# Time space study of visits to small sets

Françoise Pène
Université de Brest and Institut Universitaire de France
Laboratoire de Mathématiques de Bretagne Atlantique

Work in collaboration with Benoît Saussol

Conference on *Non-uniformly and partially hyperbolic dynamical systems; coupling and renewal*
CIRM, February 24th, 2017

# Introduction and partial bibliography

- $(\Omega, \mathcal{F}, \mu, f)$ ergodic probability-preserving dynamical system.

# Introduction and partial bibliography

- $(\Omega, \mathcal{F}, \mu, f)$ ergodic probability-preserving dynamical system.
- small sets: $A_\varepsilon \in \mathcal{F}$, $\lim_{\varepsilon\to 0} \mu(A_\varepsilon) = 0+$. Visits to $A_\varepsilon$

# Introduction and partial bibliography

- $(\Omega, \mathcal{F}, \mu, f)$ ergodic probability-preserving dynamical system.
- small sets: $A_\varepsilon \in \mathcal{F}$, $\lim_{\varepsilon\to 0} \mu(A_\varepsilon) = 0+$. Visits to $A_\varepsilon$
  $\mathbb{E}[\#\{k = 1, ..., n : f^k(\cdot) \in A_\varepsilon\}] = \sum_{k=1}^n \mu(f^{-k}(A_\varepsilon)) = n\mu(A_\varepsilon)$.

# Introduction and partial bibliography

- $(\Omega, \mathcal{F}, \mu, f)$ ergodic probability-preserving dynamical system.
- small sets: $A_\varepsilon \in \mathcal{F}$, $\lim_{\varepsilon\to 0} \mu(A_\varepsilon) = 0+$. Visits to $A_\varepsilon$
  $\mathbb{E}[\#\{k = 1, ..., n : f^k(\cdot) \in A_\varepsilon\}] = \sum_{k=1}^n \mu(f^{-k}(A_\varepsilon)) = n\mu(A_\varepsilon)$.
- Classical behaviour :
  $\#\{k = 1, ..., \lfloor t/\mu(A_\varepsilon)\rfloor : f^k(\cdot) \in A_\varepsilon\} \Rightarrow P_t$ Poisson.

# Introduction and partial bibliography

- $(\Omega, \mathcal{F}, \mu, f)$ ergodic probability-preserving dynamical system.
- small sets: $A_\varepsilon \in \mathcal{F}$, $\lim_{\varepsilon\to 0}\mu(A_\varepsilon) = 0+$. Visits to $A_\varepsilon$
- Classical behaviour :
  $\#\{k = 1, ..., \lfloor t/\mu(A_\varepsilon)\rfloor : f^k(\cdot) \in A_\varepsilon\} \Rightarrow \mathcal{P}_t$ Poisson.
- cylinder sets: [Hirata1993], [Hirata,Saussol,Vaienti1999], [Bruin,Vaienti03], [Abadi,Vergne08], [Haydn,Vaienti04].
- Uniformly expanding maps [Collet,Galves1995].
- Non-uniformly expanding maps (intermittent maps) [Collet,Galves1993], [Bruin,Saussol03], [Bruin,Vaienti03], [Collet01], [Freitas,Freitas,Todd10], [Holland,Nicol,Török15].
- some partially hyperbolic systems [Dolgopyat04].
- Sinai billiard [Collet,Chazottes13].
- weakly hyperbolic [Haydn,Wasilewska14], [Pène,Saussol15].
- Bunimovich billiard: [Freitas,Haydn,Nicol14], [Pène,Saussol15].
- [Carvalho,Moreira Freitas,Freitas,Holland,Nicol15] periodic points

# Space-time point process

- $(\Omega, \mathcal{F}, \mu, f)$ ergodic probability-preserving dynamical system.
- small sets: $A_\varepsilon \in \mathcal{F}$, $\lim_{\varepsilon \to 0} \mu(A_\varepsilon) = 0+$. Visits to $A_\varepsilon$
  $\mathbb{E}[\#\{k = 1, ..., n : f^k(\cdot) \in A_\varepsilon\}] = n\mu(A_\varepsilon)$.

# Space-time point process

- $(\Omega, \mathcal{F}, \mu, f)$ ergodic probability-preserving dynamical system.
- small sets: $A_\varepsilon \in \mathcal{F}$, $\lim_{\varepsilon\to 0} \mu(A_\varepsilon) = 0+$. Visits to $A_\varepsilon$
$\mathbb{E}[\#\{k = 1, ..., n : f^k(\cdot) \in A_\varepsilon\}] = n\mu(A_\varepsilon)$.
- spatial renormalization functions $H_\varepsilon : A_\varepsilon \to V \subset \mathbb{R}^D$.

# Space-time point process

- $(\Omega, \mathcal{F}, \mu, f)$ ergodic probability-preserving dynamical system.
- small sets: $A_\varepsilon \in \mathcal{F}$, $\lim_{\varepsilon\to 0}\mu(A_\varepsilon) = 0+$. Visits to $A_\varepsilon$ $\mathbb{E}[\#\{k = 1, ..., n : f^k(\cdot) \in A_\varepsilon\}] = n\mu(A_\varepsilon)$.
- spatial renormalization functions $H_\varepsilon : A_\varepsilon \to V \subset \mathbb{R}^D$.
- Space-time Point process for visits to $A_\varepsilon$:

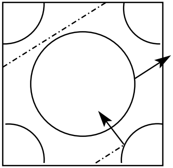
$$\mathcal{N}_\varepsilon : y \mapsto \sum_{n\ :\ f^n(y)\in A_\varepsilon} \delta_{(n\mu(A_\varepsilon),H_\varepsilon(f^n y))}, \quad \mathcal{N}_\varepsilon(B)?$$

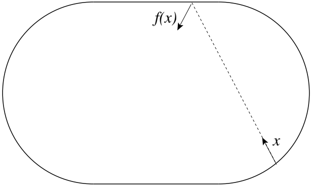
# Space-time point process

- $(\Omega, \mathcal{F}, \mu, f)$ ergodic probability-preserving dynamical system.
- small sets: $A_\varepsilon \in \mathcal{F}$, $\lim_{\varepsilon\to 0} \mu(A_\varepsilon) = 0+$. Visits to $A_\varepsilon$
  $\mathbb{E}[\#\{k = 1, ..., n : f^k(\cdot) \in A_\varepsilon\}] = n\mu(A_\varepsilon)$.
- spatial renormalization functions $H_\varepsilon : A_\varepsilon \to V \subset \mathbb{R}^D$.
- Space-time Point process for visits to $A_\varepsilon$:

$$\mathcal{N}_\varepsilon : y \mapsto \sum_{n\ :\ f^n(y)\in A_\varepsilon} \delta_{(n\mu(A_\varepsilon), H_\varepsilon(f^n y))},$$

- Motivation: Sinai billiard and Bunimovich billiard






$A_\varepsilon$ = ball for the configuration or for the position.

# Approximation by a Poisson Point Process

- $(\Omega, \mathcal{F}, \mu, f)$, $A_\varepsilon \in \mathcal{F}$, $\mu(A_\varepsilon) \to 0+$, $H_\varepsilon : A_\varepsilon \to V \subset \mathbb{R}^D$.

$$\mathcal{N}_\varepsilon := \sum_{n \,:\, f^n(\cdot) \in A_\varepsilon} \delta_{(n\mu(A_\varepsilon), H_\varepsilon(f^n \cdot))}.$$

# Approximation by a Poisson Point Process

- $(\Omega, \mathcal{F}, \mu, f)$, $A_\varepsilon \in \mathcal{F}$, $\mu(A_\varepsilon) \to 0+$, $H_\varepsilon : A_\varepsilon \to V \subset \mathbb{R}^D$.

$$\mathcal{N}_\varepsilon := \sum_{n \,:\, f^n(\cdot) \in A_\varepsilon} \delta_{(n\mu(A_\varepsilon), H_\varepsilon(f^n \cdot))}.$$

- Set $\mu_\varepsilon := \mu\left(H_\varepsilon^{-1}(\cdot) | A_\varepsilon\right)$ and $m_\varepsilon := \lambda \times \mu_\varepsilon$.

# Approximation by a Poisson Point Process

- $(\Omega, \mathcal{F}, \mu, f)$, $A_\varepsilon \in \mathcal{F}$, $\mu(A_\varepsilon) \to 0+$, $H_\varepsilon : A_\varepsilon \to V \subset \mathbb{R}^D$.
$$\mathcal{N}_\varepsilon := \sum_{n\,:\, f^n(\cdot)\in A_\varepsilon} \delta_{(n\mu(A_\varepsilon), H_\varepsilon(f^n\cdot))}.$$
- Set $\mu_\varepsilon := \mu\left(H_\varepsilon^{-1}(\cdot)|A_\varepsilon\right)$ and $m_\varepsilon := \lambda \times \mu_\varepsilon$.
- **Theorem** [Pène,Saussol17+] *Let $\mathcal{V}$ be a $\pi$-system of relatively compact open subsets of $V$ that generates $\mathcal{B}(V)$ such that for any finite subset $\mathcal{V}_0 \subset \mathcal{V}$ and any $t > 0$,*
$$\sup_{A\in H_\varepsilon^{-1}\mathcal{V}_0, B\in\bigcup_{n=1}^{t/\mu(A_\varepsilon)} \sigma(f^{-n}H_\varepsilon^{-1}\mathcal{V}_0)} |\mu(B\cap A) - \mu(B)\mu(A)| = o(\mu(A_\varepsilon)).$$
*Assume that there exists a measure $\mu_0$ on $V$ s.t. $\forall F \in \mathcal{V}$, $\mu_0(\partial F) = 0$ and $\mu_\varepsilon(F) \to \mu_0(F)$.*
*Then $\mathcal{N}_\varepsilon \Longrightarrow_{\varepsilon\to 0} \mathcal{P}_0$ : $PPP(m = \lambda \times \mu_0)$ (wrt any $\mathbb{P} \ll \mu$), i.e. $\forall B$, s.t. $m(\partial B) = 0$,*
*$\mathcal{N}_\varepsilon(B) \Longrightarrow_{\varepsilon\to 0} \mathcal{P}_0(B) \hookrightarrow$ Poisson $m(B)$.*

PPP(m)

# Approximation by a Poisson Point Process

- $(\Omega, \mathcal{F}, \mu, f)$, $A_\varepsilon \in \mathcal{F}$, $\mu(A_\varepsilon) \to 0+$, $H_\varepsilon : A_\varepsilon \to V \subset \mathbb{R}^D$.

$$\mathcal{N}_\varepsilon := \sum_{n\ :\ f^n(\cdot)\in A_\varepsilon} \delta_{(n\mu(A_\varepsilon), H_\varepsilon(f^n\cdot))}.$$

- Set $\mu_\varepsilon := \mu\left(H_\varepsilon^{-1}(\cdot)|A_\varepsilon\right)$ and $m_\varepsilon := \lambda \times \mu_\varepsilon$.
- **Theorem** [Pène,Saussol17+] *Let $\mathcal{V}$ be a $\pi$-system of relatively compact open subsets of $V$ that generates $\mathcal{B}(V)$ such that for any finite subset $\mathcal{V}_0 \subset \mathcal{V}$ and any $t > 0$,*

$$\sup_{A\in H_\varepsilon^{-1}\mathcal{V}_0, B\in\bigcup_{n=1}^{t/\mu(A_\varepsilon)} \sigma(f^{-n}H_\varepsilon^{-1}\mathcal{V}_0)} |\mu(B\cap A) - \mu(B)\mu(A)| = o(\mu(A_\varepsilon)).$$

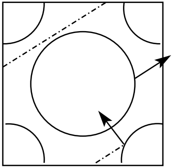
*Assume that there exists a measure $\mu_0$ on $V$ s.t. $\forall F \in \mathcal{V}$, $\mu_0(\partial F) = 0$ and $\mu_\varepsilon(F) \to \mu_0(F)$.*

*Then $\mathcal{N}_\varepsilon \Longrightarrow_{\varepsilon\to 0} \mathcal{P}_0$ : $PPP(m = \lambda \times \mu_0)$ (wrt any $\mathbb{P} \ll \mu$), i.e. $\forall B$, s.t. $m(\partial B) = 0$,*
*$\mathcal{N}_\varepsilon(B) \Longrightarrow_{\varepsilon\to 0} \mathcal{P}_0(B) \hookrightarrow$ Poisson $m(B)$.*

Billiard in a compact domain Q

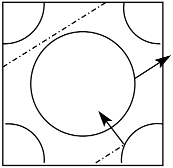
# Results for the Sinai billiard

- Billiard domain: $Q = \mathbb{T}^2 \setminus \bigcup_{i=1}^{I} O_i$, finite horizon.



# Results for the Sinai billiard

- Billiard domain: $Q = \mathbb{T}^2 \setminus \bigcup_{i=1}^{I} O_i$, finite horizon.

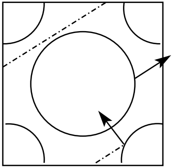


- Billiard map $(M = \partial Q \times [-\frac{\pi}{2}, \frac{\pi}{2}], \mu, f)$, $\frac{d\mu}{d\lambda}(q, \varphi) \sim \cos\varphi$.

  **Theorem**
  - For $\mu$-a.e. $x_0 = (q_0, \varphi_0) \in M$, if $A_\varepsilon = B(x_0, \varepsilon)$, $H_\varepsilon(q, \varphi) = (q - q_0, \varphi - \varphi_0)/\varepsilon$, $V = [-1, 1]^2$, then $\mathcal{N}_\varepsilon \Longrightarrow PPP(\lambda_3)$. Picture

# Results for the Sinai billiard

- Billiard domain: $Q = \mathbb{T}^2 \setminus \bigcup_{i=1}^{I} O_i$, finite horizon.



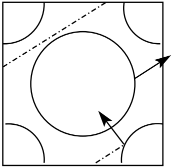
- Billiard map $(M = \partial Q \times [-\frac{\pi}{2}, \frac{\pi}{2}], \mu, f)$, $\frac{d\mu}{d\lambda}(q, \varphi) \sim \cos\varphi$.

**Theorem**

  - For $\mu$-a.e. $x_0 = (q_0, \varphi_0) \in M$, if $A_\varepsilon = B(x_0, \varepsilon)$, $H_\varepsilon(q, \varphi) = (q - q_0, \varphi - \varphi_0)/\varepsilon$, $V = [-1, 1]^2$, then $\mathcal{N}_\varepsilon \Longrightarrow PPP(\lambda_3)$. Picture
  - $\forall q_0 \in \partial Q$, if $A_\varepsilon = B(q_0, \varepsilon) \times [-\pi/2, \pi/2]$, $H_\varepsilon(q, \varphi) = (\varepsilon^{-1}(q - q_0), \varphi)$, $V = [-1, 1] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$, then $\mathcal{N}_\varepsilon \Longrightarrow PPP(K \cos(\varphi)\lambda_3)$. Picture

# Results for the Sinai billiard

- Billiard domain: $Q = \mathbb{T}^2 \setminus \bigcup_{i=1}^{I} O_i$, finite horizon.



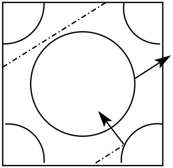
- Billiard map $(M = \partial Q \times [-\frac{\pi}{2}, \frac{\pi}{2}], \mu, f)$, $\frac{d\mu}{d\lambda}(q, \varphi) \sim \cos\varphi$.

  **Theorem**

  - For $\mu$-a.e. $x_0 = (q_0, \varphi_0) \in M$, if $A_\varepsilon = B(x_0, \varepsilon)$, $H_\varepsilon(q, \varphi) = (q - q_0, \varphi - \varphi_0)/\varepsilon$, $V = [-1, 1]^2$, then $\mathcal{N}_\varepsilon \Longrightarrow PPP(\lambda_3)$. Picture
  - $\forall q_0 \in \partial Q$, if $A_\varepsilon = B(q_0, \varepsilon) \times [-\pi/2, \pi/2]$, $H_\varepsilon(q, \varphi) = (\varepsilon^{-1}(q - q_0), \varphi)$, $V = [-1, 1] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$, then $\mathcal{N}_\varepsilon \Longrightarrow PPP(K \cos(\varphi)\lambda_3)$. Picture

- Idem for the billiard flow and the same targets.

# Billiard flow: visits to a ball



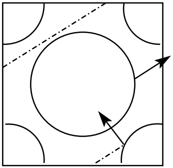
Billiard flow $(\mathcal{M}, \nu, (Y_t)_t)$, $\mathcal{M} = Q \times S^1$, $\nu = Leb(\cdot|\mathcal{M})$.
Let $q_0 \in Q$.

$$\mathfrak{N}_{\varepsilon,q_0} = \sum_{t\,:\,Y_t \text{ enters } B(q_0,\varepsilon)\times S^1} \delta_{\left(\frac{2\pi\varepsilon t}{Area(Q)},(\varepsilon^{-1}(\Pi_Q(Y_t(y))-q_0)),\Pi_V(Y_t(y))\right)}.$$

Picture

**Theorem** $\mathfrak{N}_{\varepsilon,q_0} \Rightarrow PPP(K\cos\varphi\lambda_3\mathbf{1}_{[0,+\infty)\times S^1\times[-\frac{\pi}{2},\frac{\pi}{2}]}dtdrd\varphi)$,
*$\varphi$ incident angle made with the inward normal vector.*

# Billiard flow: visits to a ball



Billiard flow $(\mathcal{M}, \nu, (Y_t)_t)$, $\mathcal{M} = Q \times S^1$, $\nu = Leb(\cdot|\mathcal{M})$.
Let $q_0 \in Q$.

$$\mathfrak{N}_{\varepsilon,q_0} = \sum_{t:\, Y_t \text{ enters } B(q_0,\varepsilon)\times S^1} \delta_{\left(\frac{2\pi\varepsilon t}{Area(Q)},(\varepsilon^{-1}(\Pi_Q(Y_t(y))-q_0)),\Pi_V(Y_t(y))\right)}.$$

Picture

**Theorem** $\mathfrak{N}_{\varepsilon,q_0} \Rightarrow PPP(K\cos\varphi\lambda_3 \mathbf{1}_{[0,+\infty)\times S^1\times[-\frac{\pi}{2},\frac{\pi}{2}]}dtdrd\varphi)$,
*$\varphi$ incident angle made with the inward normal vector.*

Our results apply also to visits to entrance in a corner for the billiard flow in a flat diamond.

# Proof of the technical assumption: A general context

Let $(M, \mu, f)$ metric, ergodic. Let $\alpha, \beta > 0$.

1. $\exists \tilde{\Pi} : (\tilde{M}, \tilde{\mu}, \tilde{f}) \to (M, \mu, f)$ and a sequence of partitions $(\mathcal{Q}_k)_{k\geq 0}$ of $\tilde{M}$ such that:
   (I) either $\sup_{Q\in\mathcal{Q}_k} \mathrm{diam}(\tilde{\Pi}(Q)) = O(k^{-\alpha})$, (non-invertible case)
   (II) or $\sup_{Q\in\mathcal{Q}_{2k}} \mathrm{diam}(\tilde{\Pi}\tilde{f}^k Q) = O(k^{-\alpha})$. (invertible case)

# Proof of the technical assumption: A general context

Let $(M, \mu, f)$ metric, ergodic. Let $\alpha, \beta > 0$.

1. $\exists \tilde\Pi : (\tilde M, \tilde\mu, \tilde f) \to (M, \mu, f)$ and a sequence of partitions $(\mathcal{Q}_k)_{k\ge 0}$ of $\tilde M$ such that:
   - (I) either $\sup_{Q\in\mathcal{Q}_k} \mathrm{diam}(\tilde\Pi(Q)) = O(k^{-\alpha})$, (non-invertible case)
   - (II) or $\sup_{Q\in\mathcal{Q}_{2k}} \mathrm{diam}(\tilde\Pi \tilde f^k Q) = O(k^{-\alpha})$. (invertible case)
2. $\exists C' > 0$ s.t., $\forall A \in \sigma(\mathcal{Q}_k)$, $\forall B \in \sigma\left(\bigcup_{m\ge 0} \mathcal{Q}_m\right)$,

$$\left| Cov_{\tilde\mu}\left(\mathbf{1}_A, \mathbf{1}_B \circ \tilde f^n\right)\right| \le C' n^{-\beta} \tilde\mu(A).$$

# Proof of the technical assumption: A general context

Let $(M, \mu, f)$ metric, ergodic. Let $\alpha, \beta > 0$.

1. $\exists \tilde{\Pi} : (\tilde{M}, \tilde{\mu}, \tilde{f}) \to (M, \mu, f)$ and a sequence of partitions $(\mathcal{Q}_k)_{k\geq 0}$ of $\tilde{M}$ such that:
   (I) either $\sup_{Q\in\mathcal{Q}_k} \mathrm{diam}(\tilde{\Pi}(Q)) = O(k^{-\alpha})$, (non-invertible case)
   (II) or $\sup_{Q\in\mathcal{Q}_{2k}} \mathrm{diam}(\tilde{\Pi}\tilde{f}^k Q) = O(k^{-\alpha})$. (invertible case)
2. $\exists C' > 0$ s.t., $\forall A \in \sigma(\mathcal{Q}_k)$, $\forall B \in \sigma\left(\bigcup_{m\geq 0} \mathcal{Q}_m\right)$,

$$\left| Cov_{\tilde{\mu}}\left(\mathbf{1}_A, \mathbf{1}_B \circ \tilde{f}^n\right)\right| \leq C' n^{-\beta} \tilde{\mu}(A).$$

3. $\sigma > 0$ so that $\varepsilon^{-\sigma}\mu(A_\varepsilon) \to 0$ and $\mu(\tau_{A_\varepsilon} \leq \varepsilon^{-\sigma} | A_\varepsilon) = o(1)$.
   Note that $\mathrm{diam}\, \tilde{\Pi} Q_K, \mathrm{diam}\, \tilde{\Pi}\tilde{f}^K Q_K < \varepsilon^{\alpha\sigma}$ for $K \sim c\varepsilon^{-\sigma}$.

# Proof of the technical assumption: A general context

Let $(M,\mu,f)$ metric, ergodic. Let $\alpha,\beta>0$.

1. $\exists\tilde\Pi:(\tilde M,\tilde\mu,\tilde f)\to(M,\mu,f)$ and a sequence of partitions $(\mathcal{Q}_k)_{k\ge 0}$ of $\tilde M$ such that:
   - (I) either $\sup_{Q\in\mathcal{Q}_k}\operatorname{diam}(\tilde\Pi(Q))=O(k^{-\alpha})$, (non-invertible case)
   - (II) or $\sup_{Q\in\mathcal{Q}_{2k}}\operatorname{diam}(\tilde\Pi\tilde f^kQ)=O(k^{-\alpha})$. (invertible case)
2. $\exists C'>0$ s.t., $\forall A\in\sigma(\mathcal{Q}_k)$, $\forall B\in\sigma\left(\bigcup_{m\ge 0}\mathcal{Q}_m\right)$,

$$\left|Cov_{\tilde\mu}\left(\mathbf{1}_A,\mathbf{1}_B\circ\tilde f^n\right)\right|\le C'n^{-\beta}\tilde\mu(A).$$

3. $\sigma>0$ so that $\varepsilon^{-\sigma}\mu(A_\varepsilon)\to 0$ and $\mu(\tau_{A_\varepsilon}\le\varepsilon^{-\sigma}|A_\varepsilon)=o(1)$.
   Note that $\operatorname{diam}\tilde\Pi Q_K,\operatorname{diam}\tilde\Pi\tilde f^KQ_K<\varepsilon^{\alpha\sigma}$ for $K\sim c\varepsilon^{-\sigma}$.
4. $\mu((\partial A_\varepsilon)^{[\varepsilon^{\alpha\sigma}]})=o(\mu(A_\varepsilon))$.

# Proof of the technical assumption: A general context

Let $(M, \mu, f)$ metric, ergodic. Let $\alpha, \beta > 0$.

1. $\exists \tilde{\Pi} : (\tilde{M}, \tilde{\mu}, \tilde{f}) \to (M, \mu, f)$ and a sequence of partitions $(\mathcal{Q}_k)_{k\geq 0}$ of $\tilde{M}$ such that:
   (I) either $\sup_{Q\in\mathcal{Q}_k} \mathrm{diam}(\tilde{\Pi}(Q)) = O(k^{-\alpha})$, (non-invertible case)
   (II) or $\sup_{Q\in\mathcal{Q}_{2k}} \mathrm{diam}(\tilde{\Pi}\tilde{f}^k Q) = O(k^{-\alpha})$. (invertible case)
2. $\exists C' > 0$ s.t., $\forall A \in \sigma(\mathcal{Q}_k)$, $\forall B \in \sigma\left(\bigcup_{m\geq 0} \mathcal{Q}_m\right)$,
$$\left| Cov_{\tilde{\mu}}\left(\mathbf{1}_A, \mathbf{1}_B \circ \tilde{f}^n\right)\right| \leq C' n^{-\beta} \tilde{\mu}(A).$$
3. $\sigma > 0$ so that $\varepsilon^{-\sigma}\mu(A_\varepsilon) \to 0$ and $\mu(\tau_{A_\varepsilon} \leq \varepsilon^{-\sigma} | A_\varepsilon) = o(1)$.
   Note that $\mathrm{diam}\, \tilde{\Pi} Q_K, \mathrm{diam}\, \tilde{\Pi}\tilde{f}^K Q_K < \varepsilon^{\alpha\sigma}$ for $K \sim c\varepsilon^{-\sigma}$.
4. $\mu((\partial A_\varepsilon)^{[\varepsilon^{\alpha\sigma}]}) = o(\mu(A_\varepsilon))$.
5. for all $F \in \mathcal{V}$, $\mu((\partial(H_\varepsilon^{-1}F))^{[\varepsilon^{\alpha\sigma}]} | A_\varepsilon) = o(1)$.

# Proof of the technical assumption: A general context

Let $(M, \mu, f)$ metric, ergodic. Let $\alpha, \beta > 0$.

1. $\exists \tilde{\Pi} : (\tilde{M}, \tilde{\mu}, \tilde{f}) \to (M, \mu, f)$ and a sequence of partitions $(\mathcal{Q}_k)_{k \geq 0}$ of $\tilde{M}$ such that:
   - (I) either $\sup_{Q \in \mathcal{Q}_k} \mathrm{diam}(\tilde{\Pi}(Q)) = O(k^{-\alpha})$, (non-invertible case)
   - (II) or $\sup_{Q \in \mathcal{Q}_{2k}} \mathrm{diam}(\tilde{\Pi}\tilde{f}^k Q) = O(k^{-\alpha})$. (invertible case)
2. $\exists C' > 0$ s.t., $\forall A \in \sigma(\mathcal{Q}_k)$, $\forall B \in \sigma\left(\bigcup_{m \geq 0} \mathcal{Q}_m\right)$,
$$\left| Cov_{\tilde{\mu}} \left( \mathbf{1}_A, \mathbf{1}_B \circ \tilde{f}^n \right) \right| \leq C' n^{-\beta} \tilde{\mu}(A).$$
3. $\sigma > 0$ so that $\varepsilon^{-\sigma} \mu(A_\varepsilon) \to 0$ and $\mu(\tau_{A_\varepsilon} \leq \varepsilon^{-\sigma} | A_\varepsilon) = o(1)$.
   Note that $\mathrm{diam}\, \tilde{\Pi} Q_K, \mathrm{diam}\, \tilde{\Pi}\tilde{f}^K Q_K < \varepsilon^{\alpha\sigma}$ for $K \sim c\varepsilon^{-\sigma}$.
4. $\mu((\partial A_\varepsilon)^{[\varepsilon^{\alpha\sigma}]}) = o(\mu(A_\varepsilon))$.
5. for all $F \in \mathcal{V}$, $\mu((\partial(H_\varepsilon^{-1} F))^{[\varepsilon^{\alpha\sigma}]} | A_\varepsilon) = o(1)$.

Proof of the technical condition.

# Billiard in the stadium

Conditions 1 and 2:
$\exists\tilde\Pi:(\tilde M,\tilde\mu,\tilde f)\to(M,\mu,f)$, and a sequence of partitions $(\mathcal{Q}_k)_{k\ge0}$ of $\tilde M$ such that: $\sup_{Q\in\mathcal{Q}_{2k}}\mathrm{diam}(\tilde\Pi\tilde f^kQ)=O(k^{-\alpha})$ and
$\forall A\in\sigma(\mathcal{Q}_k)$, $\forall B\in\sigma\left(\bigcup_{m\ge0}\mathcal{Q}_m\right)$,

$$\left|Cov_{\tilde\mu}\left(\mathbf{1}_A,\mathbf{1}_B\circ\tilde f^n\right)\right|\le C'n^{-\beta}\tilde\mu(A).$$

are satisfied by invertible systems that can be modeled by a Gibbs-Markov-Young towers with $\mathcal{Q}_k:=\bigvee_{i=0}^k\tilde f^{-i}\mathcal{Q}_0$, $\mathrm{diam}(f^n\gamma^s)+\mathrm{diam}(f^{-n}\gamma^u)\le Cn^{-\alpha}$; $Leb_\gamma(R>n)=O(n^{-\beta-1})$.
(see [Young1998], [Young1999], [Alves-Pinheiro08], [Alves-Azevedo13]).

# Billiard in the stadium

Conditions 1 and 2:
$\exists\tilde\Pi:(\tilde M,\tilde\mu,\tilde f)\to(M,\mu,f)$, and a sequence of partitions $(\mathcal{Q}_k)_{k\ge 0}$ of $\tilde M$ such that: $\sup_{Q\in\mathcal{Q}_{2k}}\mathrm{diam}(\tilde\Pi\tilde f^k Q)=O(k^{-\alpha})$ and $\forall A\in\sigma(\mathcal{Q}_k)$, $\forall B\in\sigma\left(\bigcup_{m\ge 0}\mathcal{Q}_m\right)$,

$$\left|Cov_{\tilde\mu}\left(\mathbf{1}_A,\mathbf{1}_B\circ\tilde f^n\right)\right|\le C'n^{-\beta}\tilde\mu(A).$$

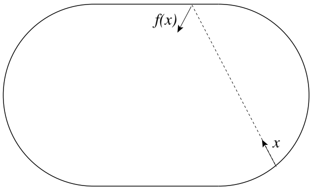
are satisfied by invertible systems that can be modeled by a Gibbs-Markov-Young towers with $\mathcal{Q}_k:=\bigvee_{i=0}^k\tilde f^{-i}\mathcal{Q}_0$, $\mathrm{diam}(f^n\gamma^s)+\mathrm{diam}(f^{-n}\gamma^u)\le Cn^{-\alpha}$; $Leb_\gamma(R>n)=O(n^{-\beta-1})$. (see [Young1998], [Young1999], [Alves-Pinheiro08], [Alves-Azevedo13]).

For the Sinai billiard: true $\forall\alpha,\beta>0$.

Conditions 1 and 2: are satisfied by invertible systems that can be modeled by a Gibbs-Markov-Young towers with $\mathcal{Q}_k := \bigvee_{i=0}^{k} \tilde{f}^{-i}\mathcal{Q}_0$, $\mathrm{diam}(f^n\gamma^s) + \mathrm{diam}(f^{-n}\gamma^u) \le Cn^{-\alpha}$; $Leb_\gamma(R > n) = O(n^{-\beta-1})$.

For the Sinai billiard: true $\forall \alpha, \beta > 0$.

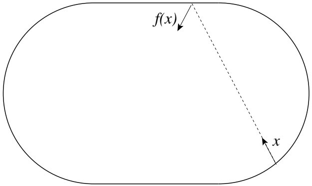
Billiard in the stadium




**Theorem (Billiard in the stadium).**[Pène,Saussol17+]
*For $\mu$-a.e. $x_0 = (q_0, \varphi_0) \in M$, if $A_\varepsilon = B(x_0, \varepsilon)$ and $F_\varepsilon(x) = \frac{x - x_0}{\varepsilon}$, $\mathcal{N}_\varepsilon \overset{w.r.t.\ \nu}{\Longrightarrow}_{r\to 0}$ Poisson process$(\lambda_3)$.* *Picture*

# Billiard in the stadium

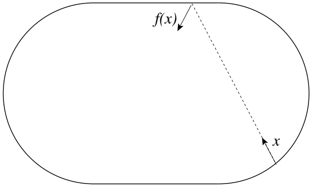
Conditions 1 and 2: are satisfied by invertible systems that can be modeled by a Gibbs-Markov-Young towers with $\mathcal{Q}_k := \bigvee_{i=0}^{k} \tilde{f}^{-i}\mathcal{Q}_0$, $\mathrm{diam}(f^n\gamma^s) + \mathrm{diam}(f^{-n}\gamma^u) \le Cn^{-\alpha}$; $Leb_\gamma(R > n) = O(n^{-\beta-1})$.




**Theorem (Billiard in the stadium).**[Pène,Saussol17+]
*For $\mu$-a.e. $x_0 = (q_0, \varphi_0) \in M$, if $A_\varepsilon = B(x_0, \varepsilon)$ and $F_\varepsilon(x) = \frac{x - x_0}{\varepsilon}$,*
$\mathcal{N}_\varepsilon \overset{w.r.t.\ \nu}{\Longrightarrow}_{r\to 0}$ *Poisson process*$(\lambda_3)$. *Picture*
**Proof.** $\mu(A_\varepsilon) \sim c\varepsilon^2$; so $\varepsilon^{-\sigma}\mu(A_\varepsilon) \to 0$ means $\sigma < 2$.

# Billiard in the stadium

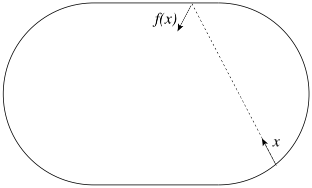
Conditions 1 and 2: are satisfied by invertible systems that can be modeled by a Gibbs-Markov-Young towers with $\mathcal{Q}_k := \bigvee_{i=0}^{k} \tilde{f}^{-i}\mathcal{Q}_0$, $\mathrm{diam}(f^n\gamma^s) + \mathrm{diam}(f^{-n}\gamma^u) \le Cn^{-\alpha}$; $Leb_\gamma(R > n) = O(n^{-\beta-1})$.




**Theorem (Billiard in the stadium).**[Pène,Saussol17+]
*For $\mu$-a.e. $x_0 = (q_0, \varphi_0) \in M$, if $A_\varepsilon = B(x_0, \varepsilon)$ and $F_\varepsilon(x) = \frac{x - x_0}{\varepsilon}$,*
*$\mathcal{N}_\varepsilon \overset{w.r.t.\ \nu}{\Longrightarrow}_{r\to 0}$ Poisson process$(\lambda_3)$.* *Picture*

**Proof.** $\mu(A_\varepsilon) \sim c\varepsilon^2$; so $\varepsilon^{-\sigma}\mu(A_\varepsilon) \to 0$ means $\sigma < 2$.

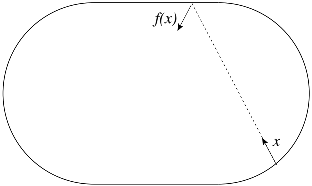
- Tower: [Markarian,04],[Chernov,Zhang05] with $\mu(R > n) = O(n^{-1-\beta})$ with $\beta = 2 > 1$.




**Theorem (Billiard in the stadium).**[Pène,Saussol17+]
*For $\mu$-a.e. $x_0 = (q_0, \varphi_0) \in M$, if $A_\varepsilon = B(x_0, \varepsilon)$ and $F_\varepsilon(x) = \frac{x - x_0}{\varepsilon}$,*
*$\mathcal{N}_\varepsilon \overset{w.r.t.\ \nu}{\Longrightarrow}_{r \to 0}$ Poisson process$(\lambda_3)$.* *Picture*

**Proof.** $\mu(A_\varepsilon) \sim c\varepsilon^2$; so $\varepsilon^{-\sigma}\mu(A_\varepsilon) \to 0$ means $\sigma < 2$.

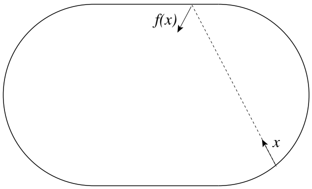
- Tower: [Markarian,04],[Chernov,Zhang05] with $\mu(R > n) = O(n^{-1-\beta})$ with $\beta = 2 > 1$.
- [Chernov,Markarian06]$\Rightarrow$ $\mathrm{diam}(f^n\gamma^s) + \mathrm{diam}(f^{-n}\gamma^u) \leq Cn^{-\alpha}$, $\alpha = 1$




**Theorem (Billiard in the stadium).**[Pène,Saussol17+]
*For $\mu$-a.e. $x_0 = (q_0, \varphi_0) \in M$, if $A_\varepsilon = B(x_0, \varepsilon)$ and $F_\varepsilon(x) = \frac{x - x_0}{\varepsilon}$,*
*$\mathcal{N}_\varepsilon \overset{w.r.t.\ \nu}{\Longrightarrow}_{r \to 0}$ Poisson process$(\lambda_3)$.* Picture

**Proof.** $\mu(A_\varepsilon) \sim c\varepsilon^2$; so $\varepsilon^{-\sigma}\mu(A_\varepsilon) \to 0$ means $\sigma < 2$.

- Tower: [Markarian,04],[Chernov,Zhang05] with $\mu(R > n) = O(n^{-1-\beta})$ with $\beta = 2 > 1$.
- [Chernov,Markarian06]$\Rightarrow$ $\mathrm{diam}(f^n\gamma^s) + \mathrm{diam}(f^{-n}\gamma^u) \le Cn^{-\alpha}$, $\alpha = 1$
- Take $1 < \sigma < 2$ $\mu((\partial A_\varepsilon)^{[\varepsilon^{\alpha\sigma}]}) \approx \varepsilon^{1+\alpha\sigma} = o(\varepsilon^2) = o(\mu(A_\varepsilon))$.




**Theorem (Billiard in the stadium).**[Pène,Saussol17+]
*For $\mu$-a.e. $x_0 = (q_0, \varphi_0) \in M$, if $A_\varepsilon = B(x_0, \varepsilon)$ and $F_\varepsilon(x) = \frac{x - x_0}{\varepsilon}$,*
*$\mathcal{N}_\varepsilon \overset{w.r.t.\ \nu}{\Longrightarrow}_{r\to 0}$ Poisson process$(\lambda_3)$.* *Picture*

**Proof.** $\mu(A_\varepsilon) \sim c\varepsilon^2$; so $\varepsilon^{-\sigma}\mu(A_\varepsilon) \to 0$ means $\sigma < 2$.

- Tower: [Markarian,04],[Chernov,Zhang05] with $\mu(R > n) = O(n^{-1-\beta})$ with $\beta = 2 > 1$.
- [Chernov,Markarian06]$\Rightarrow$ $\mathrm{diam}(f^n\gamma^s) + \mathrm{diam}(f^{-n}\gamma^u) \le Cn^{-\alpha}$, $\alpha = 1$
- Take $1 < \sigma < 2$ $\mu((\partial A_\varepsilon)^{[\varepsilon^{\alpha\sigma}]}) \approx \varepsilon^{1+\alpha\sigma} = o(\varepsilon^2) = o(\mu(A_\varepsilon))$.

Same result for the billiard flow with the same $A_\varepsilon$.

# Application to hyperbolic fixed points

- $(\Omega, \mathcal{F}, \mu, f)$, $f(x_0) = x_0$, $A_\varepsilon \in \mathcal{F}$,
  $A_\varepsilon := B(x_0, \varepsilon) \setminus f^{-1}(B(x_0, \varepsilon))$.

$$\mathcal{N}_\varepsilon := \sum_{n \,:\, f^n(\cdot) \in A_\varepsilon} \delta_{(n\mu(A_\varepsilon), H_\varepsilon(f^n \cdot))}$$

$$\widetilde{\mathcal{N}}_\varepsilon := \sum_{n \,:\, f^n(\cdot) \in B(x_0,\varepsilon)} \delta_{(n\mu(A_\varepsilon), H_\varepsilon(f^n \cdot))}.$$

- Set $\mu_\varepsilon := \mu\left(H_\varepsilon^{-1}(\cdot) | A_\varepsilon\right)$ and $m_\varepsilon := \lambda \times \mu_\varepsilon$.

# Application to hyperbolic fixed points

- $(\Omega, \mathcal{F}, \mu, f)$, $f(x_0) = x_0$, $A_\varepsilon \in \mathcal{F}$, $A_\varepsilon := B(x_0, \varepsilon) \setminus f^{-1}(B(x_0, \varepsilon))$.

$$\mathcal{N}_\varepsilon := \sum_{n\ :\ f^n(\cdot)\in A_\varepsilon} \delta_{(n\mu(A_\varepsilon), H_\varepsilon(f^n\cdot))}$$

$$\widetilde{\mathcal{N}}_\varepsilon := \sum_{n\ :\ f^n(\cdot)\in B(x_0,\varepsilon)} \delta_{(n\mu(A_\varepsilon), H_\varepsilon(f^n\cdot))}.$$

- Set $\mu_\varepsilon := \mu\left(H_\varepsilon^{-1}(\cdot)|A_\varepsilon\right)$ and $m_\varepsilon := \lambda \times \mu_\varepsilon$.
- **Theorem** [Pène,Saussol2016+] *Let $\Omega$ be a riemannian manifold, and $f$ a $C^1$-diffeomorphism between two neighbourhoods of $x_0$ with $Df_{x_0}$ hyperbolic. Assume that our first general Theorem(i) applies to $A_\varepsilon$ (+some other assumptions). Then $\mathcal{N}_\varepsilon \Longrightarrow \mathcal{P}$ and $\widetilde{\mathcal{N}}_\varepsilon \Longrightarrow \Psi(\mathcal{P})$, with $\ell_y := \inf\{k \ge 0\ :\ Df_{x_0}^{-k}(y) \notin B(0,1)\}$ and $\Psi : \sum_m \delta_{(t_m,x_m)} = \sum_m \sum_{k=0}^{\ell_{x_m}} \delta_{(t_m, Df_{x_0}^{-k}(x_m))}$.*