Non trivial Ruelle spectrum in hyperbolic dynamics

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Let $\mathbb{T}$ be the unit circle in $\mathbb{C}$. A map $\tau : \mathbb{T} \to \mathbb{T}$ is analytic expanding iff τ is real analytic on $\mathbb{T}$ and

$$\inf_{z \in \mathbb{T}} |\tau'(z)| > 1.$$

The simplest examples being $z \mapsto z^p$ for $p \geq 2$.

Analytic expanding maps admit a *unique* absolutely continuous (analytic) invariant probability measure: the SRB measure μ. Given $f, g \in C^0(\mathbb{T})$, set for all $n \geq 0$

$$C_{f,g}(n) := \int_{\mathbb{T}} (f \circ \tau^n) g d\mu - \int_{\mathbb{T}} f d\mu \int_{\mathbb{T}} g d\mu.$$

The SRB measure is mixing: we have

$$\lim_{n \to +\infty} C_{f,g}(n) = 0.$$

Problem: can one give a precise asymptotic expansion of $C_{f,g}$ for smooth enough observables ? Assume that f, g are analytic and $\int f d\mu = 0$ for simplicity. The Fourier transform $\widehat{C}_{f,g}(z)$ is then defined (for $|z|$ small) by

$$\widehat{C}_{f,g}(z) := \sum_{n\geq 0} C_{f,g}(n) z^n = \int_{\mathbb{T}} f(I - z\mathcal{L})^{-1}(h.g)dm,$$

where $\mathcal{L}$ is the Ruelle transfer operator, defined locally by

$$\mathcal{L}(g)(x) := \sum_{Ty=x} |T'(y)|^{-1} g(y),$$

and $d\mu = hdm$.

The fourier transform $\widehat{C}_{f,g}(z)$ is obviously holomorphic in $\{|z|<1\}$ and we have the (Cauchy) inversion formula:

$$C_{f,g}(n) = \frac{1}{2i\pi}\int_{|z|=\rho} \frac{\widehat{C}_{f,g}(z)}{z^{n+1}}dz,$$

for all $0<\rho<1$. Assume that $\widehat{C}_{f,g}(z)$ has a meromorphic extension to a larger disc $\{|z|<R\}$ with $R>1$, with poles denoted by $\mathcal{P}_R$, then for all $\epsilon>0$, one has as $n\to+\infty$,

$$C_{f,g}(n) = \sum_{z^*\in\mathcal{P}_R} (z^*)^{-n} P_{f,g,z^*}(n) + O(R^{-n+\epsilon}),$$

where P_{f,g,z^*} are polynomials.

Theorem (1, Ruelle)

When acting on Holomorphic function spaces, $\mathcal{L}$ is a compact nuclear operator. Hence for all g real analytic on $\mathbb{T}$, $\widehat{C}_{f,g}(z)$ has a meromorphic extension to $\mathbb{C}$, whose (inverses of) poles are called Ruelle resonances, denoted by
$\lambda_0(\tau) = 1 > |\lambda_1(\tau)| \geq \ldots \geq |\lambda_n(\tau)| \geq \ldots \geq 0.$

- The Ruelle resonances are the eigenvalues of the compact operator $\mathcal{L}$ acting on a suitable function space.
- The rate of mixing is at least exponential.
- Correlations functions for analytic observables have asymptotic expansions with arbitrarily small error term.

As a byproduct of nuclearity, we have also:

Theorem (2, Ruelle)

Let $\tau : \mathbb{T} \to \mathbb{T}$ *be analytic expanding. There exist* $C, \alpha > 0$ *such that for all* $n \in \mathbb{N}$,

$$|\lambda_n(\tau)| \leq Ce^{-\alpha n}.$$

Main issue: For the obvious examples $z \mapsto z^p$, The Ruelle spectrum is $\{0\} \cup \{1\}$! The rate of mixing is actually super exponential, which can be proved using Fourier series.

This not typical as expected and we prove the following statement.

Theorem (3, Bandtlow-Naud)

Let $\tau : \mathbb{T} \to \mathbb{T}$ *be analytic expanding. For all* $\varepsilon > 0$*, one can find* $\tau_\varepsilon : \mathbb{T} \to \mathbb{T}$ *analytic expanding such that* $\sup_{z \in \mathbb{T}} |\tau(z) - \tau_\varepsilon(z)| \leq \varepsilon$ *and for all* $\eta > 0$*,*

$$\limsup_{n \to \infty} |\lambda_n(\tau_\varepsilon) e^{n^{1+\eta}}| > 0.$$

- The upper bound is therefore optimal.
- The set of expanding maps with infinitely many distincts Ruelle eigenvalues is dense (in a suitable space).

Let $\mathbb{T}^2 = \mathbb{R}^2 \backslash \mathbb{Z}^2$ be the flat torus. Any Hyperbolic matrix $M \in SL_2(\mathbb{Z})$ induces an Anosov map $A_M : \mathbb{T}^2 \to \mathbb{T}^2$. Anosov means that we have a continuous $D_x A$-invariant splitting

$$T_x \mathbb{T}^2 = E_x^- \oplus E_x^+,$$

with for all $n \geq 0$,

$$\| D_x A^{\pm n}|_{E_x^\pm} \| \leq C e^{-\lambda n},$$

for some uniform $C, \lambda > 0$. Small C^1-perturbations of A_M are still Anosov.

Theorem (4, Faure-Roy)

Assume that A is a real analytic perturbation of a linear hyperbolic map M. Then there exists a Hilbert space $\mathcal{H}_{r,M}$ on which the Koopman operator $T_A : f \mapsto f \circ A$ is a compact, trace class operator. The trace is given by Lefschetz formula:

$$Tr(T_A) = \sum_{Ax=x} \frac{1}{|\det(I - D_x A)|}.$$

The spectrum of T_A, denoted by $\lambda_0(A) = 1 > |\lambda_1(A)| \geq |\lambda_2(A)| \geq \ldots \geq 0$ is called Ruelle spectrum.

- For linear hyperbolic maps, the spectrum is trivial $\{0\} \cup \{1\}$.

This behaviour is *non typical* again.

Theorem (5, A. Adam)

- *Generic real analytic perturbations of linear hyperbolic maps have non-trivial Ruelle spectrum.*
- *Generic volume preserving perturbations of linear hyperbolic maps have non-trivial Ruelle spectrum.*

The proof is based on a taylor expansion of $Tr(T_{A+\epsilon B})$ for $\epsilon \simeq 0$, but does not provide effective estimates on the location of non-trivial eigenvalues.

On the other hand, a recent result of Slipantschuk-Bandtlow-Just gives a family of Anosov maps for which the spectrum can be explicitly computed. We view the torus $\mathbb{T}^2$ as

$$\mathbb{T}^2 = \{(z_1, z_2) \in \mathbb{C}^2 \ : \ |z_1| = |z_2| = 1\}.$$

Let $T_\lambda : \mathbb{T}^2 \to \mathbb{T}^2$ be defined by ($|\lambda| < 1$)

$$T(z_1, z_2) = \left(z_1 \frac{z_1 - \lambda}{1 - \overline{\lambda} z_1} z_2, \frac{z_1 - \lambda}{1 - \overline{\lambda} z_1} z_2 \right).$$

Theorem (6, SBJ)

There exists a Hilbert space $\mathcal{H}_a$ on which $K_T : f \mapsto f \circ T_\lambda$ is a compact trace class operator, with spectrum

$$\sigma(K_T) = \{0, 1\} \cup \{(-\lambda)^n,\ n \in \mathbb{N}^*\} \cup \{(-\overline{\lambda})^n,\ n \in \mathbb{N}^*\}.$$

By the work of Faure-Roy, for Anosov maps, Ruelle eigenvalues enjoy a *streched* exponential estimate:

$$|\lambda_n(A)| \leq Ce^{-\beta\sqrt{n}}.$$

- The examples of SBJ have infinitely many non-trivial eigenvalues, but they do not saturate the bounds (exponential decay versus streched exponential decay)
- Not enough eigenvalues to obtain the analog of Theorem 3 of Bandtlow-Naud !
- A heavier machinery is required....

Given a smooth Anosov map $A : \mathbb{T}^2 \to \mathbb{T}^2$, one can build a partially hyperbolic map through an "extension" (G is a compact connected Lie group) $\widehat{A} : \mathbb{T}^2 \times G \to \mathbb{T}^2 \times G$ by

$$\widehat{A}(x, g) := (Ax, \tau(x)g),$$

where $\tau : \mathbb{T}^2 \to G$ is a smooth map. A natural invariant measure is the product of SRB-measure μ with Haar measure m. It is still an open conjecture that $\widehat{A}$ is exponentially mixing for generic choices of τ. However the following is known.

Theorem (7, Dolgopyat)

For a generic choice of τ, for observables $f, g \in C^\infty(\mathbb{T}^2 \times G)$, we have rapid mixing i.e. for all k, as $n \to +\infty$

$$\left| C_{f,g}(n) - \int f \int g \right| = O_k(n^{-k}).$$

There exists nevertheless a universal lower bound on the rate of mixing.

Theorem (8, Naud)

Assume that A is a real analytic perturbation of a linear hyperbolic map. Assume that G is a linear Algebraic connected compact Lie group and $\tau : \mathbb{T}^2 \to G$ *is real analytic. Then there exist* $N(G) > 0$ *such that for all* $\epsilon > 0$*, one can find real analytic observables* f, g *such that*

$$\limsup_{n\to+\infty} \left| C_{f,g}(n) - \int f \int g \right|^{1/n} \geq e^{N(G)P(-2\log J^u)-\epsilon},$$

where $P(.)$ *is the topological pressure and* J^u *is the unstable Jacobian.*

- If $G = S^1$, then $N(G) = \frac{5}{2}$, while if $G = SU_2(\mathbb{C})$, $N(G) = \frac{3}{2}$.
- In the circle case $G = S^1$, it is possible to find trigonometric polynomials τ such that the extension is rapidly mixing and

$$\limsup_{n\to+\infty} \left| C_{f,g}(n) - \int f \int g \right|^{1/n} \geq e^{\frac{1}{2}P(-2\log J^u)-\epsilon}.$$

- The proof is probabilistic (based on random trigonometric series) and does not provide explicit examples.
- In the case of $G = SU_2(\mathbb{C})$, it is possible to find locally constant, exponentially mixing extensions with explicit estimates on the rate !

Here we indicate ideas in the proof of Thm 3 on expanding maps. **First step: explicit computations.** Set $\mathbb{D} = \{z \in \mathbb{C} \ : \ |z| < 1\}$. For all $a = (a_1, a_2, \ldots, a_p) \in \mathbb{D}^p$, consider the Blaschke product

$$B_a(z) := \prod_{k=1}^{p} \left(\frac{z - a_k}{1 - \overline{a_k} z} \right).$$

If $\sum_k \frac{1-|a_k|}{1+|a_k|} > 1$ then $B_a : \mathbb{T} \to \mathbb{T}$ is analytic expanding. Blaschke products B_a act on the Riemann sphere $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ and have exactly two attracting fixed points $z_0 \in \mathbb{D}$ and $1/\overline{z_0} \in \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Set $\mu = B'_a(z_0)$, the multiplier at z_0.

Theorem (3, Bandtlow-Just-Slipantschuk)

Let B_a be an expanding Blaschke product. Then the Ruelle spectrum is exactly

$$\{1\} \cup \{\mu^n \ : \ n \geq 1\} \cup \{\overline{\mu}^n \ : \ n \geq 1\},$$

where $|\mu| < 1$.

This result provides a large set of analytic expanding maps for which the spectrum is known explicitly, answering numerous questions. It is actually a key ingredient in the proof of *our main result*.

Step two: determinants and Growth order. Most of our analysis rests on the following fact. Let $r < 1 < R$ and set $A_{r,R} := \{r < |z| < R\}$. A map $\tau : A_{r,R} \to \mathbb{C}$ is called a holomorphic expanding map of the annulus $A_{r,R}$ if τ is holomorphic on $\overline{A_{r,R}}$ and $\tau(A_{r,R}) \supset \overline{A_{r,R}}$.

Proposition

Let τ be holomorphic expanding on $A_{r,R}$. Then there exists a Hilbert space $\mathcal{H}_{r,R}$ of hyperfunctions on $A_{r,R}$ such that $C_\tau(f) := f \circ \tau$ acts boundedly on $\mathcal{H}_{r,R}$. Moreover,

- *$C_\tau : \mathcal{H}_{r,R} \to \mathcal{H}_{r,R}$ is a compact trace class operator.*
- *If τ is a circle map, then the spectrum of C_τ coincides with the Ruelle spectrum.*

To study the Ruelle spectrum, we will study the order of growth of determinants

$$\mathcal{Z}_\tau(w) := \det(I - e^w C_\tau).$$

Set

$$\rho(\tau) := \limsup_{r\to\infty} \frac{\log(\sup_{|w|\leq r} \max\{\log|\mathcal{Z}_\tau(w)|, 0\})}{\log r}.$$

Lemma

For all τ holomorphic expanding on $A_{r,R}$, we have $\rho(\tau) \leq 2$. Assume that for all $n \geq 0$, we have $|\lambda_n(\tau)| \leq Ce^{-\alpha n^\beta}$, then

$$\rho(\tau) \leq 1 + 1/\beta.$$

To prove Theorem 3, we need to find τ_ϵ ϵ-close to any τ such that $\rho(\tau_\epsilon) = 2$.

Assume that τ is an analytic expanding circle map of degree $d \geq 2$. Pick any Blaschke product B_a with same degree and non-trivial Ruelle spectrum. We have the following.

Proposition

There exist $r < 1 < R$, and an open set $\mathbb{C} \supset \mathcal{U} \supset [0, 1]$, a holomorphic map $T : \mathcal{U} \times A_{r,R} \to \mathbb{C}$ such that

- $T(0, \zeta) = \tau(\zeta), \ \forall \zeta \in A_{r,R}$.
- $T(1, \zeta) = B_a(\zeta), \ \forall \zeta \in A_{r,R}$
- *For all $z \in \mathcal{U}$, $T(z, .)$ is an expanding holomorphic map of the annulus $A_{r,R}$.*
- *For all $z \in [0, 1]$, $T(z, .)$ is a circle map.*

Step three: potential theory. We recall that a map on a domain Ω, $f : \Omega \subset \mathbb{C} \to [-\infty, \infty)$ is subharmonic iff we have:

- f is upper semi-continuous.
- For all $z_0 \in \Omega$, $r > 0$ small enough, we have

$$f(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta.$$

If $f \not\equiv -\infty$ on Ω, then $\{z \in \Omega \; : \; f(z) = -\infty\}$ is very thin: it has Hausdorff dimension 0. There is a corresponding extension to several complex variables called plurisubharmonicity.

For all $z \in \mathcal{U}$, set $\rho(z) := \rho(T(z,.))$. Because

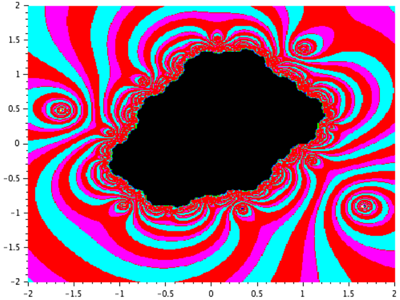
$$\mathcal{U} \times \mathbb{C} \ni (z, w) \mapsto \log |\mathcal{Z}_{T(z,.)}(w)|$$

is plurisubharmonic, there is some hope that $\rho(z)$ is itself subharmonic on $\mathcal{U}$. *If it was*, we would be done by the maximum principle: indeed, we have $\rho(z) \leq 2$ for all z and $\rho(1) = 2$ because B_a has a known spectrum.

The maximum principle for subharmonic functions implies that $\rho(z) = 2$, except on a polar set which have Hausdorff dimension 0. The appropriate tools to achieve this goal come from Lelong-Ferrand's "implicit subharmonic function theorem".

It is worth noticing that when $z \in \mathcal{U} \setminus [0,1]$, $\zeta \mapsto T(z,\zeta)$ is a priori no longer a circle map. Instead, it has an invariant Julia set which may be a quasi-circle has depicted by the explicit example given by

$$T(z,\zeta) := \zeta \left(\frac{2\zeta - z}{2 - z\zeta} \right).$$



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