Noise induced order, a computer aided proof.

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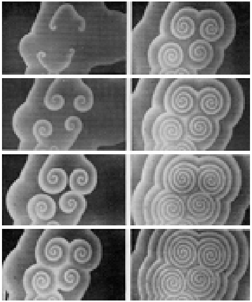
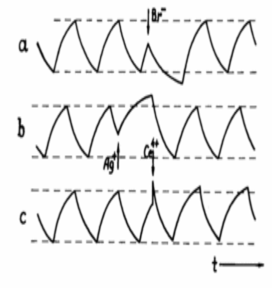
Overview

- We will see a computer aided approach to the rigorous proof of the existence of **Noise Induced Order.** A phenomemon for which in certain random dynamical system (of interest in applications) the adding of noise stabilizes the behavior of the system.
- The proof is based on the **rigorous** (certified) estimate of the properties of the stationary measure of the system, and its convergence to equilibrium.
- The approach used is quite general and could be applied to any map of th interval, perturbed by additive noise.
- From a joint work with M. Monge and I. Nisoli. (appearing on arXiv tomorrow!)

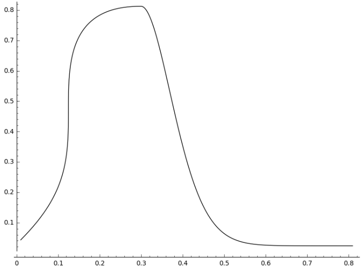
The Belosuv-Zabotinsky reaction.





- A chemical reaction with a chaotic behavior (spiral waves evolving).
- In 1983 Matsumoto and Tsuda discovered by numerical simulations that the behavior of a model of the reaction is less chaotic when a certain quantity of noise is added.
- The discovery was confirmed by real experiments with the actual chemical reaction
- Such kind of phenomena were found in several other systems, also of biological origin.

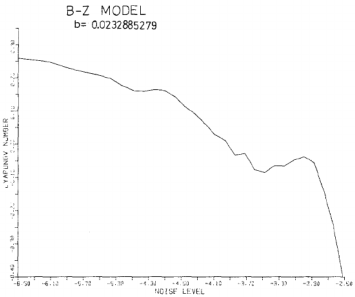
Noise induced order, the model



$$T(x) = \begin{cases} (a + (x - \frac{1}{8})^{1/3})e^{-x} + b & (0 \le x \le 0.3), \\ c(10xe^{-10x/3})^{19} + b & (x > 0.3). \end{cases}$$

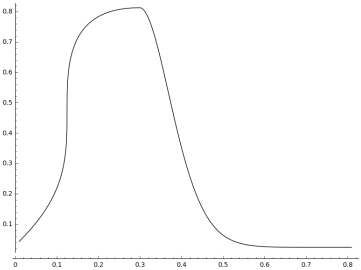
- $a = 0.5060735690368223\ldots$ (so that $T'(0.3) = 0$),
- $b = 0.0232885279$ (for some chemical reason),
- $c = 0.1212056927389751\ldots$ (so that $T(0.3^-) = T(0.3^+)$),
- Uniformly distributed additive noise on $[-\xi, \xi]$

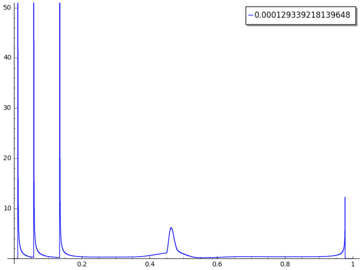
The behavior of the model



- Matsumoto and Tsuda made mumerical simulations obtaining this behavior for the Lyapunov exp. ($\int \log(T')\ d\mu$, where μ is the stationary measure)
- In practice that would be an indicator of exponential divergence of nearby starting orbits supposing they are subjected to the same noise.
- Notice that Lyap. exp. > 0 when the noise very small and then it becomes $<< 0$ for larger noise. (the *N. I. O.*).

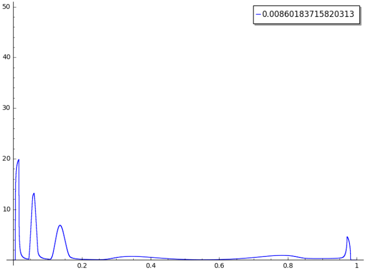
A rigorous approach

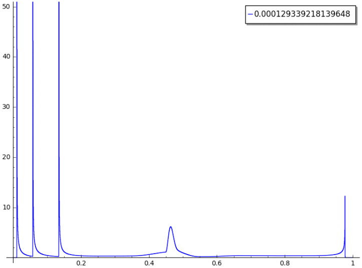




- The mathematical appoach to the problem is complicated by the structure of the deterministic part of the dynamics. In the map, strongly expanding and strongly contracting regions cohexist.
- With zero noise the map seems to have a singular invariant measure.
- The global dynamics is made in a way that with small noise, the dynamics visit the expanding part often enough to have a positive exponent.

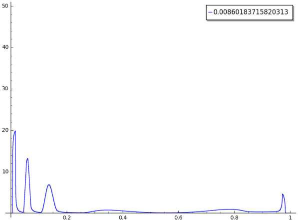
A rigorous approach, the noise

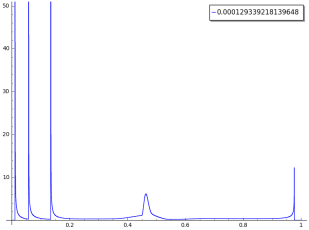




- The presence of the noise *smooth out* all the fine (singular) details in the statistical behavior of the system, allowing it to be well approximable by a finite dimensional one.
- This allow the rigorous estimate of the stationary measure with a small error in L^1, and other of its regularity features. And by this the Lyapunov exponent is easily estimated.
- We need a suitable computational framework where the numerical error can be certified (interval arithmetics e.g.), and a suitable approximation strategy.

A rigorous approach, the noise





- Naive, rigorous approximation strategies cannot work for the small noises need to be used to see the positive exponent ($\xi \sim 10^{-4}$).
- To make it, some mathematical ideas are needed, to find fastly converging approximation schemes. We will use a kind of boostrap argument.
- A fine approximation of the stationary measure is found by some preliminary information, coming from a coarse approximation of it.
- Needed information on the behavior of the (finely approximatd) transfer operator are found by a coarse approximation of it.

The transfer operator for a deterministic system

Let us consider a metric space X with a dynamics defined by $T : X \to X$. Let us also consider the space $PM(X)$ of probability measures on X. Define the function

$$L : PM(X) \to PM(X)$$

in the following way: if $\mu \in PM(X)$ then:

$$L\mu(A) = \mu(T^{-1}(A))$$

- Considering measures with sign ($SM(X)$) or complex valued measures we have a vector space and L is linear.
- Invariant measures are fixed points of the tranfer operator L.
- If you search for invariant measures having some regularity, restrict to a subspace .

The transfer operator with noise

- If after applying T we add some noise, the transfer operator is composed by a convolution.
- Let $\xi > 0$,radius
- $\rho_\xi(x) = \frac{1}{\xi}\rho(\frac{1}{\xi}x)$ noise kernel (with $\rho \in BV$ and $\int \rho = 1$)
- $N_\xi f = \rho_\xi * f$,
- Transfer operator $L_\xi : SM([0,1]) \to L^1[0,1]$ is defined by

$$L_\xi = N_\xi L.$$

The approximated transfer operator

- Space X discretized by a partition I_δ
- Let F_δ be the $\sigma-$algebra associated to I_δ, then consider

$$\pi_\delta \quad : \quad SM(X) \to L^1(X) \tag{1}$$
$$\pi_\delta(g) \quad = \quad \mathbf{E}(g|F_\delta) \tag{2}$$

- The approximated operator we are going to use is defined by a kind of Ulam discretization
- $$L_{\delta,\xi} = \pi_\delta N_\xi \pi_\delta L \pi_\delta$$
- The fixed points and other properties of L_ξ will be approximated by the ones of $L_{\delta,\xi}$.

Approximation of the fixed point

Let us suppose $f_{\delta,\xi}$ and f_{ξ} be fixed points of $L_{\delta,\xi}$ and L_{ξ}

$$\begin{aligned}
\|f_{\delta,\xi} - f_{\xi}\|_{L^1} &= \|L^n_{\delta,\xi} f_{\delta,\xi} - L^n_{\xi} f_{\xi}\|_{L^1} \\
&= \|L^n_{\delta,\xi} f_{\delta,\xi} - L^n_{\delta,\xi} f_{\xi} + L^n_{\delta,\xi} f_{\xi} - L^n_{\xi} f_{\xi}\|_{L^1} \\
&\leq \|L^n_{\delta,\xi}(f_{\delta,\xi} - f_{\xi})\|_{L^1} + \|(L^n_{\delta,\xi} - L^n_{\xi}) f_{\xi}\|_{L^1}.
\end{aligned}$$

Therefore if (we verfy computionally that)

$$||L^{\overline{n}}_{\delta,\xi}|_V||_{L^1\to L^1} \leq \alpha$$

Where $V = \{f \in BV, \int f \ dm = 0\}$ (zero avg. measures)
And...

Approximation of the fixed point

Let us suppose $f_{\delta,\xi}$ and f_ξ be fixed points of $L_{\delta,\xi}$ and L_ξ

$$\|f_{\delta,\xi} - f_\xi\|_{L^1} \leq \|L^n_{\delta,\xi}(f_{\delta,\xi} - f_\xi)\|_{L^1} + \|(L^n_{\delta,\xi} - L^n_\xi)f_\xi\|_{L^1}.$$

Therefore if (we verfy computionally that)

$$||L^{\overline{n}}_{\delta,\xi}|_V||_{L^1\to L^1} \leq \alpha$$

Where $V = \{f \in BV, \int f \; dm = 0\}$ (zero avg. measures). Then

$$\|f_{\delta,\xi} - f_\xi\|_{L^1} \leq \alpha\|f_{\delta,\xi} - f_\xi\|_{L^1} + \|(L^n_{\delta,\xi} - L^n_\xi)f_\xi\|_{L^1}.$$

and

$$\|f_\xi - f_{\delta,\xi}\|_{L^1} \leq \frac{1}{1-\alpha}\left\|(L^{\overline{n}}_{\delta,\xi} - L^{\overline{n}}_\xi)f_\xi\right\|_{L^1}.$$

(find computationally a good compromise between α and $\overline{n}$)

Approximation of the fixed point

Expanding $\left\|(L^{\overline{n}}_{\delta,\xi} - L^{\overline{n}}_{\xi})f_\xi\right\|_{L^1}$ in a telescopic sum and applying triangular ineq.:

$$\begin{aligned}\left\|(L^{\overline{n}}_{\delta,\xi} - L^{\overline{n}}_{\xi})f_\xi\right\|_{L^1} &= \left\|\left[(\pi_\delta N_\xi \pi_\delta L)^{\overline{n}}\pi_\delta - (N_\xi L)^{\overline{n}}\right]f_\xi\right\|_{L^1} \\ &\leq \|(\pi_\delta - 1)N_\xi L f_\xi\|_{L^1} + \left(\sum_{i=0}^{\overline{n}-1}\left\|L_{\delta,\xi}|_V^i\right\|_{L^1\to L^1}\right)\times \\ &\quad \times \left(\left\|N_\xi(\pi_\delta - 1)Lf_\xi\right\|_{L^1} + \left\|N_\xi \pi_\delta L(\pi_\delta - 1)f_\xi\right\|_{L^1}\right)\end{aligned}$$

Estimate for $\left\| N_\xi(\pi_\delta - 1) L f_\xi \right\|_{L^1}$, $\left\| N_\xi(\pi_\delta - 1) L f_\xi \right\|_{L^1}$, $\left\| N_\xi \pi_\delta L (\pi_\delta - 1) N_\xi L f_\xi \right\|_{L^1}$.

Lemma

We have

$$\| N_\xi (1 - \pi_\delta) \|_{L^1 \to L^1} \leq \frac{1}{2} \delta \xi^{-1} \mathrm{Var}(\rho).$$

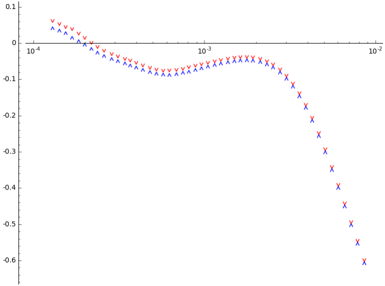
$$\| (1 - \pi_\delta) N_\xi \|_{L^1 \to L^1} \leq \frac{1}{2} \delta \xi^{-1} \mathrm{Var}(\rho).$$

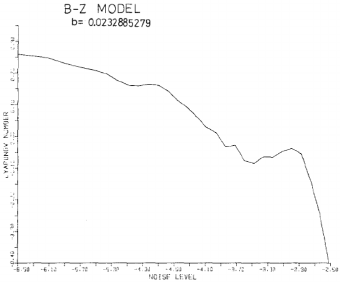
Recalling that $\|\pi_\delta\|_{L^1 \to L^1} \leq 1$, $\|N_\xi\|_{L^1 \to L^1} \leq 1$ we obtain

$$\| f_\xi - f_{\xi,\delta} \|_{L^1} \leq \frac{1 + \sum_{i=0}^{\overline{n}-1} \left\| L_{\delta,\xi} |_V^i \right\|_{L^1 \to L^1}}{2(1-\alpha)} \delta \xi^{-1} \mathrm{Var}(\rho). \tag{3}$$

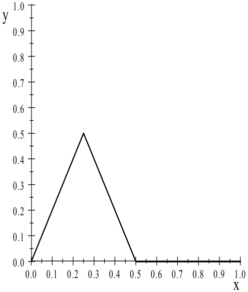
The results of the computation

$\tilde{n}$ about 50-70,
α about $0.2 - 0.5$,
δ up to 2^{-26},
ξ up to 0.129×1^{-3}.



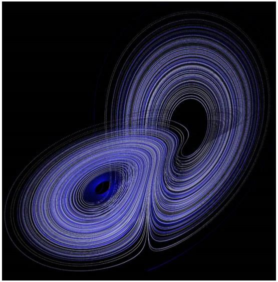


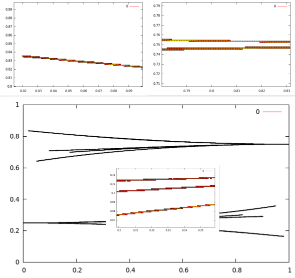
A toy model

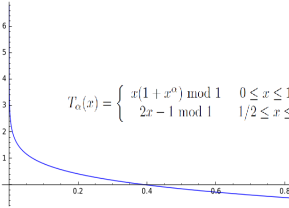


$$T(x) = \begin{cases} 2x & \text{if } x \in [0, \frac{1}{4}) \\ -2(x - \frac{1}{2}) & \text{if } x \in [\frac{1}{4}, \frac{1}{2}) \\ 0 & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

Conclusion







- In many nontrivial cases, it is hard to have quantitative estimates to important indicators regarding the statistical behavior of a system:
- entropy; dimension of attractors; Lyapunov exponents; speed of convergence to equilibrium; averages of given observables; diffusion coefficients; linear response; regularity of the invariant measure...
- Thanks to self validating numerical methods (libraries) and suitable mathematical ideas (quantitative stability estimates), the computer can help much in the mathematical understanding of the statistical properties of dynamics.

Thanks for the attention!

If I is a union of intervals of Π_δ, we have:

$$\|1-\pi_\delta\|_{\mathrm{Var}(I)\to L^1}, \quad \|1-\pi_\delta\|_{L^1(I)\to W} \leq \delta/2,$$

and

$$\|1-\pi_\delta\|_{\mathrm{Var}(I)\to W} \leq \delta^2/8,$$

while concening the noise operator

$$\|N_\xi\|_{L^1\to \mathrm{Var}}, \quad \|N_\xi\|_{W\to L^1} \leq \frac{\mathrm{Var}(\rho)}{\xi}.$$

As a consequence

$$\|(1-\pi_\delta)N_\xi\|_{L^1}, \quad \|N_\xi(1-\pi_\delta)\|_{L^1} \leq \frac{\delta \mathrm{Var}(\rho)}{2\xi},$$

and (assuming $L=1$) we would have

$$\|N_\xi(1-\pi_\delta)N_\xi\|_{L^1} \leq \frac{\delta^2 \mathrm{Var}(\rho)}{8\xi^2}.$$

Ingredient 3: local norms for L

Need to understand

$$\|N_\xi(1-\pi_\delta)LN_\xi L\tilde{f}\|_{L^1}, \quad \|N_\xi L(1-\pi_\delta)N_\xi L\tilde{f}\|_{L^1},$$

breaking $N_\xi L\tilde{f}$ along the intervals of Π_δ if necessary.
We can prove that

$$\|L\|_{W(I)\to W} \leq \|T'\|_{L^\infty(I)},$$

(spoiler: W-norm is not increased too much, unless $|T'|$ is very big) and

$$\begin{aligned}\mathrm{Var}_I(Lg) \leq \sum_i \Bigg(& \mathrm{Var}_{T_i^{-1}(I)}(g) \cdot \left\| \frac{1}{T'} \right\|_{L^\infty(T_i^{-1}(I))} \\ & + \|g\|_{L^1(T_i^{-1}(I))} \cdot \left\| \frac{T''}{T'^2} \right\|_{L^\infty(T_i^{-1}(I))} \\ & + \sum_{y\in\partial Dom(T_i):T(y)\in I} \left| \frac{g(y)}{T'(y)} \right| \Bigg)\end{aligned}$$

(spoiler: variation is not increased too much, unless $|T'|$ is close to 0).