \qquad y_\ell \to \infty\,, \quad (T_j(Y) \to 0)$$

For each $\vec{k}$ the coefficients $C_n(\vec{k})$ converge (actually, stabilize)

$$\lim_{n\to\infty} C_n(\vec{k}) = \left\langle \tau_0^{k_0} \tau_1^{k_1} \dots \tau_\ell^{k_\ell} \dots \right\rangle_{\substack{\overline{\mathcal{M}}_{g,m} \\ m=\sum k_j, \\ 3g-3=\sum(j-1)k_j}}$$

I.e. $\lim_{n\to\infty} F_n^{\mathit{Kont}}(Y) = F^{\mathit{Witten}}(T)$.

[1] They are **not** independent!

Finite n results: Isomonodromic system

SPOILER ALERT

The $Z_n^{\mathsf{Kont}}(Y,x)$ is an *isomonodromic tau function* in the sense of Jimbo-Miwa-Ueno ('80).

$$A(\lambda;x,\vec{y}) = \begin{bmatrix} 0 & i \\ -i\left(\lambda+\frac{x}{2}-u(x;\vec{y})\right) & 0 \end{bmatrix} + \sum_{j=1}^{n} \frac{A_j(x;\vec{y})}{\lambda - y_j^2},$$

$$U(\lambda;x,\vec{y}) = \begin{bmatrix} 0 & i \\ -i\left(\lambda-2u(x;\vec{y})\right) & 0 \end{bmatrix}$$

Dependence of u, A_j's on $x, \vec{y}$ is determined by the **isomonodromic** condition

$$\partial_x A - \partial_\lambda U + [A,U] \equiv 0\ , \qquad \frac{\partial_{\lambda_k} A_j}{\lambda-\lambda_j} - \frac{\partial_{\lambda_j} A_k}{\lambda-\lambda_j} + \left[\frac{A_j}{\lambda-\lambda_j}, \frac{A_k}{\lambda-\lambda_k}\right] \equiv 0$$

$$\partial_\lambda \frac{A_k}{\lambda-\lambda_k} - \partial_{\lambda_k} A + \left[\frac{A_k}{\lambda-\lambda_k}, A\right] \equiv 0\ , \qquad \frac{\partial_x A_k}{\lambda-\lambda_k} - \partial_{\lambda_k} U + \left[\frac{A_k}{\lambda-\lambda_k}, U\right] \equiv 0.$$

$$\frac{1}{2y_k}\partial_{y_k} F_n^{\mathsf{Kont}}(x;\vec{y}) = \underset{\lambda=y_k^2}{\mathrm{res}} \mathrm{Tr}\, A^2\, \mathrm{d}\lambda\ ; \qquad \partial_x^2 F_n^{\mathsf{Kont}}(x;\vec{y}) = u(x;\vec{\lambda}) + \frac{x}{2}$$

Remark

The gap probability of Airy's point field (Tom's talk yesterday) is also a tau function of a system of the same form (but with monodromy around the poles).

Asymptotics: Formal statements

Reduction of KdV: the first Painlevé hierarchy I

Choose $N \geqslant 2$ and set $T_j = 0 \ j \geqslant N+1$. Then Witten's formal generating function is $\ln \tau(T_1, \ldots, T_N)$ becomes a (formal) solution of the N-th member of the **first Painlevé** hierarchy.

(String equation) + (KdV) + (Reduction) = N-th member of Painlevé hierarchy

Reminder of the Korteweg deVries hierarchy

We use different scaling and labelling

$$t_{2j+1} = -\frac{2^{\frac{2j+1}{3}}(T_j - \delta_{j,1})}{(2j+1)!!}\ , \qquad x := t_1;$$

Define recursively the following differential polynomials (Lenard–Magri)

$$\frac{\partial}{\partial x}\mathcal{L}_{n+1}[u] = \left(\frac{1}{4}\frac{\partial^3}{\partial x^3} + u(x)\frac{\partial}{\partial x} + \frac{1}{2}u_x(x)\right)\mathcal{L}_n[u], \quad \mathcal{L}_0[u] = 1, \quad \mathcal{L}_n[0] = 0 \quad (1)$$

The KdV hierarchy

$$\frac{\partial u}{\partial t_{2n+1}} = 2\frac{\partial}{\partial x}\mathcal{L}_{n+1}[u], \quad n \in \mathbb{Z}_{\geqslant 0}; \quad u = u(\vec{t}), \quad \vec{t} = (t_1, t_3, t_5, \ldots). \qquad (2)$$

and the String equation takes the form

$$\sum_{\ell \geqslant 0}(2\ell+1)t_{2\ell+1}\mathcal{L}_\ell[u] + x \equiv 0\ , \qquad u = 2\partial_x^2 \ln \tau(\vec{t})$$

Special case $t_j = \delta_{j,2N+1} t_{2N+1} + \delta_{j,1} x$

$$(2N+1) t_{2N+1} \mathcal{L}_N[u(x;t)] + u(x;t) + x = 0 \ , \quad u(x,t) = 2\partial_x^2 \ln \tau(x;t).$$

Example

For $N = 2, 3$ the equation reads

$$N = 2; \qquad \frac{5}{8} t_5 \left(u'' + 3u^2\right) + u + x = 0$$

$$N = 3; \qquad \frac{7}{32} t_7 \left(u^{(4)} + 10uu'' + 5(u')^2 + 10u^3\right) + u + x = 0 \ .$$

The case $N = 2$ above is, up a scaling the standard first Painlevé 1 equation $U'' + 3U^2 = X$.

- It is hard to make non-formal sense of convergence statement;
- Can we fix $T_j = 0$ for $j \geqslant N$? In this case we have seen that the KdV hierarchy reduces to the N-th member of the Painlevé I hierarchy. But what does it mean to fix infinitely $t_j = 0$ if they are not independent?
- If we prove convergence, what analytic solutions of PI_N we can get? (i.e.; which solutions of PI_N are relevant for enumerative geometry?).
- Are they uniquely determined? In which region of the parameters the convergence is guaranteed?

OUR ANSWER

We provide a (almost) complete answer for the case $t_{2N+1} = t, t_0 = x$ and all other $t_j = 0$.

CAVEAT

Not the way we state in the paper but equivalent!

PROBLEM

Choose $Y_{(n)}$ sequence of diagonal matrices so that $\mathrm{Tr}\,(Y_{(n)}{}^{-2J-1}) \to t\delta_{J,2}$ and $Z_n^{\mathrm{Kont}}(x; Y_{(n)}) \to \tau(x,t)$
The poles of the Lax matrix $A(\lambda; x, \vec{y}_{(n)})$ will distribute themselves in such a way that –uniformly over compact sets of $\lambda \in \mathbb{C}$;

$$\lim_{n\to\infty} A(\lambda; x, \vec{y}_{(n)}) = A_I(\lambda; x, t) \qquad \lim_{n\to\infty} U(\lambda; x, \vec{y}_{(n)}) = U_I(\lambda; x, t)$$

$$A_I(\lambda; x, t) = \lambda^2 \begin{bmatrix} 0 & 0 \\ -\frac{5i}{2}t & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & \frac{5it}{2} \\ -5itu(x,t) - i & 0 \end{bmatrix} + A_0$$

$$U_I(\lambda; x, t) = \begin{bmatrix} 0 & i \\ -i(\lambda - 2u) & 0 \end{bmatrix}$$

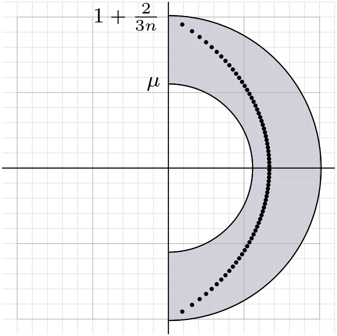
Caveat

For technical reasons, n is a multiple of $2N+1=5$.

Let P_r be the r-th Padé approximant to e^{-z}

$$\mathrm{e}^{-z} = \frac{P_r(z)}{P_r(-z)} + \mathcal{O}(z^{2r+1}), \quad z \to 0.$$

Zero distribution is known [SaffVarga78]; all in the region $\Re z > 0$



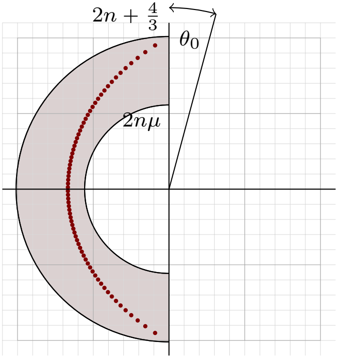


FIGURE: The zeroes of $P_r(2nz)$ for

FIGURE: The poles of the Padé

Set

$$\mathcal{Y} = \{y_1, \ldots, y_n\} = \left\{y : \quad P_r(ty^5) = 0\right\} \ , \quad n = 5r.$$

Theorem

For $N = 2$ the Kontsevich's tau function converges as $n \to \infty$ to the tau function $\tau(x;t)$ of the tritronquée solution of PI equation The particular solution $u(x;t) = -2\partial_x^2 \ln \tau(x;t)$ has no poles for $|t|$ sufficiently small within the sector $|\arg(t)| < \frac{\pi}{2}$

$$\frac{5}{8} t \left(u'' + 3u^2\right) + u + x = 0$$

Nonlinear Stokes

For $N \geqslant 2$ there are **several** tronquée solution that have the same asymptotic behaviours; i.e. the Witten tau function is a formal asymptotic expansion of several possible analytic (tronquée) tau functions.

We need to describe *which* Tronquée solution; → Riemann–Hilbert problem (i.e. associated linear ODE).

About the proof

The proof is a based on a formulation in terms of a Riemann–Hilbert problem and rigorous asymptotic analysis.

A proof working for the general PI_N and for the full sequence n requires to study the zeros/poles of the Padé approximations of

$$\exp\left(\sum_{\ell=0}^{N} t_{2\ell+1} z^{2\ell+1}\right) = \frac{P_n(z)}{P_n(-z)} + \mathcal{O}(z^{2n+1}).$$

Part II: The universality of the (Kontsevich's) Airy and the Bessel Matrix integrals in Random Matrix Theory.

- Brezin-Hikami (2000) **duality** formula for characteristic functions

$$\frac{1}{Z_n}\int_{\mathcal{H}_n} dM \prod_{i=1}^{k} \det(\lambda_i - M) e^{-\frac{n}{2}\operatorname{Tr} M^2 + N \operatorname{Tr} MA} =$$

$$= \frac{1}{Z_k}\int_{\mathcal{H}_k} dB \prod_{j=1}^{n} \det(a_j - iB) e^{-\frac{n}{2}\operatorname{Tr}(B - i\Lambda)^2}.$$

  This formula exchanges k-points correlation functions on $n \times n$ matrices with n-points correlation functions on $k \times k$ matrices: note that here both the models are Gaussian with external potentials $A := \operatorname{diag}(a_1, \ldots, a_n)$ and $\Lambda := \operatorname{diag}(\lambda_1, \ldots, \lambda_k)$.

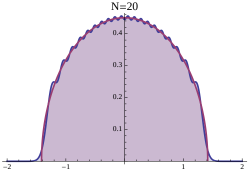
- Okounkov-Pandharipande ('01) used this to show the appearance of the Kontsevich model at the **edge of the spectrum** in GUE.
- The duality exists for Gaussian models in arbitrary β ensemble (not just Unitary ($\beta = 2$) Symplectic ($\beta = 4$) and Orthogonal ($\beta = 1$). (Desrosiérs '09)

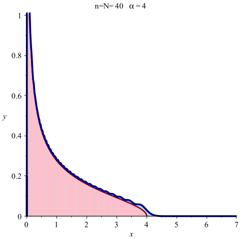
Problem

The results are only for **Gaussian** models. ....Or are they?

Unitary Matrix Model	Laguerre Matrix Model
$\mathrm{d}\mu(M) \propto \mathrm{d}M \mathrm{e}^{-n \operatorname{Tr} V(M)}$ $M \in H_n$	$\mathrm{d}\mu(M) \propto \mathrm{d}M (\det M)^{\nu} \mathrm{e}^{-n \operatorname{Tr} V(M)}$ $M \in H_n^{\geqslant 0}$
$\liminf_{\lvert x\rvert \to \infty} \frac{V(x)}{\ln \lvert x\rvert} = +\infty$	$\liminf_{x \to +\infty} \frac{V(x)}{\ln \lvert x\rvert} = +\infty$ $\inf_{x \in \mathbb{R}_+} V(x) > -\infty$

In either cases, in the limit $n \to \infty$, the spectral density becomes confined; we are interested at the **edges**





- Soft Edge Case:

$$\left\langle \prod_{j=1}^{2S} \mathrm{e}^{-\frac{n}{2}V(\xi_j)} \det(\xi_j - M) \right\rangle_{\mathcal{H}_n},$$

here the average is taken with respect to the measure $\mathrm{d}\mu(M) = \mathrm{e}^{-n\,\mathrm{Tr}\,V(M)}dM$ on the space of $n \times n$ Hermitian matrices.

- Hard Edge Case:

$$\left\langle \prod_{j=1}^{2S} \xi_j^{\frac{\nu}{2}} \mathrm{e}^{-\frac{n}{2}V(\xi_j)} \det(\xi_j - M)^{\pm 1} \right\rangle_{\mathcal{H}_n^+},$$

here the average is taken with respect to the measure $\mathrm{d}\mu(M) = M^{\nu}\mathrm{e}^{-n\,\mathrm{Tr}\,V(M)}dM$ on the space of $n \times n$ semi–positive definite Hermitian matrices, and $\mathrm{Re}(\nu) > -1$.

Studied by Fyodorov-Strahov ('03), Baik-Deift-Strahov ('03), Akemann-Fyodorov ('03); formulas exist involving Orthogonal Polynomials of Christoffel–Uvarov type.

Theorem (Universality of the Matrix Airy (Kontsevich) function)

Let $V(x)$ *generic;* a *right endpoint of the support of the equilibrium measure. Let* $\xi_j = a + C^{-1} n^{-\frac{2}{3}} y_j^2, \quad j = 1 \dots, 2S$. *Then*

$$\lim_{n\to\infty} \frac{C^{S^2} n^{\frac{2S^2}{3}}}{\prod_{\ell=n}^{n+S-1} h_\ell} \left\langle \prod_{j=1}^{2S} \mathrm{e}^{-\frac{n}{2} V(\xi_j)} \det(\xi_j - M) \right\rangle_{\mathcal{H}_n} =$$

$$= \frac{\prod_{j<k} (y_j + y_k) \prod_{j=1}^{2S} y_j^{-\frac{1}{2}}}{2^{2S} \pi^S} \mathrm{e}^{-\frac{2}{3} \operatorname{Tr} Y^3} Z_{2S}^{Kont}(Y),$$

where $Y = \mathrm{diag}(y_1, \dots, y_{2S})$ *and*

$$Z_{2S}^{Kont}(Y) = \frac{\int_{\mathcal{H}_{2S}} \mathrm{d}H \mathrm{e}^{\operatorname{Tr}\left(i \frac{H^3}{3} - Y H^2\right)}}{\int_{\mathcal{H}_{2S}} \mathrm{d}H \mathrm{e}^{-\operatorname{Tr}(Y H^2)}}.$$

Theorem (Universality of the Matrix Bessel J, K functions)

Let $V(x)$ *be a regular potential and let* $\xi_j = C^{-1} n^{-2} y_j,\ j = 1, \dots, 2S$. *Then*

$$\lim_{n\to\infty} \frac{C^{S^2} n^{2S^2}}{\prod_{\ell=n}^{n+S-1} h_\ell} \left\langle \prod_{j=1}^{2S} \xi_j^{\frac{\nu}{2}} \mathrm{e}^{-\frac{n}{2}V(\xi_j)} \det(\xi_j - M) \right\rangle_{\mathcal{H}_n^+} =$$

$$= \frac{\det(Y)^{\frac{\nu}{2}} (2\pi)^K}{\pi^{S(2S-1)}} \int_{CUE_{2S,\hat{\gamma}}} (\det H)^{\nu-1} \mathrm{e}^{\,\mathrm{Tr}\,(-YH + H^{-1})} \frac{\mathrm{d}H}{(2i\pi)^{2S}}$$

where $Y = \mathrm{diag}(y_1, \dots, y_{2S})$. *Similarly* $\xi_j = -Cn^{-2} y_j$, $y_j \notin \mathbb{R}_-$*:*

$$\lim_{n\to\infty} \left(\prod_{\ell=n-S}^{n-1} h_\ell \right) C^{S^2} n^{2S^2} \left\langle \prod_{j=1}^{2S} \frac{(\mathrm{e}^{i\pi}\xi_j)^{-\frac{\nu}{2}} \mathrm{e}^{\frac{n}{2}V(\xi_j)}}{\det(\xi_j - M)} \right\rangle_{\mathcal{H}_n^+} =$$

$$= \frac{\det(Y)^{\frac{\nu}{2}} (2\pi)^S}{\pi^{S(2S-1)}} \int_{\mathcal{H}_{2S}^+} (\det H)^{\nu-1} \mathrm{e}^{\,\mathrm{Tr}\,(-YH - H^{-1})} \,\mathrm{d}H.$$

On formal level: Painlevé hierarchy allows to extract **explicit** formulæ for intersection numbers (joint with Dubrovin, Yang '16)

$$F_n^{WK}(\lambda_1,\ldots,\lambda_n) := \sum_{\vec{k}\in\mathbb{N}^n}^{\infty} \langle \tau_{k_1}\ldots\tau_{k_n}\rangle \frac{(2k_1+1)!!}{\lambda_1^{k_1+1}}\cdots\frac{(2k_n+1)!!}{\lambda_n^{k_n+1}}, \quad n\geqslant 1$$

Recipe

- Take matrix solution of Airy equation (bare problem)[a];

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Psi(\lambda) = \begin{bmatrix} 0 & 1 \\ \lambda & 0 \end{bmatrix}\Psi(\lambda)$$

- Define the Master Matrix (asymptotic series at $\lambda=\infty$)

$$M(\lambda) := \sqrt{\lambda}\Psi(\lambda)\sigma_3\Psi^{-1}(\lambda) =$$
$$= \begin{pmatrix} -\sum_{g=1}^{\infty}\frac{(6g-5)!!}{2\cdot 24^{g-1}\cdot(g-1)!}\lambda^{-3g+2} & -\sum_{g=0}^{\infty}\frac{(6g-1)!!}{24^g\cdot g!}\lambda^{-3g} \\ \sum_{g=0}^{\infty}\frac{6g+1}{6g-1}\frac{(6g-1)!!}{24^g\cdot g!}\lambda^{-3g+1} & \sum_{g=1}^{\infty}\frac{(6g-5)!!}{2\cdot 24^{g-1}\cdot(g-1)!}\lambda^{-3g+2} \end{pmatrix}.$$

[a] Numerical normalizations are incorrect

- Then

$$F_1(\lambda) = \frac{1}{2}\operatorname{Tr}\left(\Psi^{-1}(\lambda)\Psi'(\lambda)\sigma_3\right) - \lambda$$

$$F_n(\lambda_1, \ldots, \lambda_n) = -\frac{1}{n}\sum_{r \in S_n} \frac{\operatorname{Tr}\left(M(\lambda_{r_1})\cdots M(\lambda_{r_n})\right)}{\prod_{j=1}^{n}(\lambda_{r_j} - \lambda_{r_{j+1}})} - \delta_{n,2}\frac{\lambda_1 + \lambda_2}{(\lambda_1 - \lambda_2)^2}$$

$n \geqslant 2.$

Similar ideology leads to a 3×3 isomonodromic system; $n \to \infty$ limit leads to Painlevé type RHP and bare system

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Psi(\lambda) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ k+1 & \lambda & 0 \end{bmatrix} \Psi(\lambda)$$

$$F^{\text{W-P}}(T_0, T_{1/2}, \dots,) := \sum_{k_0, k_{1/2} k_1, \dots \geqslant 0} \sum_{b \geqslant 0} \left\langle \prod_{\ell \geqslant 0} \tau_\ell^{k_\ell} \sigma_\ell^{k_{\ell+\frac{1}{2}}} \right\rangle_{\overline{\mathcal{M}}_{g,m,b}} K^b \prod_{j \in \frac{1}{2}\mathbb{N}} \frac{T_j^{k_j}}{k_j!}$$

where b=number of boundaries.
Identification of which isomonodromic system is relevant for this model allows explicit formulæ for the intersection numbers.

EXAMPLE

The one–point function

$$F_1^{KP}(\lambda) := \sum_{k \in \mathbb{N}} \sum_{b \geqslant 0} \langle \tau_k \rangle_{\mathcal{M}_{g,n,b}} K^b \frac{(2k+1)!!}{\lambda^{k+1}}$$

expressible in terms of nested sums of combinations of factorials.

Thank you!

If time permits... some details on the proof

Kontsevich proved (it is not hard) that

$$Z_n(x;Y) = 2^n \pi^{\frac{n}{2}} \mathrm{e}^{\frac{2}{3}\operatorname{Tr} Y^3 + x \operatorname{Tr} Y} \frac{\det\left[\mathrm{Ai}^{(j-1)}(y_k^2 + x)\right]_{k,j\leqslant n} \prod_{j=1}^n (y_j)^{\frac{1}{2}}}{\prod_{j<k}(y_j - y_k)} \quad \Re y_j > 0.$$

We need a **suitable** extension (not the analytic continuation!) when some y_j's are negative; the extension must admit a **regular** asymptotic expansion as $y_j \to \infty$ in given sectors. This is a multi-dimensional Stokes' phenomenon.

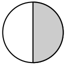
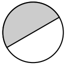
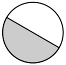
$$\mathrm{e}^{\frac{2}{3}y^3 + xy}\mathrm{Ai}(y^2 + x) = \frac{\mathrm{e}^{\frac{4}{3}y^3 + 2xy}}{2\sqrt{\pi}\sqrt{y}}(\mathbf{1} + \mathcal{O}(y^{-3})), \ |\arg\lambda| < \pi$$

$$\mathcal{Y} = \mathcal{Y}_0 \sqcup \mathcal{Y}_1 \sqcup \mathcal{Y}_2$$

$$Z_n(x;\mathcal{Y}^{(0)},\mathcal{Y}^{(1)},\mathcal{Y}^{(2)}) = C_n \frac{\mathrm{e}^{\frac{2}{3}\,\mathrm{Tr}\, Y^3 + x\,\mathrm{Tr}\, Y}}{\prod_{j<k}(y_j - y_k)} \det \left[\begin{array}{c} \left[\mathbf{Ai}_0^{(k-1)}(y_j^2+x)\right]_{\substack{y_j\in\mathcal{Y}^{(0)}\\ 1\leqslant k\leqslant n}} \\ \hline \left[\mathbf{Ai}_1^{(k-1)}(y_j^2+x)\right]_{\substack{y_j\in\mathcal{Y}^{(1)}\\ 1\leqslant k\leqslant n}} \\ \hline \left[\mathbf{Ai}_2^{(k-1)}(y_j^2+x)\right]_{\substack{y_j\in\mathcal{Y}^{(2)}\\ 1\leqslant k\leqslant n}} \end{array} \right].$$

$$\mathbf{Ai}_s(\lambda) := \mathrm{Ai}(\omega^s \lambda)\ ,\quad \omega := \mathrm{e}^{\frac{2i\pi}{3}}.$$

These determinants admit all a regular asymptotic expansion (the same expansion!) if $\mathcal{S}_a \ni y_j \to \infty, \quad \arg(y_j) \in \mathcal{S}_a$.

$\arg \mathcal{S}_0$:  ; $\arg \mathcal{S}_1$:  $\arg \mathcal{S}_2$: 

It is a RHP in the auxiliary plane $\lambda = y^2$; define

$$\mathbf{d}(\lambda) := \mathrm{e}^{2g(\lambda)} = \prod_{j=1}^{n_1} \frac{y_j + \sqrt{\lambda}}{y_j - \sqrt{\lambda}} \prod_{j=1}^{n_2} \frac{y_j - \sqrt{\lambda}}{y_j + \sqrt{\lambda}}. \qquad \Re y_{1,\dots,n_1} < 0; \quad \Re y_{n_1+1,\dots,n} > 0$$

PROBLEM

Find a 2×2 matrix valued $\Gamma = \Gamma_n(\lambda; \Lambda)$ such that it satisfies the *jump conditions*

$$\Gamma_+(\lambda) \; = \Gamma_-(\lambda) M_n(\lambda) \, , \qquad \lambda \in \varpi_{0,\pm}$$

$$M_n(\lambda) = \begin{cases} \mathbf{1} + \mathbf{d}(\lambda) \mathrm{e}^{-\frac{4}{3}\lambda^{\frac{3}{2}} - 2x\lambda^{\frac{1}{2}}} \sigma_+ & \lambda \in \varpi_0 := \mathrm{e}^{i\theta_0}\mathbb{R}_+ \\ \mathbf{1} + \frac{1}{\mathbf{d}(\lambda)} \mathrm{e}^{\frac{4}{3}\lambda^{\frac{3}{2}} + 2x\lambda^{\frac{1}{2}}} \sigma_- & \lambda \in \varpi_\pm := \mathrm{e}^{i\theta_\pm}\mathbb{R}_+ \\ i\sigma_2 & \lambda \in \mathbb{R}_- \end{cases}$$

$$\Gamma(\lambda) \; = \lambda^{-\frac{\sigma_3}{4}} \frac{\mathbf{1} + i\sigma_1}{\sqrt{2}} \left(\mathbf{1} + \frac{a^{(n)}(x; \Lambda)}{\sqrt{\lambda}} \sigma_3 + \mathcal{O}(\lambda^{-1}) \right)$$

The matrix

$$
\begin{aligned}
\Psi &= \Psi(\lambda; x, \Lambda) := \Gamma_n(\lambda) e^{-\vartheta(\lambda;x)\sigma_3} D^{-1}(\lambda) \\
D(\lambda) &:= \begin{bmatrix} \prod_{j=1}^{n_2}(y_j+\sqrt{\lambda}) \prod_{j=1}^{n_1}(-y_j-\sqrt{\lambda}) & 0 \\ 0 & \prod_{j=1}^{n_2}(y_j-\sqrt{\lambda}) \prod_{j=1}^{n_1}(-y_j+\sqrt{\lambda}) \end{bmatrix}
\end{aligned}
$$

satisfies the system of "monodromy preserving" deformation in the sense of [Jimbo Miwa, Ueno, '80]

A couple of remarks

1. For $n=0$ the RHP is the familiar RHP for the so–called Airy parametrix; the corresponding Tau function is easily computed as $\tau_0 = \frac{x^3}{12}$; this gives the correct "initial condition" for the WK KdV tau function. **Dressing**
2. The choice of poles is such that

$$
\mathbf{d}(\lambda) \to e^{t\lambda^{\frac{2N+1}{2}}}
$$