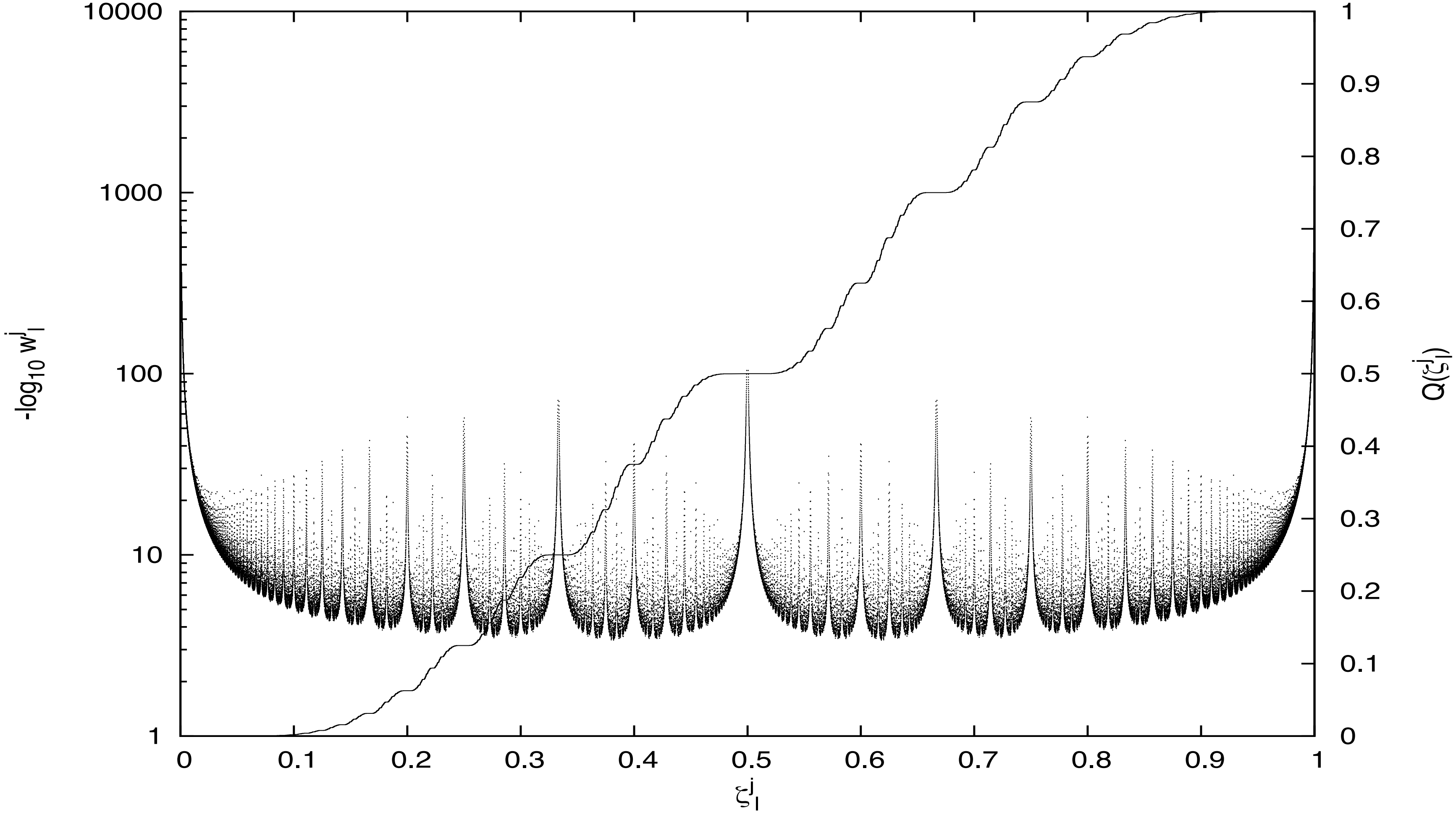
Center for Nonlinear and Complex Systems
Dipartimento di Fisica e Matematica

Regularity properties of Minkowski's ? measure



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Summary

- *Minkowski's ? function and measure: definition and relevance in dynamics*
- *Moebius Iterated Function Systems*
- *Regularity in logarithmic potential theory*
- *Proof of regularity of Minkowski's ? measure and its consequences*
- *A further regularity conjecture*

Center for Nonlinear and Complex Systems

Dipartimento di Fisica e Matematica

VERHANDLUNGEN

DES DRITTEN INTERNATIONALEN

MATHEMATIKER-KONGRESSES

Zur Geometrie der Zahlen.

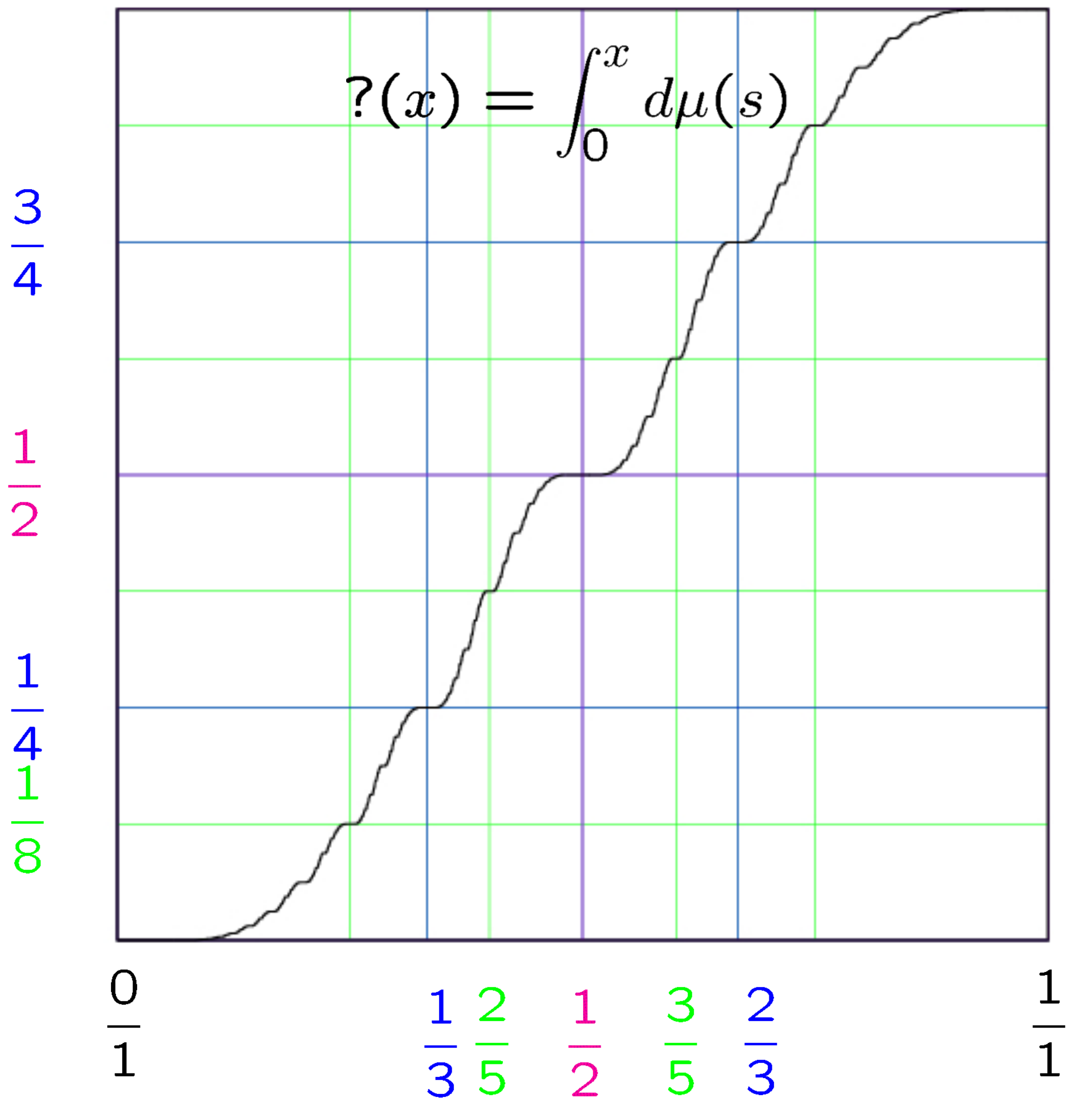
(Mit Projektionsbildern auf einer Doppeltafel.)

Von

H. Minkowski aus Göttingen.

Endlich möchte ich noch einige Worte über Kriterien für algebraische Zahlen hinzufügen.

(Fig. 7.) Durch diese Figur suche ich dem bekannten Lagrangeschen Kriterium für eine reelle quadratische Irrationalzahl eine neue Seite abzugewinnen. In einem Quadrat von der Seitenlänge 1 sind hier auf der linken vertikalen Seite, der y-Achse, fortgesetzt Halbierungen vorgenommen, so daß sukzessive alle Punkte erhalten werden, deren Ordinate eine rein dyadische Zahl, d. h. eine rationale Zahl mit einer Potenz von 2 als Nenner ist. Jedem auf der y-Achse auftretenden Intervall oder Teilpunkt wird nun ein Intervall bez. ein rationaler Teilpunkt auf der x-Achse, der unteren horizontalen Seite, dadurch zugeordnet, daß zunächst den Endwerten $y = 0$ und $y = 1$ die Endwerte $x = 0$ und $x = 1$ entsprechen sollen, und weiter, so oft dort ein Intervall halbiert wird, hier zwischen die Endpunkte a/b, a'/b' des zugeordneten Intervalls, a und b, ferner a' und b' als relativ prim gedacht, ein neuer Teilpunkt in $x = (a + a')/(b + b')$ eingeschaltet wird. Auf der

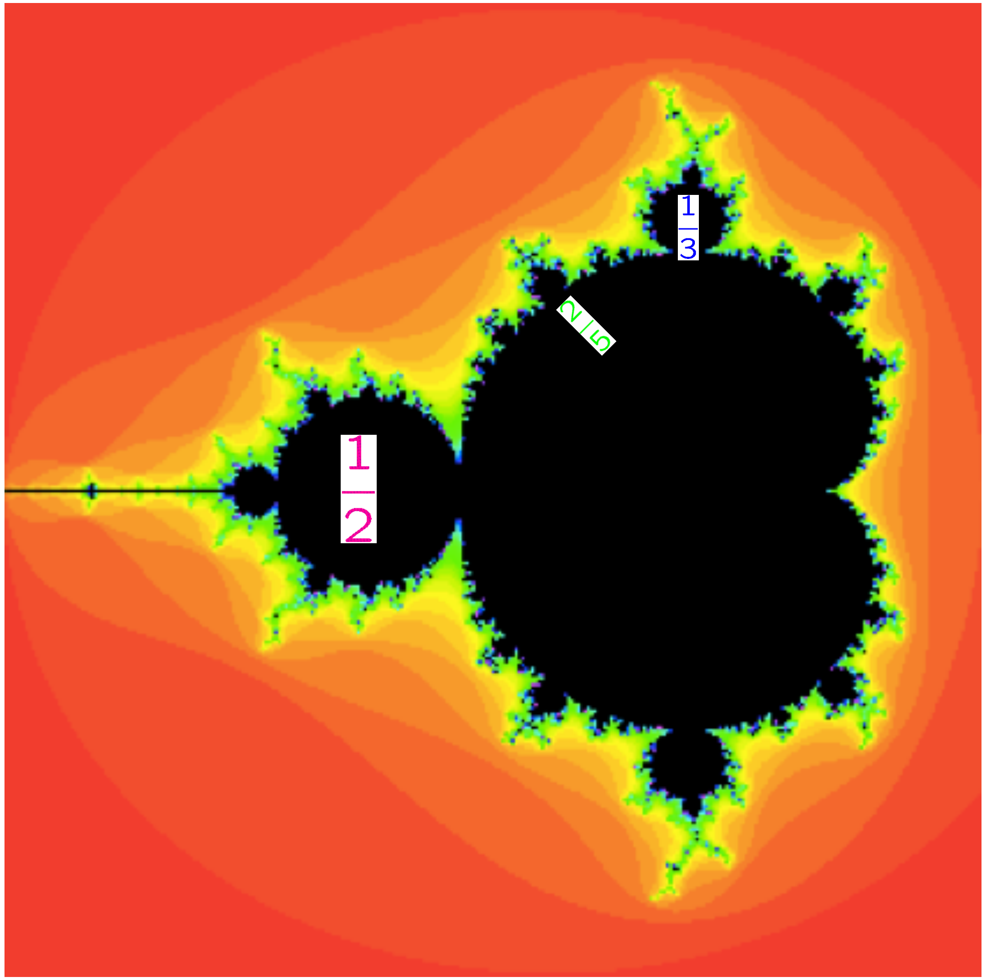

$?(x) = \int_0^x d\mu(s)$
3/4
1/2
1/4
1/8
0/1
1/3
2/5
1/2
3/5
2/3
1/1

ein neuer Teilpunkt in $x = (a + a')/(b + b')$ eingeschaltet wird. Auf der horizontalen Seite treten so als Teilpunkte sukzessive alle Punkte mit rationaler Abszisse auf, und die Zuordnung der gleichzeitig konstruierten Abszissen und Ordinaten liefert uns das Bild einer beständig wachsenden Funktion $y = ?(x)$, zunächst für alle rationalen x, dann durch die Forderung der Stetigkeit erweitert auf beliebige reelle Argumente im Intervalle $0 \leqq x \leqq 1$, während gleichzeitig y dieses Intervall beliebig durchläuft. Wenn nun x eine quadratische Irrationalzahl ist und daher auf eine periodische Kettenbruchentwickelung führt, so entspricht dadurch dem Werte $y = ?(x)$ eine periodische Dualentwickelung und erweist sich infolgedessen y als rational. Wir erhalten dadurch die Sätze:

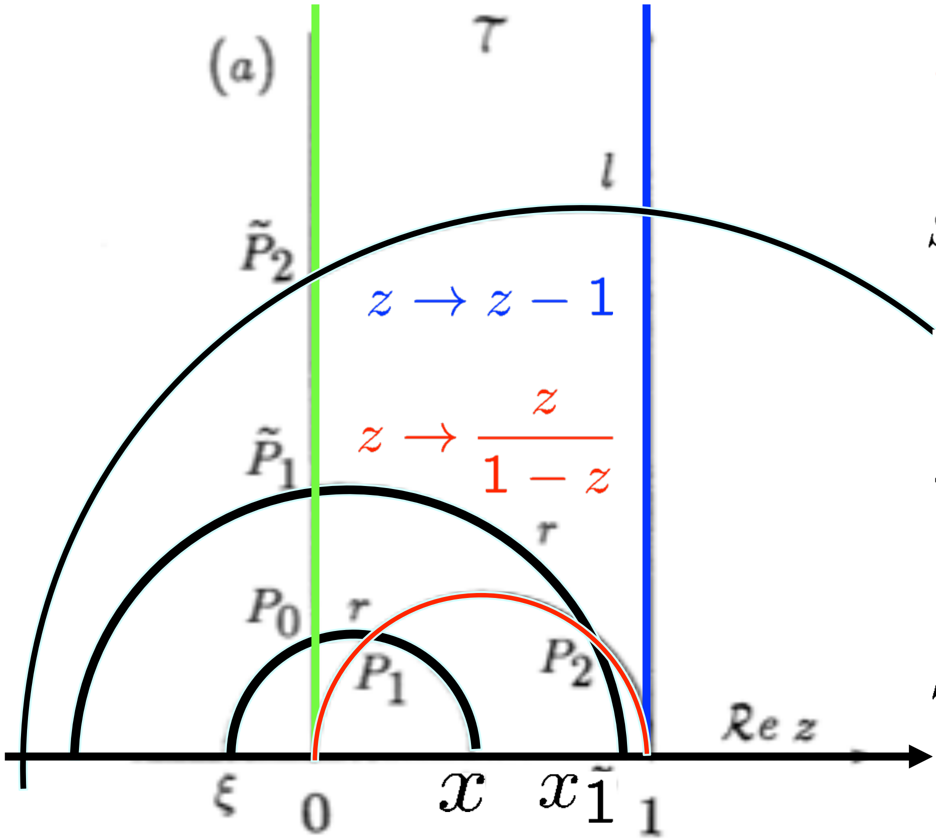
Ist x eine quadratische Irrationalzahl, so ist y rational, aber nicht rein dyadisch. Ist x rational, so ist y rein dyadisch. Und diese Sätze sind völlig umkehrbar.

If x is a quadratic irrational number, then y is rational, but not purely dyadic. If x is rational, then y is purely dyadic. And these statements are completely reversible.

Zur Geometrie der Zahlen



Hyperbolic polygonal billiards



$r \to \sigma = 0$

$l \to \sigma = 1$

$$S(x) = \sum_{j=0}^{\infty} \sigma(j) 2^{-j}$$

Kettenbruchentwickelung

$$x = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cdots}}}$$

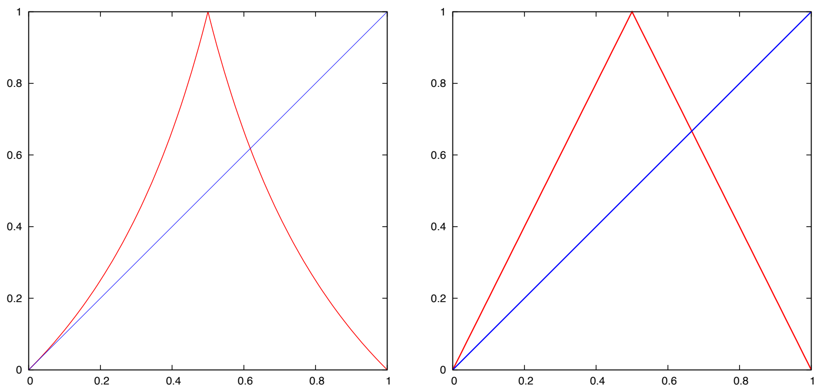
$$S(x) = \sum_{j=1}^{\infty} (-1)^{j+1} 2^{1-\sum_{l=1}^{j} n_l}$$

$\Updownarrow$

Invariant Multifractal Measures in Chaotic Hamiltonian Systems, and Related Structures

Martin C. Gutzwiller and Benoit B. Mandelbrot [a]
IBM T. J. Watson Research Center, Yorktown Heights, New York 10598
(Received 27 October 1987)

Farey Map



$$F(x) = \begin{cases} \frac{x}{1-x} & 0 \leq x < \frac{1}{2} \\ \frac{1-x}{x} & \frac{1}{2} \leq x \leq 1 \end{cases}$$

$$T(x) = \begin{cases} 2x & 0 \leq x < \frac{1}{2} \\ 2 - 2x & \frac{1}{2} \leq x \leq 1 \end{cases}$$

$$\begin{array}{ccc} x & \xrightarrow{F} & F(x) \\ \downarrow ? & & \downarrow ? \\ x & \xrightarrow{T} & T(x) \end{array}$$

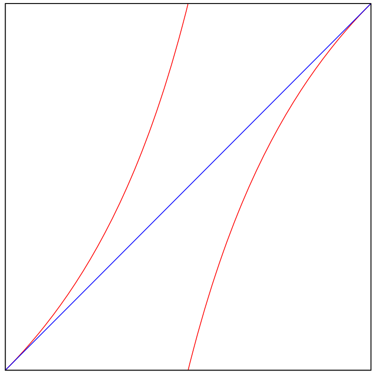
$$\mu(A) = \mu(F^{-1}(A))$$

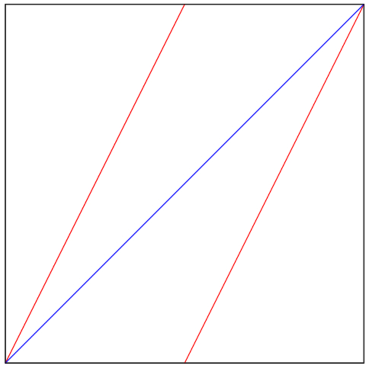
? = ?

?(x) = Minkowski's ? Function

μ = Minkowski's ? measure

Moebius Map





$$M(x) = \begin{cases} \frac{x}{1-x} & 0 \le x < \frac{1}{2} \\ \frac{2x-1}{x} & \frac{1}{2} \le x \le 1 \end{cases}$$

$$D(x) = \begin{cases} 2x & 0 \le x < \frac{1}{2} \\ 2x-1 & \frac{1}{2} \le x \le 1 \end{cases}$$

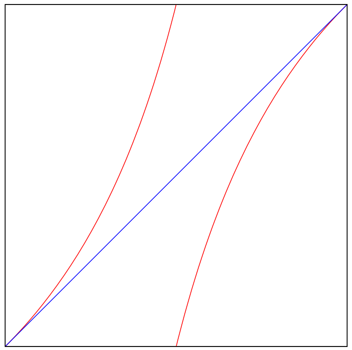
$$\begin{array}{ccc} x & \xrightarrow{D} & D(x) \\ \downarrow ? & & \downarrow ? \\ x & \xrightarrow{M} & M(x) \end{array}$$

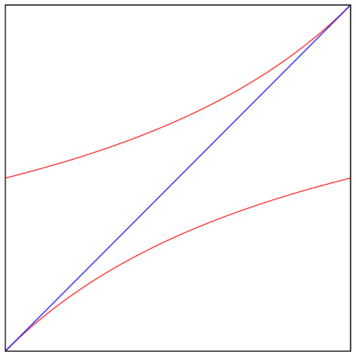
?(x) = Minkowski's ? Function

$$\mu(A) = \mu(M^{-1}(A))$$

μ = Minkowski's ? measure

Moebius Iterated Functions Systems





$$M(x) = \begin{cases} \frac{x}{1-x} & 0 \leq x < \frac{1}{2} \\ \frac{2x-1}{x} & \frac{1}{2} \leq x \leq 1 \end{cases}$$

$$\mu(A) = \mu(M^{-1}(A))$$

$$?(x) = \int_0^x d\mu(s)$$

$$M_{0,1}(x) = \begin{matrix} \frac{x}{x+1} \\ \frac{1}{2-x} \end{matrix} \qquad T^* : \mathcal{M}_1([0,1]) \rightarrow \mathcal{M}_1([0,1])$$

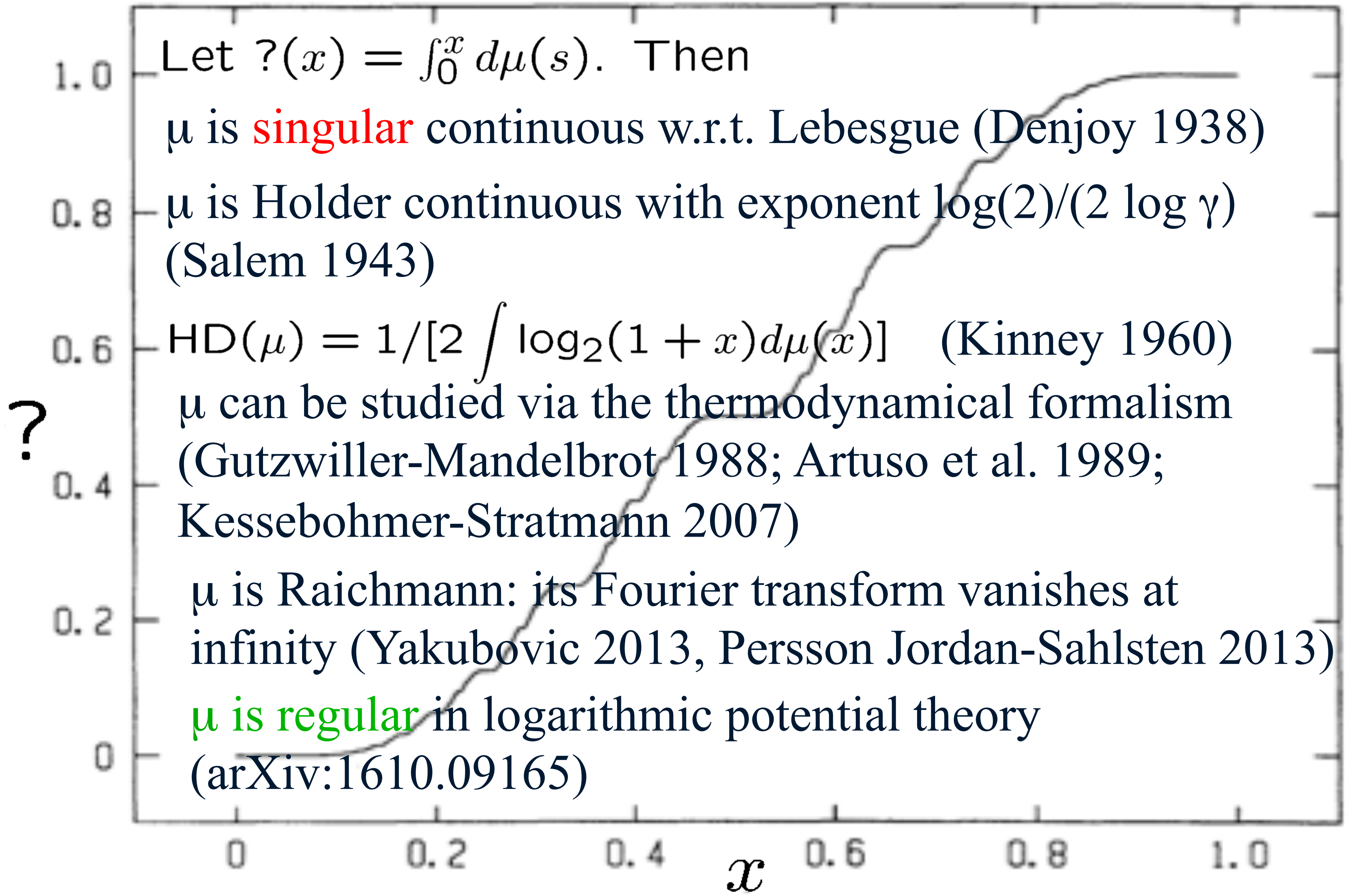
$$\int f d(T^* \mu) = \frac{1}{2} \sum_{i=0,1} \int (f \circ M_i) d\mu, \ \ \forall f \in C([0,1])$$

$$\exists! \ \mu \in \mathcal{M}_1([0,1]) \ \text{ s.t. } \ T^* \mu = \mu$$

Theorem: Minkowski's question mark measure is the balanced measure of the Moebius IFS above

G.M. & D. Bessis, PRL **66** 2939-2942 (1991)

Properties Minkowski's ? measure



Let $?(x) = \int_0^x d\mu(s)$. Then

μ is singular continuous w.r.t. Lebesgue (Denjoy 1938)

μ is Holder continuous with exponent log(2)/(2 log γ) (Salem 1943)

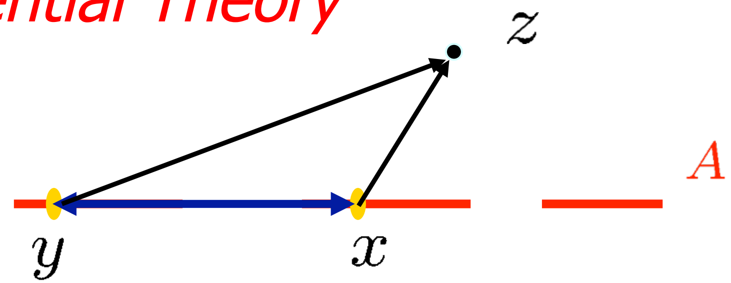
$\mathrm{HD}(\mu) = 1/[2 \int \log_2(1+x) d\mu(x)]$ (Kinney 1960)

μ can be studied via the thermodynamical formalism (Gutzwiller-Mandelbrot 1988; Artuso et al. 1989; Kessebohmer-Stratmann 2007)

μ is Raichmann: its Fourier transform vanishes at infinity (Yakubovic 2013, Persson Jordan-Sahlsten 2013)

μ is regular in logarithmic potential theory (arXiv:1610.09165)

Logarithmic Potential Theory



$$V_\mu(z) = -\int_A \log|(z-x)|d\mu(x)$$

$$E_\mu = \int_A V_\mu(y) d\mu(y)$$

Thm: if $\inf\{E_\mu, \mathrm{Supp}(\mu) \subset A, \mu(A) = 1\} < \infty$

$\exists!\ \nu_A$ s.t. $E_{\nu_A} = \inf\{E_\mu, \mathrm{Supp}(\mu) \subset A, \mu(A) = 1\} = -\log \mathrm{Cap}(A)$

$G_A(z) \geq 0$, harmonic in $\mathbf{C} \setminus (A \cup \{\infty\})$

$G_A(z) \sim \log|z| - \log \mathrm{Cap}(A)$ as $z \to \infty$

$G_A(z) \to 0$ as $z \to A$

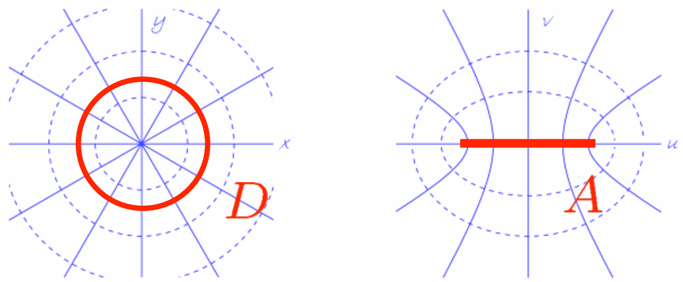
$G_A(z) = -V_{\nu_A}(z) - \log \mathrm{Cap}(A)$

$G_A(z) = \log|\Phi_A(z)|$, Φ_A conformal mapping $A \to D$

Example: $A = [0,1]$

$$d\nu_A(x) = \frac{1}{\pi\sqrt{x(1-x)}} dx$$

$$\Phi_A(z) = 2z - 1 + 2\sqrt{z^2 - z}$$



Regularity in potential theory

$\{p_j(\mu;x)\}_{j\in\mathbf{N}}$ s.t. $\int_A d\mu(x)\, p_l(\mu;x)\, p_m(\mu;x) = \delta_{l,m}$

uniformly on K compact, $K \cap A = \emptyset$

$\mu \in \text{Reg}(A) \iff |p_j(\mu;z)|^{1/j} \to e^{G_A(z)} = |\Phi_A(z)|$ Ullman-Saff-Stahl-Totik

$\mu \in N(A) \iff p_{j+1}(\mu;z)/p_j(\mu;z) \to \Phi_A(z)$ Nevai

$\mu \in S(A) \iff p_j(\mu;z)/\Phi^j_A(z) \to f(z)$ Szegö

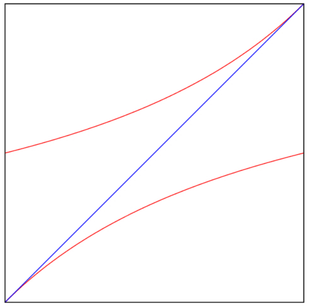
Stahl – Totik λ* Criterion for A = [0,1]

$\mu \in \text{Reg}(A) \Leftarrow \lambda(\{x \text{ s.t. } \mu(x-\epsilon, x+\epsilon) \leq e^{-a/\epsilon}\}) \to 0, \text{ as } \epsilon \to 0, \ \forall a > 0$

Theorem: Minkowski's question mark measure satisfies S-T λ criterion, hence it is USST regular.*

arXiv:1610.09165

Proof of Theorem



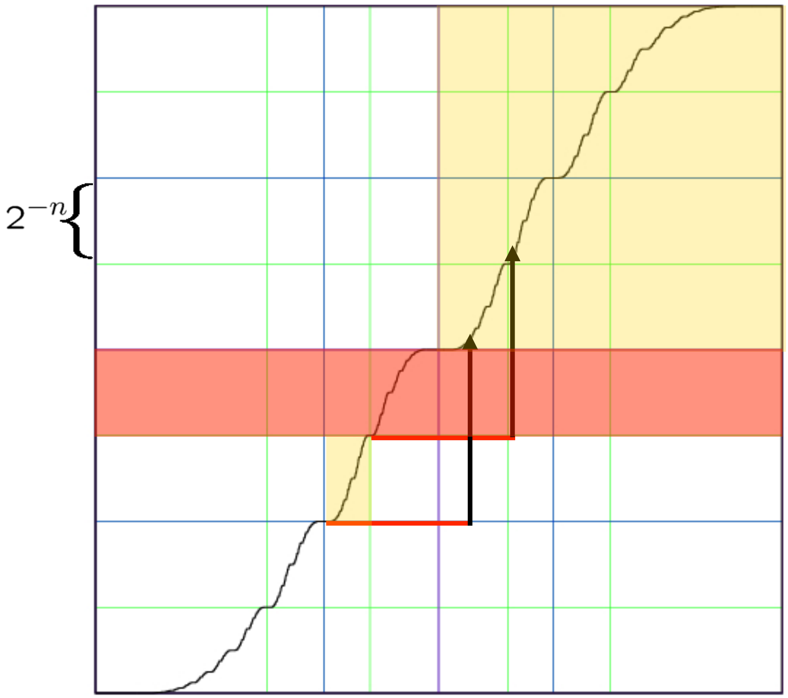
$$M_{0,1}(x) = \begin{cases} \frac{x}{x+1} \\ \frac{1}{2-x} \end{cases}$$

$\Sigma^n = \{n$ letter words in 0,1$\}$

$M_\sigma = M_{\sigma_1} \circ M_{\sigma_2} \circ \cdots \circ M_{\sigma_n}$

$I_\sigma = M_\sigma([0,1]); \;\; \mu(I_\sigma) = 2^{-|\sigma|}$

$I_\sigma = [x_\sigma, x_{\hat{\sigma}}]$



$$B_3 = \{0, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, 1\}$$

Lemma 0

$\{x_\sigma, \; \sigma \in \Sigma^n\} = B^n = \{x^n_j, j = 0,.,2^n\}$
is the n-th Stern–Brocot sequence.

Stahl Totik Criterion

$\Lambda^n(\alpha) = \{x \text{ s.t. } \mu([x, x+\alpha/n]) \geq 2^{-n}\}$

$\lambda(\Lambda^n(\alpha)) \to 1$, as $n \to \infty$, $\forall \alpha > 0$

Lemma 1

$$x^n_{j+2} - x^n_j \leq \frac{\alpha}{n} \Rightarrow [x^n_j, x^n_{j+1}] \subset \Lambda^n(\alpha)$$

$$\Updownarrow$$

$$j \in A^n(\alpha)$$

$$\sum_{j \in A^n(\alpha)} |x^n_{j+1} - x^n_j| \leq \lambda(\Lambda^n(\alpha))$$

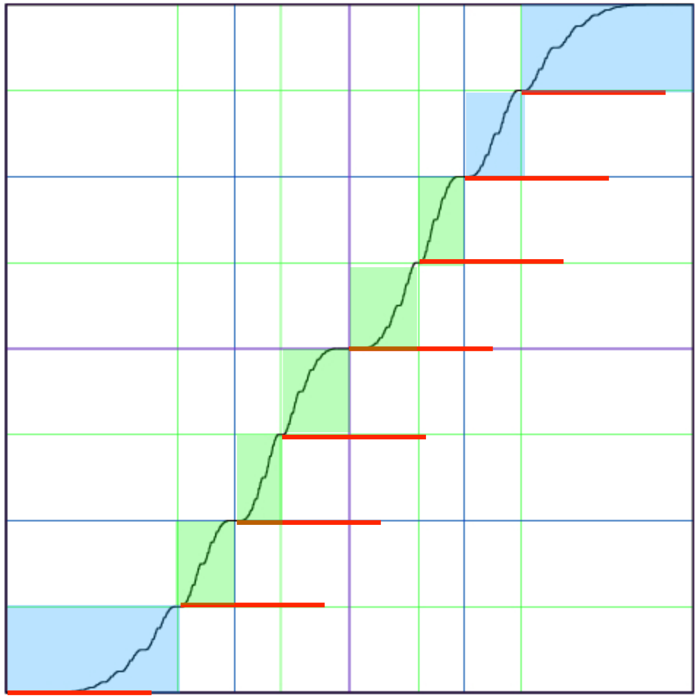
Proof of Main Theorem ctd.

Stahl Totik Criterion

$$\Lambda^n(\alpha) = \{x \text{ s.t. } \mu([x, x+\alpha/n]) \geq 2^{-n}\}$$

$$\lambda(\Lambda^n(\alpha)) \to 1, \text{ as } n \to \infty, \ \forall \alpha > 0$$

$$B_3 = \{x_j^3, j = 0, \ldots, 8\}$$



$$B_3 = \{0, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{2}{5}, \tfrac{1}{2}, \tfrac{3}{5}, \tfrac{2}{3}, \tfrac{3}{4}, 1\}$$

Lemma 1

$$x_{j+2}^n - x_j^n \leq \frac{\alpha}{n} \Rightarrow [x_j^n, x_{j+1}^n] \subset \Lambda^n(\alpha)$$

$$\Longleftrightarrow \quad j \in A^n(\alpha)$$

$$\textstyle\sum_{j \in A^n(\alpha)} |x_{j+1}^n - x_j^n| \leq \lambda(\Lambda^n(\alpha))$$

$$\textstyle\sum_{j \notin A^n(\alpha)} |x_{j+1}^n - x_j^n| \geq 1 - \lambda(\Lambda^n(\alpha))$$

$$1 - \lambda(\Lambda^n(\alpha)) \leq \#\overline{A^n(\alpha)} \max\{|x_{j+1}^n - x_j^n|\}$$

$$\leq \#\overline{A^n(\alpha)} \frac{1}{n+1} \quad \stackrel{?}{\longrightarrow} 0$$

$$j \notin A^n(\alpha) \Rightarrow (x_{j+1}^n - x_j^n \geq \tfrac{\alpha}{2n}) \vee$$

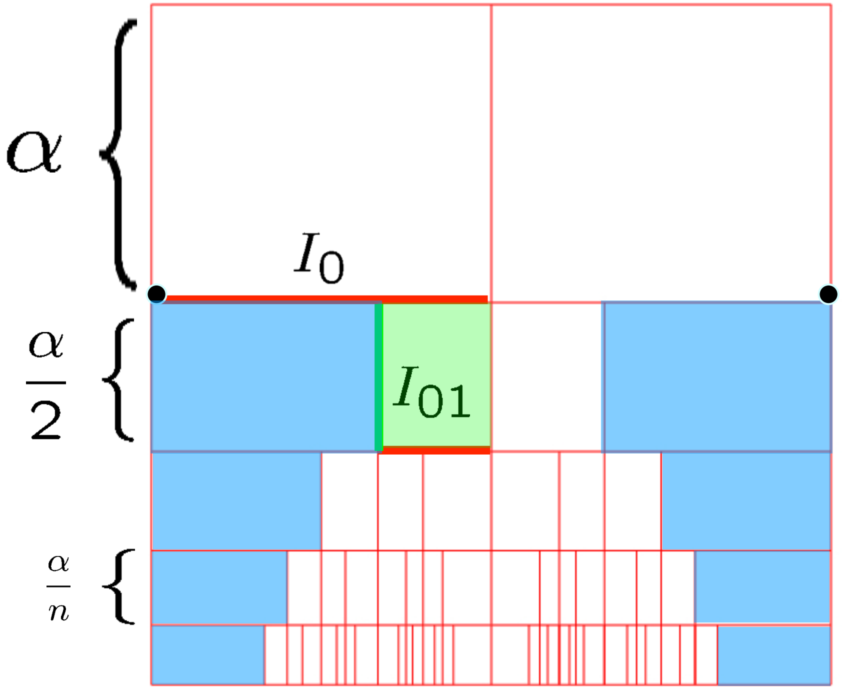
$$\vee (x_{j+2}^n - x_{j+1}^n \geq \tfrac{\alpha}{2n}) \vee (j = 2^n - 1)$$

$$\#\overline{A^n(\alpha)} \leq 2\#\{j \text{ st } x_{j+1}^n - x_j^n \geq \tfrac{\alpha}{2n}\} + 1$$

$$\#\{j \text{ st } x_{j+1}^n - x_j^n \geq \tfrac{\alpha}{2n}\} \stackrel{?}{<} C_\alpha$$

Proposition: For any $\alpha > 0$ the cardinality of $L^n(\alpha)$ is superiorly bounded, independent of n

$$L^n(\alpha) = \{\sigma \in \Sigma^n \text{ s.t. } \lambda(I_\sigma) \geq \frac{\alpha}{n}\} = \overline{S^n(\alpha)}$$



$$I_\sigma = [x_\sigma, x_{\hat{\sigma}}] = [\frac{p_\sigma}{q_\sigma}, \frac{p_{\hat{\sigma}}}{q_{\hat{\sigma}}}]$$

$$\mathbf{Q}_\alpha = \{\zeta = \frac{p}{q} \text{ s.t. } p \perp q, q^2 \leq \frac{1}{\alpha}\}$$

$$\mathcal{E}_\alpha = \{\sigma \in \Sigma \text{ s.t. } x_\sigma, x_{\hat{\sigma}} \notin \mathbf{Q}_\alpha\}$$

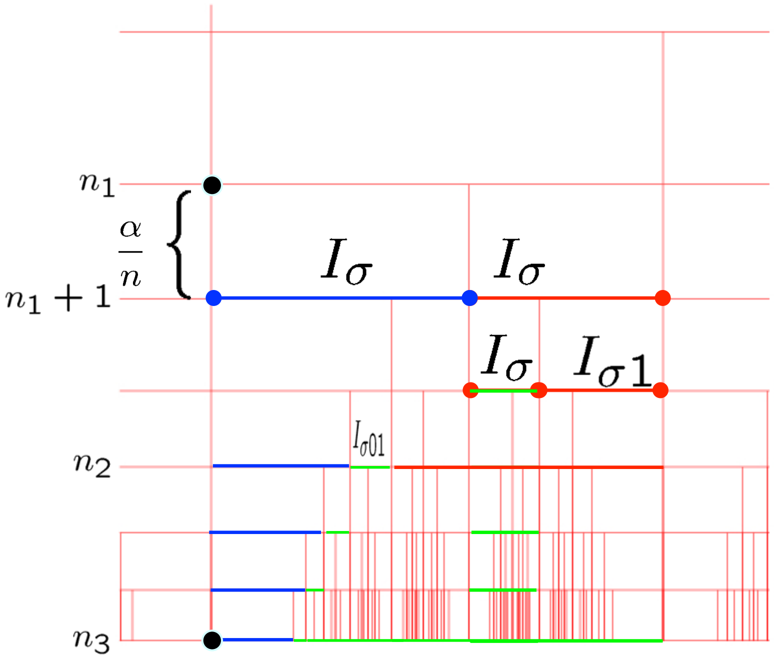
$$\mathcal{E}\mathcal{S}^n_\alpha = \mathcal{E}_\alpha \bigcap S^n(\alpha)$$

Proof of Proposition

$$L^n(\alpha) = \{\sigma \in \Sigma^n \text{ s.t. } \lambda(I_\sigma) \geq \frac{\alpha}{n}\} = \overline{S^n(\alpha)}$$

$$I_\sigma = [x_\sigma, x_{\hat{\sigma}}] = [\frac{p_\sigma}{q_\sigma}, \frac{p_{\hat{\sigma}}}{q_{\hat{\sigma}}}]; \qquad \mathbf{Q}_\alpha = \{\zeta = \frac{p}{q} \text{ s.t. } p \perp q, q^2 \leq \frac{1}{\alpha}\}$$

$$\mathcal{E}_\alpha = \{\sigma \in \Sigma \text{ s.t. } x_\sigma, x_{\hat{\sigma}} \notin \mathbf{Q}_\alpha\}; \qquad \mathcal{ES}^n_\alpha = \mathcal{E}_\alpha \cap S^n(\alpha)$$



Lemma 1

$\sigma \in \mathcal{E}_\alpha \Rightarrow \sigma\eta \in \mathcal{E}_\alpha, \ \forall \eta \in \Sigma,$

$\sigma \in \mathcal{ES}^n_\alpha \Rightarrow \sigma\eta \in \mathcal{ES}^n_\alpha, \ \forall \eta \in \Sigma$

Lemma 2

$\sigma \in \mathcal{E}_\alpha \Rightarrow \ \exists\, k_1(\sigma) \in \mathbf{N} \text{ s.t. } \sigma\eta \in \mathcal{ES}^n_\alpha$
$\forall \eta, |\eta| \geq k_1(\sigma)$

Lemma 3

$\sigma \in \Sigma \Rightarrow \ \exists\, k_2(\sigma) \in \mathbf{N} \text{ s.t.}$
$\sigma 0^k 1, \sigma 1^k 0 \in \mathcal{ES}^{|\sigma|+k+1}_\alpha$
$\forall k \geq k_2(\sigma)$

Proposition $\Rightarrow$ Criterion $\lambda^* \Rightarrow \mu \in \text{Reg}(A)$

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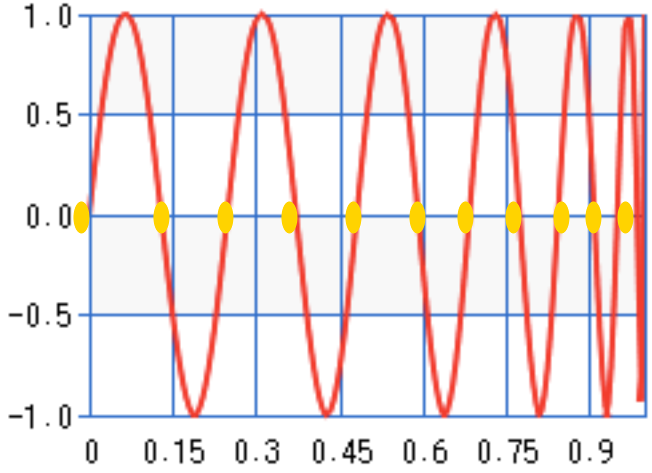
Regularity (potential theoretic) collides with singularity (Lebesgue)

$$\mu \in \mathrm{Reg}(A) \iff |p_j(\mu; z)|^{1/j} \to e^{G_A(z)} = |\Phi_A(z)|$$

Uniformly on K compact, $K \cap A = \emptyset$.



The measure of the zeros

ζ_j^i s.t. $p_j(\mu; \zeta_j^i) = 0, i = 1, \ldots, j$

$$\mu \in \mathrm{Reg}(A) \iff \frac{1}{j} \sum_{i=1}^{j} \delta_{\zeta_j^i} \to \nu_A$$

$$d\nu_A(x) = \frac{1}{\pi\sqrt{x(1-x)}} dx$$

Gaussian integration measure

$$\mu \in \mathcal{M}([0,1]) \Longrightarrow \sum_{i=1}^{j} \lambda_j^i \delta_{\zeta_j^i} \to \mu \qquad ?(x) = \int_0^x d\mu(s)$$

$$\lambda_j^i = 1 / \sum_{l=0}^{j-1} p_l(\mu; \zeta_j^i)^2$$

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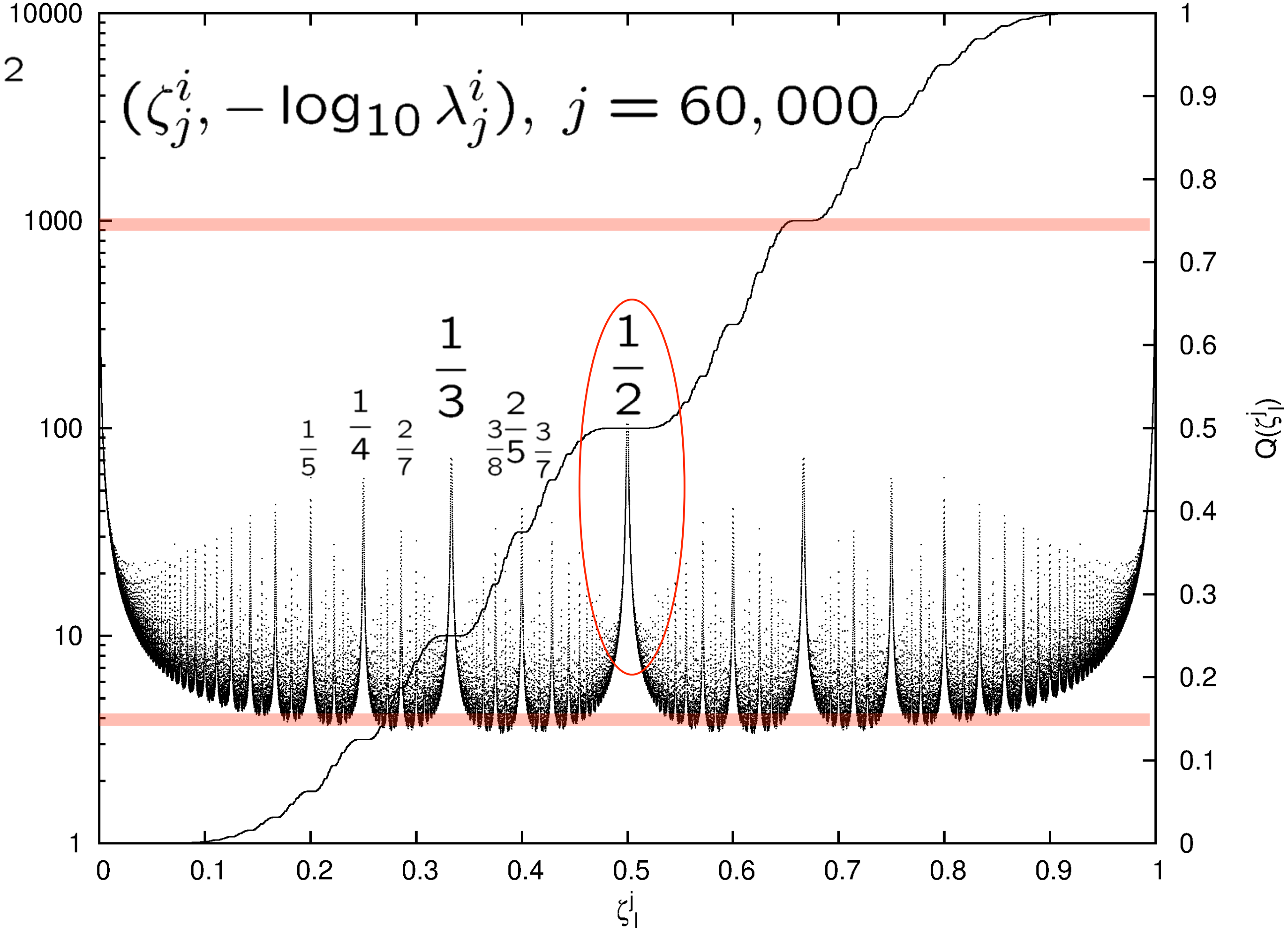
Regularity vs. singularity

$$\mu \in \mathrm{Reg}(A) \iff \frac{1}{j}\sum_{i=1}^{j} \delta_{\zeta_j^i} \to \nu_A \qquad \mu \in \mathcal{M}([0,1]) \Longrightarrow \sum_{i=1}^{j} \lambda_j^i \delta_{\zeta_j^i} \to \mu$$

$$\lambda_j^i = 1/\sum_{l=0}^{j-1} p_l(\mu;\zeta_j^i)^2$$

$\lambda_j^i \simeq 10^{-1000} \Rightarrow$

$\lambda_j^i \simeq 10^{-4} \Rightarrow$



arXiv:1603.05815

Christoffel functions

$$\lambda_j(x) = 1/\sum_{l=0}^{j-1} p_l(\mu; x)^2$$

Proposition. The logarithmic amplitude of the Christoffel function $\lambda_j(x)$ of order j at the point $x = \frac{1}{q} + y$ is given by an asymptotic formula that comprises the sum of four contributions:

$$\log(\lambda_j(\frac{1}{q} + y)) \sim \Lambda_j(q; y) = \Lambda^1(q; y) + \Lambda^2(q) + \Lambda^3(q; y) + \Lambda^4(j)$$

$$\begin{aligned}
\Lambda^1(q; y) &= \tfrac{1}{2}\log[\tfrac{1}{q} + y - (\tfrac{1}{q} + y)^2] + \log 2 \\
\Lambda^2(q) &= (-q + 2 - \tfrac{1}{q})\log(2) + \log(\log(2)) \\
\Lambda^3(q; y) &= -\frac{\log 2}{q^2 y} - 2\log(qy) \\
\Lambda^4(j) &= -\log(j)
\end{aligned}$$

Proof: arXiv:1603.05815 (to appear in J. Approx. Theory)

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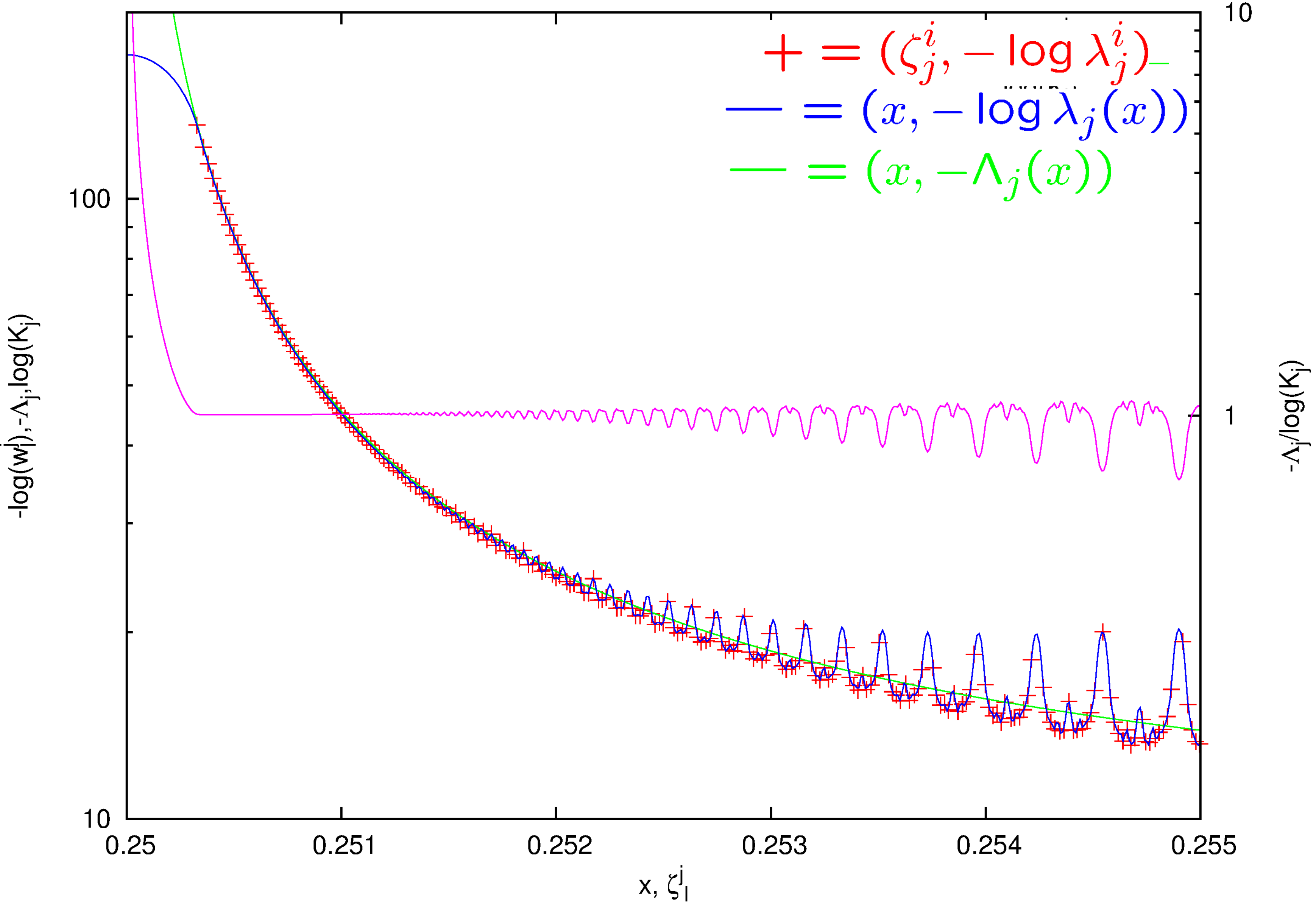

Jacobi Matrix: the double helix of measures

$$\mu \in \mathcal{M}_1([0,1]) \Longrightarrow \int d\mu(x)\, p_l(\mu;x)\, p_m(\mu;x) = \delta_{l,m}$$

$$x\, p_m(\mu;x) = a_{m+1}\, p_{m+1}(\mu;x) + a_m\, p_{m-1}(\mu;x) + b_m\, p_n(\mu;x)$$

Jacobi Matrix

$$J_\mu := \begin{bmatrix} b_0 & a_1 & 0 & \dots \\ a_1 & b_1 & a_2 & 0 \\ 0 & a_2 & b_2 & a_3 \\ \dots & 0 & a_3 & b_3 \end{bmatrix}$$

Uniqueness

$$\mu \Longleftrightarrow \{p_j(\mu;x)\} \Longleftrightarrow J_\mu$$

USST Regularity

$$\mu \in \mathsf{Reg}(A) \Longleftrightarrow |p_j(\mu;z)|^{1/j} \to e^{G_A(z)} = |\Phi_A(z)|$$

$$\mu \in \mathsf{Reg}(A) \Longleftrightarrow [\prod_{i=1}^{j} a_i]^{\frac{1}{j}} \to \mathsf{Cap}(A)$$

Nevai class

$$\mu \in N(A) \Longleftrightarrow p_{j+1}(\mu;z)/p_j(\mu;z) \to \Phi_A(z)$$

$$\mu \in N(A) \Longleftrightarrow a_j \to a = \mathsf{Cap}(A),\; b_j \to b$$

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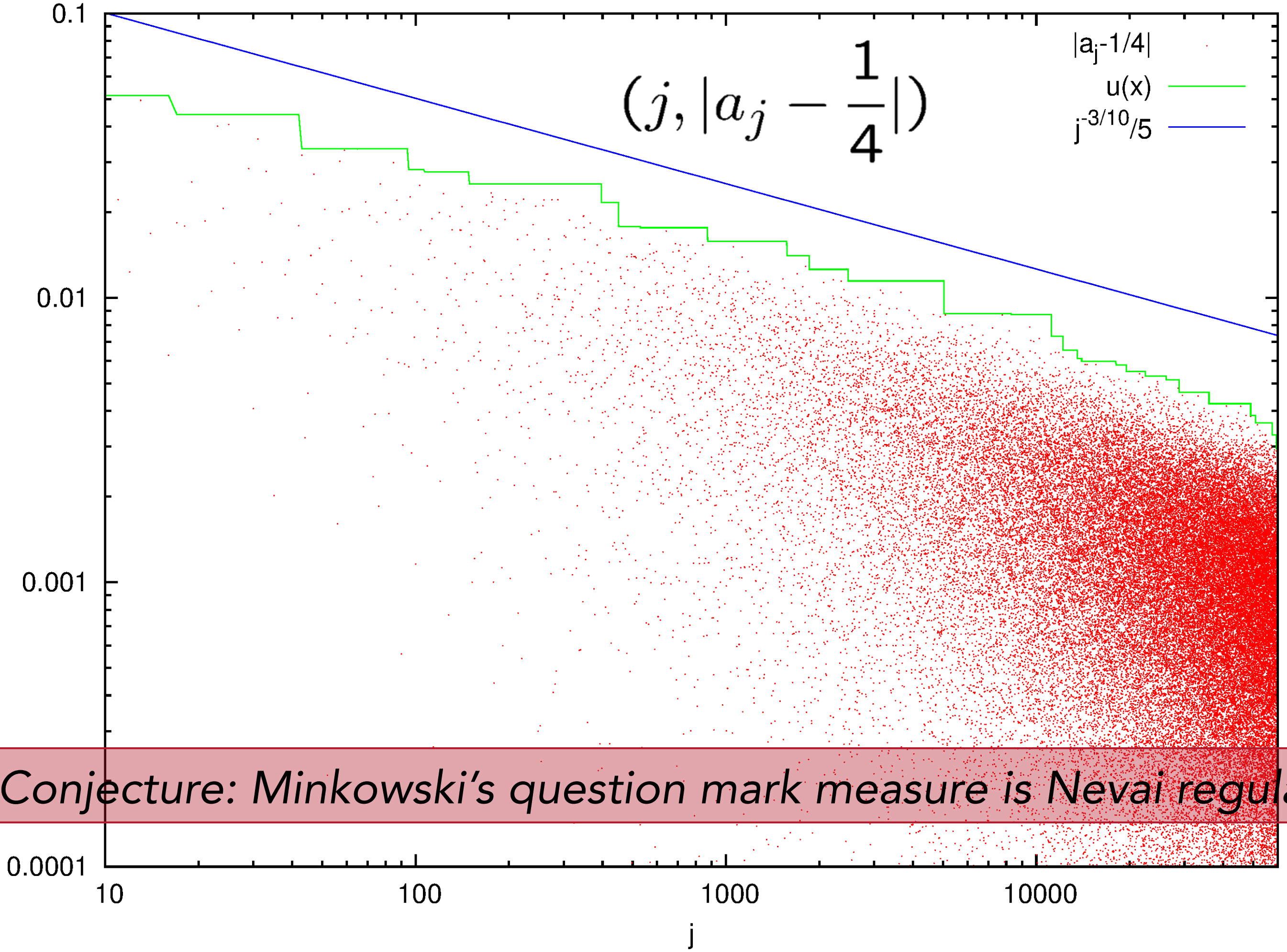
Nevai regularity !

$$\mu \in N(A) \Longleftrightarrow a_j \to a = \text{Cap}(A) = \frac{1}{4}$$



Conjecture: Minkowski's question mark measure is Nevai regular.

Conclusions. Regularity: where does it come from ?

$\Lambda^n(\alpha) = \{x \text{ s.t. } \mu([x, x+\alpha/n]) \geq 2^{-n}\}$ Stahl Totik Criterion

$\lambda(\Lambda^n(\alpha)) \to 1, \forall \alpha > 0$

$\Downarrow$

Theorem: Minkowski's question mark measure is USST regular.

$\Uparrow$

Conjecture: Minkowski's question mark measure is Nevai regular.

$\Uparrow$

Rakhmanov's Thm: $\frac{d\mu}{dx} > 0 \ \lambda$ a.e.

$\Uparrow$ X

$?'(x) = 0$ Lebesgue a.e.

$\Uparrow$?

$\mu([x, x+\delta]) = ?(x+\delta) - ?(x) > 0 \quad \forall x, \delta > 0$
+ something

The end

Conclusions

- *Minkowski's ? measure is UST regular*
- *Compelling numerical evidence suggests that it belongs to the Nevai class*
- *Asymptotic expansion of Christoffel weights reveal the hierarchical structure of Minkowski's measure and its encoding in the Jacobi matrix*
- *detail available at http://arxiv.org/abs/1603.05815*

Computing the Jacobi Matrix

$$\eta \longrightarrow T^*\eta$$

$$\Updownarrow \qquad \Updownarrow$$

Measure level

$$J_\eta \rightarrow \mathcal{T}(J_\eta) = J_{T^*\eta}$$

Jacobi matrix level

$$\mathcal{T}(J_\eta) = \frac{1}{2}\left(\frac{J_\eta}{I + J_\eta} \oplus \frac{I}{2I - J_\eta}\right)$$

The algorithm

$$d\eta(x) = \chi_{[0,1]}(x)\, dx \Rightarrow J_\eta$$

η is the Lebesgue measure

$$\mathcal{T}^n(J_\eta) = \text{Tridiag}([b_j^n, a_j^n])$$

$$\Delta_j^n(J_\eta) = |a_j^n - a_j^{n-1}|$$

gauge convergence !

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Hausdorff dimension of ? measure

$$\mathrm{HD}(\mu) = 1/[2 \int \log_2(1+x) d\mu(x)]$$

$f(x) = \log(1+x)$ is totally monotone on $[0, \infty)$

$(-1)^n f^{(n)}(x) \geq 0$ on $[0, \infty)$ for all $n \geq 0$

The first and second Gauss integration formulae give rigorous upper and lower bounds to the integral of f

These formulae can be computed starting from the Jacobi matrix J_μ

Difference between the Gaussian formulae and the exact value can be exactly estimated as

$$\frac{f^{2j}(\xi)}{k_j^2 (2j)!}, \frac{f^{2j+1}(\eta)}{h_j^2 (2j+1)!}$$

The Hausdorff dimension of μ can be exactly estimated.

Hausdorff dimension of ? measure

$$\mathrm{HD}(\mu) = 1/[2\int \log_2(1+x)\,d\mu(x)]$$

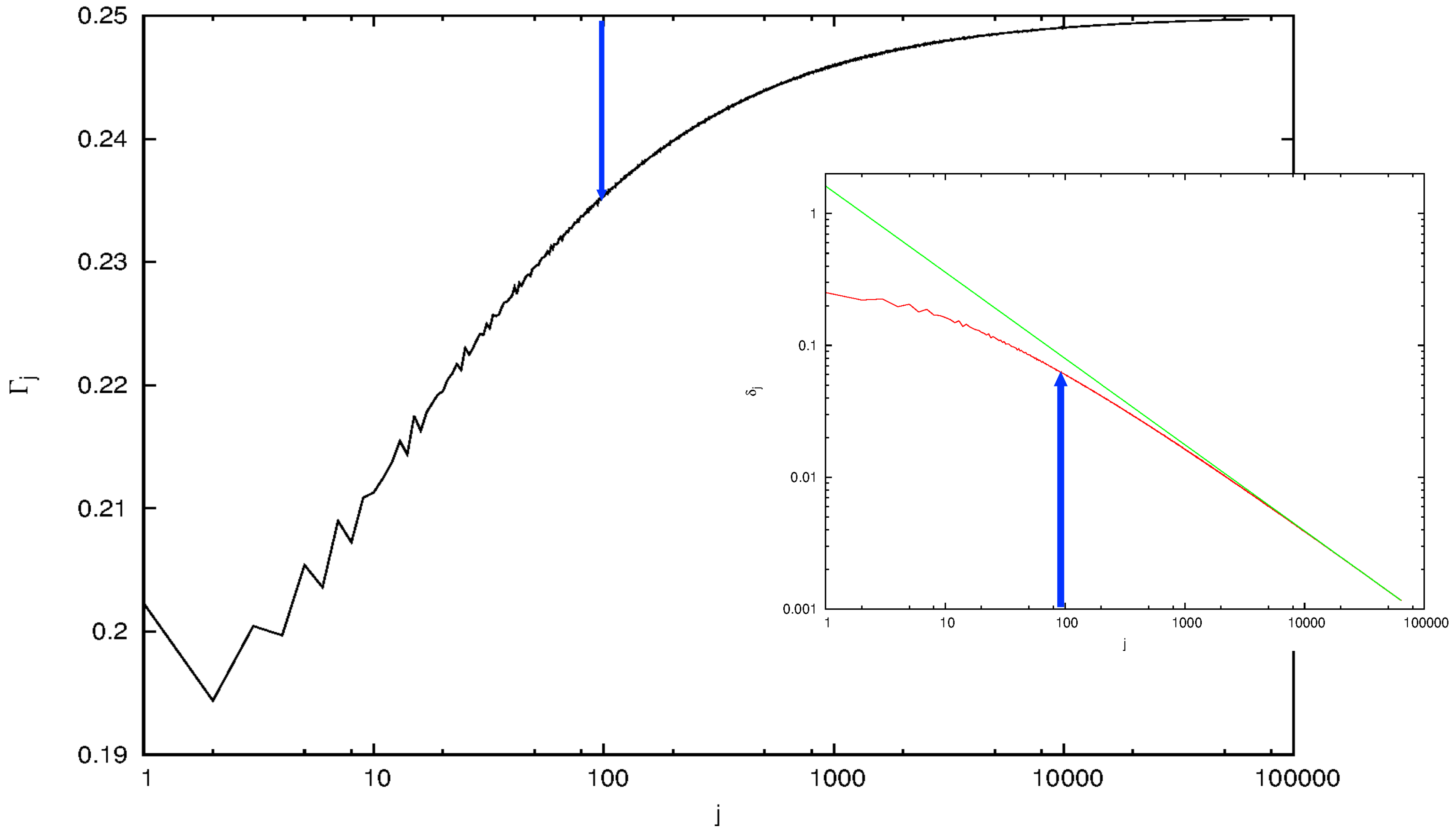
The Hausdorff dimension of μ can be exactly estimated.

2	0.874761611261160	0.874552879123086	0.000208732138074
3	0.874716939422290	0.874714034545017	0.000002904877273
4	0.874716314143510	0.874716274367535	0.000000039775975
5	0.874716305274063	0.874716304510136	0.000000000763927
6	0.874716305110859	0.874716305099384	0.000000000011475
7	0.874716305108267	0.874716305108003	0.000000000000264
8	0.874716305108213	0.874716305108207	0.000000000000006
9	0.874716305108212	0.874716305108211	0.000000000000001

If B is such that $\mathrm{HD}(B) < \mathrm{HD}(\mu)$, then $\mu(B) = 0$;
There exists a set C such that $\mathrm{HD}(C) = \mathrm{HD}(\mu)$ and $\mu(C) = 1$

UST regularity !

$$\mu \in \mathrm{Reg}(A) \Longleftrightarrow [\prod_{i=1}^{j} a_i]^{\frac{1}{j}} \to \mathrm{Cap}(A) = \frac{1}{4}$$

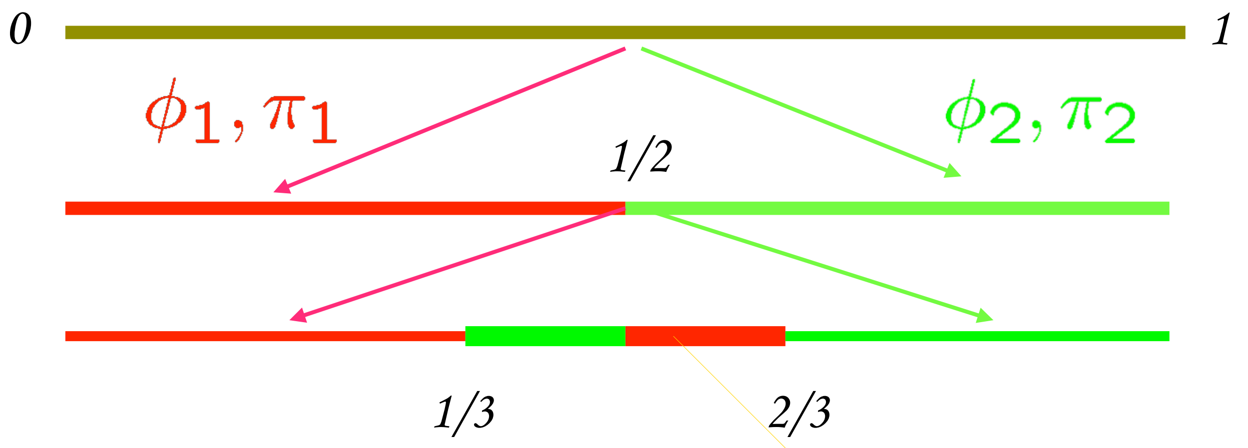


Moebius IFS

$$\{\phi_i : X \to X, \quad i \in \mathcal{I}\} \qquad \pi_i > 0, \quad i \in \mathcal{I} \quad \sum_{i \in \mathcal{I}} \pi_i = 1$$

$$X = [0:1], \quad \mathcal{I} = \{1, 2\}, \quad \pi_1 = \pi_2 = \frac{1}{2}$$

$$\phi_1(x) = \frac{x}{x+1}, \quad \phi_2(x) = \frac{1}{2-x}$$



$$\exists! \mu \text{ s.t. } T^* \mu = \mu, (T^*)^n \eta \to \mu, \forall \eta \in \mathcal{M}([0,1])$$

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Quick review of Iterated Functions Systems

$$\{\phi_i : X \to X, \quad i \in \mathcal{I}\}$$

X cpct metric space
ϕ_i continuous, contractive

$$\pi_i > 0, \quad i \in \mathcal{I} \quad \sum_{i\in\mathcal{I}} \pi_i = 1$$

π_i map probabilities

$$T(f) = \sum_{i\in\mathcal{I}} \pi_i (f \circ \phi_i)$$

T : C(X) --> C(X)

$$\int f d(T^*\mu) = \sum_{i\in\mathcal{I}} \pi_i \int (f \circ \phi_i) d\mu$$

T* : M(X) --> M(X)

$$T^*\mu = \mu$$

μ is an invariant measure

Thm: (Hutchinson 1981)

$$\exists! \mu \text{ s.t. } T^*\mu = \mu,$$
$$(T^*)^n \eta \to \mu, \forall \eta \in \mathcal{M}([0,1])$$

Slippery devil's staircase