Hitting Times and Escape Rates

Mark Demers

Fairfield University

Research supported in part by NSF Grant DMS-1362420

Nonuniformly Hyperbolic Dynamical Systems,
Coupling and Renewal Theory
CIRM, Luminy
February 22, 2017

joint work with Henk Bruin and Mike Todd

Hitting Times to Shrinking Balls

- Dynamical system $f : X \circlearrowleft$, with invariant measure μ.
- Nested sequence of subsets of X, $(U_r)_{r \geq 0}$, with $\cap_r U_r = \{z\}$.
- For $x \in X$, define the first hitting time to U_r:

$$\tau_r(x) = \inf\{n \geq 1 : f^n(x) \in U_r\}.$$

Q. How does $\mu(\tau_r > t)$ depend asymptotically on r and t?

To derive an exponential Hitting Time Statistics (HTS) law, one sets $t = \frac{s}{\mu(U_r)}$ for some $s > 0$, and considers the limit

$$\lim_{r \to 0} \mu(\tau_r > s/\mu(U_r)),$$

which in may cases for typical z, converges to e^{-s}. Can rewrite as

$$\lim_{r \to 0} -\frac{1}{s} \log \mu(\tau_r > s/\mu(U_r)) = 1.$$

HTS for (Nonuniformly) Expanding Maps

More generally, the limit depends on z: For nonperiodic z, it is 1, while for periodic z there is a correction depending on the period.

This quantity is also connected to Return Time Statistics and Extreme Value Theory, where it is called the extremal index. This has been studied in many systems, starting with [Galves, Schmitt '90].

- Nonuniformly expanding maps via inducing [Bruin, Saussol, Troubetzkoy, Vaienti '03] ,[Holland, Nicol, Török '12], , [Hayden, Winterberg, Zweimüller '14]
- Multimodal maps with a.c.i.p. [Bruin, Todd '09]
- α-mixing processes [Abadi, Saussol '11]
- Manneville-Pomeau maps [Freitas, Freitas, Todd, Vaienti '16]
- Connection with spectral perturbation [Keller '12]

Escape Rates for Open Systems

From the point of view of open systems, one fixes r and declares the set $U_r \subset X$ to be a hole: Once trajectories enter U_r, they are not allowed to exit.

For $x \in X$, define the escape time

$$e_r(x) = \inf\{n \geq 0 : f^n(x) \in U_r\}$$

The exponential escape rate from the open system is defined to be

$$-\log \lambda_r = \lim_{t \to \infty} -\frac{1}{t} \log \mu(e_r > t), \qquad \text{when the limit exists.}$$

While $e_r(x)$ and $\tau_r(x)$ are different when $x \in U_r$, there is a simple connection between them:

$$\{x \in X : \tau_r(x) = t\} = f^{-1}\big(\{x \in X : e_r(x) = t - 1\}\big)$$

Due to the invariance of μ, we have $\mu(\tau_r > t) = \mu(e_r > t - 1)$, so that the escape rate can be defined in terms of τ_r as well.

Derivative of the Escape Rate

Taking a nested sequence of sets U_r as before with $\cap_r U_r = \{z\}$, we can ask how the escape rate scales with the size of the hole,

$$\lim_{r \to 0} \frac{-\log \lambda_r}{\mu(U_r)}$$

In some cases with exponential escape rates, this limit has been shown to equal the extremal index from the HTS.

- Full one-sided shifts [Bunimovich, Yurchenko '11]
- Spectral approach applied to piecewise expanding maps [Keller, Liverani '09]
- Finite alphabets & conformal repellers [Ferguson, Pollicott '12]
- Nonuniformly expanding maps via inducing schemes [D., Todd '16], [Pollicott, Urbanski '16]

Goal of Present Project

Both HTS and escape rate asymptotic are special cases of

$$-\frac{1}{\mu(U_r)}\frac{1}{t}\log\mu(\tau_r > t).$$

- Open system: First limit $t \to \infty$, then $r \to 0$.
- HTS: Set $t = s/\mu(U_r)$ and take diagonal limit $r \to 0$.

Two Goals:

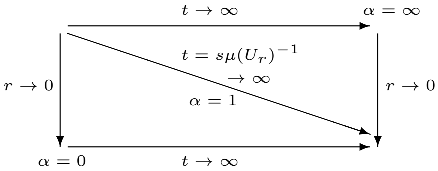
- View these expressions as two paths in a two-dimensional parameter space that allows us to move naturally between the two limits.
- Determine conditions under which the limits yield the same value and when a phase transition occurs in moving from one to the other.

Note: HTS holds much more generally than the escape rate law.

A Family of Paths in Parameter Space

For $\alpha, s \in (0, \infty)$, set $t = s\mu(U_r)^{-\alpha}$. Define

$$L_{\alpha,s}(z) := \lim_{r \to 0} \frac{-1}{s\mu(U_r)^{1-\alpha}} \log \mu(\tau_r > s\mu(U_r)^{-\alpha}).$$



- $\alpha = 1$: Diagonal limit for HTS.
- $\alpha = \infty$: Escape rate asymptotic,
$$\lim_{r \to 0} \lim_{t \to \infty} -\frac{1}{\mu(U_r)} \frac{1}{t} \log \mu(\tau_r > t).$$
- $\alpha = 0$: Reversed limits, $\lim_{t \to \infty} \lim_{r \to 0} -\frac{1}{\mu(U_r)} \frac{1}{t} \log \mu(\tau_r > t)$.
- Limit for $\alpha \neq 1$ more delicate than for $\alpha = 1$.

First Setting: Uniformly (Piecewise) Expanding Maps

Setting: $f : I \circlearrowleft$, satisfying:

- $\exists \mathcal{Z} = \{Z_i\}_i$, countable collection of intervals on which f is continuous and monotonic; set $D = I \setminus (\cup_i Z_i)$;
- $\exists$ nonatomic Borel probability measure m_φ, conformal with respect to potential φ, i.e. $\frac{dm_\varphi}{d(m_\varphi \circ f)} = e^\varphi$, and $m_\varphi(D) = 0$.
- Define transfer operator $\mathcal{L}_\varphi \psi(x) = \sum_{y \in f^{-1}x} \psi(y) e^{\varphi(y)}$.

(P1) (Bounded distortion) $\exists C_d > 0$ s.t. $|e^{S_n\varphi(x) - S_n\varphi(y)} - 1| \le C_d |f^n x - f^n y|$, whenever $f^i x$, $f^i y$ lie in same element of $\mathcal{Z}$ for all $i = 0, 1, \ldots n-1$;

(P2) $\sum_{Z \in \mathcal{Z}} \sup_Z e^\varphi < \infty$;

(P3) $\exists n_0 \in \mathbb{N}$ s.t. $\sup_I e^{S_{n_0}\varphi} < \inf_{I \setminus D} \mathcal{L}_\varphi^{n_0} 1$;

(P4) $\forall$ intervals $J \subset I \setminus D$, $\exists N$ s.t. $\inf_{I \setminus D} \mathcal{L}_\varphi^N 1_J > 0$.

φ is a contracting potential: satisfies conditions of [Rychlik '83]. (P4) is the covering property; (P1) used for perturbation argument.

Perturbations of $\mathcal{L}_\varphi$

Under (P1)-(P4), $\mathcal{L}_\varphi : BV \circlearrowright$ has a spectral gap.

$\exists$ unique invariant measure $\mu_\varphi = g m_\varphi$, $g \in BV$, $g > 0$.
[Rychlik '83], [Liverani, Saussol, Vaienti '98]

We want to consider perturbations of $\mathcal{L}_\varphi$ from the point of view of open systems.

- Fix $z \in I$, nested sequence of open sets $(U_r)_{r \geq 0}$, $\cap_r U_r = \{z\}$.
- Define punctured transfer operator

$$\mathring{\mathcal{L}}^n_{\varphi, U_r} \psi = \mathcal{L}^n_\varphi(\psi \cdot 1_{\mathring{I}^{n-1}_r})$$

  where $\mathring{I}^{n-1}_r = \cap_{i=0}^{n-1} f^{-i}(I \setminus U_r)$.
- We have $\int \mathring{\mathcal{L}}^n_{\varphi, U_r} \psi \, dm_\varphi = \int_{\mathring{I}^{n-1}} \psi \, dm_\varphi$.
- $\mathring{\mathcal{L}}_{\varphi, U_r}$ is not a small perturbation of $\mathcal{L}_\varphi$ in BV, but can be small as operator $BV \to L^1(m_\varphi)$ if we have uniform Lasota-Yorke inequalities [Keller, Liverani '99].

Perturbations of $\mathcal{L}_\varphi$: Assumptions on z

$$\text{(P3)} \implies \exists n_1 \in \mathbb{N} \text{ s.t. } (2 + 2C_d) \sup_I e^{S_{n_1}\varphi} < 1.$$

Let $\mathcal{Z}_r^n$ be the intervals of monotonicity of $f^n|_{\mathring{I}_r^{n-1}}$.

(U1) (Large images) $\exists c_0, r_0 > 0$ such that

$$\inf_{r \in [0, r_0]} \inf\{m_\varphi(f^{n_1}(J)) : J \in \mathcal{Z}_r^{n_1}\} \geq c_0.$$

(U2) If z is periodic with prime period p, assume g is continuous at z and f^p is monotonic at z.

Define $I_{cont} = \{z \in I : f^k \text{ is continuous at } z \text{ for all } k \in \mathbb{N}\}$.
Note that $m_\varphi(I_{cont}) = 1$ since $m_\varphi(D) = 0$.

Piecewise Expanding Maps: Same Law Across All Paths

Theorem 1

Suppose (f, φ) *satisfies (P1)-(P4). Let* $z \in I_{cont}$ *and* $(U_r)_{r \geq 0}$ *be a nested sequence of intervals such that* $\cap_r U_r = \{z\}$*, satisfying (U1) and in the periodic case, (U2). Then* $\forall s > 0$, $\forall \alpha \in [0, \infty]$,

$$L_{\alpha,s}(z) := \lim_{r \to 0} \frac{-1}{s\mu_\varphi(U_r)^{1-\alpha}} \log \mu_\varphi(\tau_r > s\mu_\varphi(U_r)^{-\alpha})$$

$$= \begin{cases} 1, & \text{if } z \text{ is not periodic} \\ 1 - e^{S_p \varphi(z)}, & \text{if } z \text{ has prime period } p \end{cases} .$$

Proof relies on proving that $\mathring{\mathcal{L}}_{\varphi, U_r}$ has a uniform spectral gap for r sufficiently small. The case $\alpha = \infty$ then follows by checking the conditions in [Keller, Liverani '09]. Then $L_{\alpha,s}(z)$ for $\alpha \in [0, \infty)$ uses additional estimates on the continuity of the spectral projectors of the relevant transfer operators.

Piecewise Expanding Maps: Some Examples

Ex 1: Lasota-Yorke map of the interval

- $\mathcal{Z}$ finite, $f|_Z$ satisfies $|Df| \geq \lambda > 1$ and $|D^2 f| \leq C$, for each $Z \in \mathcal{Z}$.
- $\varphi = -\log |Df|$, $m_\varphi =$ Lebesgue measure.
- Theorem 1 applies as long as we choose a sequence $(U_r)_{r \geq 0}$ satisfying (U1), (U2).

Ex 2: Gauss map, $f(x) = 1/x$ (mod 1)

- $\mathcal{Z} = \{Z_j\}_{j \geq 1}$, $Z_j = (\frac{1}{j+1}, \frac{1}{j})$.
- $\varphi = -\log |Df|$, $m_\varphi =$ Lebesgue.
- (P1) fails since distortion is only Hölder with exponent 1/2; however, the potential is monotonic on each branch and so still contracting, (P2)-(P4) still hold.
- If we choose n_1 s.t. $4|e^{S_{n_1}\varphi}|_\infty < 1$, then (U1) holds as long as z is not an endpoint of $\mathcal{Z}^{n_1}$. Theorem 1 then applies.

Piecewise Expanding Maps: Some Examples

Ex 3: Mixing Gibbs-Markov maps with large images

- $\mathcal{Z}$ is countable, but is a Markov partition for f: Each image $f(Z)$ is a union of $Z' \in \mathcal{Z}$. $|f'| \geq \lambda > 1$ on each $Z \in \mathcal{Z}$.
- (BIP) $\exists$ finite set $\{Z_j\}_{j \in \mathcal{J}} \subset \mathcal{Z}$ s.t. $\forall Z \in \mathcal{Z}$, $\exists j, k \in \mathcal{J}$ s.t. $f(Z_j) \supseteq Z$ and $f(Z) \supseteq Z_k$.
- φ is (uniformly) Lipschitz continuous on elements of $\mathcal{Z}$ and admits a nonatomic Borel probability measure m_φ with $m_\varphi(\cup_{Z \in \mathcal{Z}} Z) = 1$.
- Then (f, φ) satisfies (P1)-(P4), so Theorem 1 applies as long as we choose z satisfying (U1) and (U2).
  Notice (U1) is satisfied as long as we do not choose z to be an endpoint of $\mathcal{Z}^{n_1}$.

Second Setting: Induced Maps

Consider cases in which $f : I \circlearrowleft$ does not satisfy (P1)-(P4), but one can define an induced map F to an interval $Y \subset I$ which does.

Formal assumptions:

- For a potential φ, f admits a conformal probability measure m_φ, and a unique invariant measure μ_φ, abs. cont. w.r.t. m_φ.
- For $z \in I$ and a sequence of sets $(U_r)_{r \geq 0}$, assume we can choose $Y \subset I$ with $\mu_\varphi(Y) > 0$ and $U_r \subset Y$ such that first return map $F = f^{R_Y} : Y \circlearrowleft$ and induced potential $\Phi = \sum_{i=0}^{R_Y - 1} \varphi \circ f^i$ satisfy (P1)-(P4).

Define:

- $\mu_Y = \frac{1}{\mu_\varphi(Y)} \mu_\varphi|_Y$;
- $R_{Y,n} = \sum_{i=0}^{n-1} R_Y \circ F^i$, time of nth return to Y;
- $A_u(\varepsilon) = \left\{ y \in Y : \exists n \geq u \text{ s.t. } \left| R_{Y,n}(y) - \frac{n}{\mu_\varphi(Y)} \right| > n\varepsilon \right\}$;
- $Y_{cont} = \{ y \in Y : F^k \text{ is continuous at } y \text{ for all } k \in \mathbb{N} \}$.

Induced Maps: Exponential Large Deviations

Standing assumptions:

- $f : I \circlearrowleft$ is as above;
- $\exists Y \subset I$ and $z \in Y_{cont}$ such that $F = f^{R_Y}$ satisfies (P1)-(P4);
- the sequence of sets $(U_r)_{r \geq 0}$ satisfies (U1), and if z is periodic, (U2) as well.

Theorem 2

If for all $\varepsilon > 0$ sufficiently small, there exists $c(\varepsilon) > 0$ such that $\mu_Y(A_u(\varepsilon)) \leq e^{-c(\varepsilon)u}$ for all large u, then for all $\alpha \in [0, \infty]$,

$$L_{\alpha,s}(z) := \lim_{r \to 0} \frac{-1}{s\mu_\varphi(U_r)^{1-\alpha}} \log \mu_\varphi(\tau_r > s\mu_\varphi(U_r)^{-\alpha})$$

$$= \begin{cases} 1, & \text{if } z \text{ is not periodic} \\ 1 - e^{S_p\varphi(z)}, & \text{if } z \text{ has prime period } p \text{ (for } f\text{)} \end{cases} .$$

Theorem 3

Under the hypotheses of Theorem 2:

a) *If* $\exists \gamma \in (0,1)$ *s.t. for any small* $\varepsilon > 0$, *there exist* $C, c(\varepsilon) > 0$ *s.t.* $\mu_Y(A_u) \le Ce^{c(\varepsilon)u^\gamma}$ *for all large* u, *then for* $\alpha < \frac{1}{1-\gamma}$,

$$L_{\alpha,s}(z) = \begin{cases} 1, & \text{if } z \text{ is not periodic} \\ 1 - e^{S_p\varphi(z)}, & \text{if } z \text{ has prime period } p \end{cases} . \qquad (1)$$

b) *If* $\exists \gamma \in (0,1)$ *and* $C, c > 0$ *such that* $\mu_Y(R_Y \ge u) \ge Ce^{-cu^\gamma}$ *for all large* u, *then* $L_{\alpha,s}(z) = 0$ *for all* $\alpha > \frac{1}{1-\gamma}$.

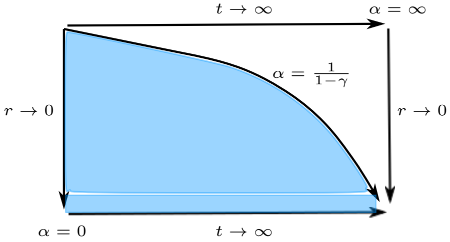
c) *If both* $\mu_Y(A_u)$ *and* $\mu_Y(R_Y \ge u)$ *decay superpolynomially in* u, *but more slowly than any stretched exponential, then* (1) *holds if* $\alpha \le 1$ *and* $L_{\alpha,s}(z) = 0$ *if* $\alpha > 1$.

Remark: Theorems 2 and 3 also hold with $\mu_Y(\tau_r > s\mu_\varphi(U_r)^{-\alpha})$ in place of $\mu_\varphi(\tau_r > s\mu_\varphi(U_r)^{-\alpha})$ in the definition of $L_{\alpha,s}(z)$.

Phase Transition in the Stretched Exponential Case

In many applications, the exponent γ governing the stretched exponential decay of $\mu_Y(R_Y \geq u)$ matches that of $\mu_Y(A_u(\varepsilon))$, so items (a) and (b) of Theorem 3 describe complementary cases.

For such maps, one has a phase transition at the path $t = s\mu_\varphi(U_r)^{-\alpha}$ when $\alpha = \frac{1}{1-\gamma}$.



- For $\alpha < \frac{1}{1-\gamma}$, $L_{\alpha,s}(z) =$ either 1 or $1 - e^{S_p\varphi(z)}$.
- For $\alpha > \frac{1}{1-\gamma}$, $L_{\alpha,s}(z) = 0$.

Applications of Inducing: Stretched Exponential

Ex 1: Generalized Farey maps

- Choose countable partition of $I = [0, 1]$, $\{A_n\}_{n \in \mathbb{N}}$, of intervals labelled in increasing order from right to left with $|A_n| = a_n$.
- Set $t_n = \sum_{k=n}^{\infty} a_k$ and for $x \in [0, 1]$, define

$$f(x) = \begin{cases} (1-x)/a_1 & \text{if } x \in A_1 \\ a_{n-1}(x - t_{n+1})/a_n + t_n & \text{if } x \in A_n,\ n \geq 2 \\ 0 & \text{if } x = 0 \end{cases}$$

  $\{A_n\}_{n \geq 1}$ is a Markov partition for f with $f(A_n) = A_{n-1}$, $n \geq 2$, and $f(A_1) = I$.
- $\varphi = -\log |Df|$, $m =$ Lebesgue is invariant measure for f

Applications of Inducing: Stretched Exponential

Ex 1: Generalized Farey maps (continued)

- F = first return map to $Y = A_1$ satisfies (P1)-(P4).
- $m(R_Y \geq u)$ is determined by $\{t_n\}_{n\in\mathbb{N}}$, which we choose to decay at a stretched exponential rate.
- F piecewise linear, so m is a Markov measure. Follows from [Gantert, Ramanan, Rembart '14] that if $m(R_Y \geq u)$ is stretched exponential with exponent $\gamma \in (0,1)$, then so is $m(A_u(\varepsilon))$ for all ε sufficiently small.
- Thus Theorem 3(a) and (b) holds for this class of maps: $L_{\alpha,s}(z)$ equals the usual HTS law for $\alpha < \frac{1}{1-\gamma}$ and $L_{\alpha,s}(z) = 0$ for $\alpha > \frac{1}{1-\gamma}$.

Applications of Inducing: Exponential Case

C^3 unimodal map $f : I \circlearrowleft$ with nonflat critical point c s.t.

- Orb$(c) = \{f^n(c) : n \geq 1\}$ is nowhere dense
- f is topologically mixing;
- f has negative Schwarzian.

Under these conditions, given $z \in I$, one can generically find an interval Y containing z such that $F = f^{R_Y}$ is Gibbs-Markov.

Ex 1: Collet-Eckmann Case

- $|Df^n(f(c))|$ grows exponentially in n;
- $\varphi_t = -t \log |Df|$: there is a unique equilbrium state μ_t for each t in a neighborhood of $[0, 1]$.
- R_Y has exponential tails and exponential large deviations w.r.t. μ_t, and so the HTS law for $L_{\alpha,s}(z)$ holds for all $\alpha \in [0, 1]$ and all t in a neighborhood of $[0, 1]$.

Applications of Inducing: Exponential Case

Ex 2: Non-Collet-Eckmann Case

- Unique equilibrium state μ_t for $\varphi_t = -t \log |Df|$, for $t \in (t_0, 1)$ and some $t_0 < 0$ [Przytycki, Rivera-Letelier '11].
- Conditions (P1)-(P4) hold for $F = f^{R_Y}$.
- R_Y has exponential tails and exponential large deviations w.r.t. μ_t, so HTS law for $L_{\alpha,s}(z)$ holds for all $\alpha \in [0, 1]$ and $t \in (t_0, 1)$.

Ex 3: Lipschitz Potentials

- φ is Lipschitz continuous and hyperbolic, i.e. $\sup_{x\in I} \frac{1}{n} S_n\varphi(x) < P(\varphi)$, where $P(\varphi)$ is variational pressure. (This follows, for example if one merely assumes $|Df^n(f(c))| \to \infty$ [Li, Rivera-Letelier '14].)
- Then $F = f^{R_Y}$ satisfies (P1)-(P4) and R_Y has exponential tails and exponential large deviations w.r.t. the unique equilibrium state μ_φ.
- The HTS law for $L_{\alpha,s}(z)$ holds for all $\alpha \in [0, 1]$.

Open Questions and Next Steps

(Q1) What about the case of polynomial tails for R_Y and polynomial large deviations?
Our proof gives that the HTS law holds for $\beta < \alpha \leq 1$ for some β depending on the polynomial rate, and for $\alpha = 0$.
Moreover, $L_{\alpha,s}(z) = 0$ for all $\alpha > 1$.
What about $\alpha \in (0, \beta)$?

(Q2) Our results for inducing schemes assume that F is a first return map. What about induced maps that are not first return maps?
[D., Todd '16] uses Young towers (not first return) to prove the case $\alpha = \infty$ for some multimodal maps and geometric potentials $\varphi_t = -t \log |Df|$, for t near 1. Would be interesting to generalize results about $L_{\alpha,s}(z)$ more fully for general inducing schemes that are not first returns.