The Dolgopyat inequality for non-Markov maps in BV.

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joint work with

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Dolgopyat inequality for the twisted transfer operator

- $F : [0,1] \to [0,1]$ is expanding non-Markov interval map;
- φ is a piecewise C^1 roof function;
- $\mathcal{L}$ is the transfer operator, with twisted version

$$\mathcal{L}_s v = \mathcal{L}(e^{s\varphi} v), \quad s = \sigma + ib.$$

Theorem: Under appropriate assumptions (to be discussed later) there exist $A, b_0 \geq 1$ and $\varepsilon, \gamma \in (0,1)$ such that

$$\|\mathcal{L}_s^n\|_b \leq \gamma^n$$

for all $|\sigma| < \varepsilon$, $|b| > b_0$ and $n \geq A \log |b|$, where $\| \; \|_b$ is a weighted version of the BV-norm.

Previous results

The tool (cancellation mechanism) comes from Chernov and Dolgopyat's work to prove exponential mixing for certain Anosov flows.

- Baladi & Vallée [2005] for general setting of suspension semiflows over p.w. C^2 Markov maps with p.w. C^1 roof.
- Avila, Gouëzel & Yoccoz [2006] for Teichmüller flows.
- Araújo & Melbourne [2015] for suspension semiflows over p.w. $C^{1+\alpha}$ Markov maps with p.w. C^1 roof (to treat the Lorenz flow).
- Eslami [2015] stretched exponential mixing for skew-products on $\mathbb{T}^2$ with non-Markov p.w. $C^{1+\alpha}$ base map and p.w. C^1 roof.
- Butterley & Eslami [2015] exponential mixing for skew-products on the torus with non-Markov base map with finitely many branches and p.w. C^2 roof.

The map F

Let $F : Y \to Y$ be an AFU map for $Y = [0, 1]$, i.e.:

- **U**niformly expanding: $|F'| \geq \rho_0 > 1$,
- **A**dler's distortion condition: $|F''|/|F'|^2$ uniformly bounded.
- possibly non-Markov, countably many branches, but with **F**inite image partition: Let α be the partition into maximal intervals of continuity. Then

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Therefore F^n has a finite image partition too, and

$$X_n = \cup\{\partial F^n a : a \in \alpha_n\}, \qquad \alpha_n = \bigvee_{i=0}^{n-1} F^{-i}\alpha$$

has cardinality $\#X_n \leq n\ \#X_1$.

Roof function φ

Let $\mathcal{H}_n$ be the collection of inverse branches of F^n.

Let $\varphi : Y \to \mathbb{R}$ be piecewise C^1 such that

- $\sup_{h\in\mathcal{H}_1} \sup_{x\in\text{dom}(h)} |(\varphi \circ h)'(x)| < \infty.$
- There is $\varepsilon_0 > 0$ such that

$$\sup_{x\in Y} \sup_{h\in\mathcal{H}_1, x\in\text{dom}(h)} |h'(x)| e^{\varepsilon_0 \varphi\circ h(x)} < \infty.$$

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This is used for "moving the contour to $\Re s > 0$" (to prove exponential mixing). Without it, one can work on imaginary axis in renewal theory context to prove polynomial mixing.

Transfer operator $\mathcal{L}$

The transfer operator associated to F is

$$\mathcal{L} : L^1(Y, \mathrm{Leb}) \to L^1(Y, \mathrm{Leb}).$$

For $s = \sigma + ib \in \mathbb{C}$, let $\mathcal{L}_s$ be the twisted version of $\mathcal{L}$:

$$\mathcal{L}_s^n v = \sum_{h \in \mathcal{H}_n} e^{s\varphi_n \circ h} |h'| v \circ h, \qquad n \geq 1,$$

for $\varphi_n = \sum_{i=0}^{n-1} \varphi \circ F^i$.

For $s = \sigma \in \mathbb{R}$, $\mathcal{L}_\sigma$ has a positive leading eigenfunction f_σ.

BV functions

Let $\mathrm{Var}_Y v$ be the total variation of $v : Y \to \mathbb{C}$.

For $b \in \mathbb{R}$ define the norm

$$\|v\|_b = \frac{1}{1+|b|}\mathrm{Var}_Y v + \|v\|_1.$$

Throughout we will work with the Banach space

$$B = \{v : Y \to \mathbb{C} : \|v\|_b < \infty\}.$$

Dolgopyat inequality

Theorem: Under the above + **additional assumptions**, including **UNI**, there exist $A, b_0 \geq 1$ and $\varepsilon, \gamma \in (0,1)$ such that

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Corollary: For every $\omega \in (0,1)$ there exists b_0 such that

$$\|(I - \mathcal{L}_s)^{-1}\|_b \leq |b|^\omega.$$

for all $|\sigma| < \varepsilon$ and $|b| > b_0$.

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3. **UNI:** For some particular constant $D > 0$, and some fixed multiple n_0 of k:

$$\forall p \in \mathcal{P}_k \; \exists h_1, h_2 \in \mathcal{H}_{n_0} \quad \inf_{x \in p} |\psi'(x)| \geq D$$

for $\psi = \varphi_{h_{n_0}} \circ h_1 - \varphi_{h_{n_0}} \circ h_2$.

Line of proof

- Analyze jump-sizes and how discontinuities are created and propagated;
- Cancellation lemma within a particular cone of pairs (u, v);
- Invariance of the cone.
- L^2 contraction in the cone.
- From outside the cone: exponential contraction to the cone
- Version of the Lasota-Yorke inequality.

Jump-sizes

The non-Markov map F generates discontinuities at certain points $x \in Y$ with jump-size defined as

$$\text{Size } v(x) := \lim_{\delta \to 0} \sup_{\xi, \xi' \in (x-\delta, x+\delta)} |v(\xi) - v(\xi')|.$$

Definition: $v : Y \to \mathbb{C}$ has exponentially decreasing jump-sizes if

$$\text{Size } v(x) \leq C_0 \rho_0^{-j/4}$$

if $x \in X_j \setminus X_{j-1}$ and v is continuous at every $x \notin \cup_j X_j$.

(Recall: $|F'| \geq \rho_0$ and C_0 is fixed in the proof.)

Jump-sizes

For λ_σ, f_σ eigenvalue resp. eigenfunction of $\mathcal{L}_\sigma$, let

$$\tilde{\mathcal{L}}_s v = \frac{1}{\lambda_\sigma f_\sigma} \mathcal{L}_s(f_\sigma v)$$

be the *normalized* version of $\mathcal{L}_s$, $s = \sigma + ib$.

Proposition: Take k large such that the additional assumptions 1 & 2 hold, and $n = 2k$. If u, v with $|v| \leq u$ have exponentially decreasing jump-sizes, then

$$\text{Size } \tilde{\mathcal{L}}_\sigma^n u(x) \ , \ \text{Size } \tilde{\mathcal{L}}_s^n v(x) \leq \frac{1}{4} \max_{p \in \mathcal{P}_k} \frac{\sup u|_p}{\inf u|_p} \ C \rho_0^{-j/4} \tilde{\mathcal{L}}_\sigma^n u(x)$$

for each $x \in X_j \setminus X_{j-1}$, $j > k$.

The cone

Define $\mathrm{Osc}_I v = \sup_{x,y\in I} |v(x) - v(y)|$ and

$$E_I(u) := \sum_{j>k} \rho_0^{-j/4} \sum_{x\in(X_j\setminus X_{j-1})\cap I^\circ} \limsup_{\xi\to x} u(\xi)$$

as intended upper bound of the sum of jumps-sizes on I.

$$\begin{aligned}Cone_b := \Big\{(u,v)\ : 0 < u\ ,\ 0 \le |v| \le u\ ,\\ &u, v \text{ have exponentially decreasing jump-sizes}\\ &\text{and } \mathrm{Osc}_I v \le C_1|b|\mathrm{Leb}(I)\sup u|_I + C_2 E_I(u)\\ &\text{for all intervals } I \subset \text{single atom of } \mathcal{P}_k\Big\}.\end{aligned}$$

(C_1 and C_2 are fixed in the proof.)

Invariance of the cone

Lemma: Assume $|b| \geq 2$, n_0 a large multiple of k. Then $Cone_b$ is invariant under

$$(u, v) \mapsto (\tilde{\mathcal{L}}_\sigma^{n_0}(\chi u), \tilde{\mathcal{L}}_s^{n_0} v),$$

where $\chi = \chi(b, u, v) \in C^1(Y, [0, 1])$ comes from the "cancellation lemma".

BV functions outside the cone.

Functions in the cone have discontinuties only in $\cup_j X_j$.
BV functions can have discontinuities at $x \notin \cup_j X_j$, but their jump-sizes descrease exponentially under iteration of $\mathcal{L}_s^{n_0}$.

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Proposition: There exists $\varepsilon \in (0,1)$ such that for all $s = \sigma + ib$, $0 \le \sigma < \varepsilon$, $|b| \ge b_0$, and all $v \in$ BV satisfying

$$\mathrm{Var}_Y v \le C_3 \, |b|^2 \, \rho_0^{mn_0/4} \, \|v\|_1,$$

there exists a pair $(u_{mn_0}, w_{mn_0}) \in Cone_b$ such that

$$\|\tilde{\mathcal{L}}_s^{mn_0} v - w_{mn_0}\|_\infty \le 2C_4 \, \rho^{-mn_0} \, |b| \, \|v\|_\infty$$

where $\|w_{mn_0}\|_\infty \le \|v\|_\infty$.

Lasota-Yorke

The spaces (BV, L^1) form an adapted pair, but for unbounded roof function φ, the operator $\mathcal{L}_s : L^1 \to L^1$ is not bounded when $\Re(s) = \sigma > 0$. Therefore, the usual Lasota-Yorke inequality fails.

Proposition: Choose k sufficiently large. Define

$$\Lambda_\sigma = \lambda_{2\sigma}^{1/2}/\lambda_\sigma \qquad \lambda_\sigma \text{ leading eigenvalue of } \mathcal{L}_\sigma.$$

Then there exist $\varepsilon > 0$ and $c > 0$ such that for all $s = \sigma + ib$ with $|\sigma| < \varepsilon$ and $b \in \mathbb{R}$,

$$\mathrm{Var}_Y(\tilde{\mathcal{L}}_s^{nk} v) \leq \rho_0^{-nk/4} \mathrm{Var}_Y v + c(1 + |b|)\Lambda_\sigma^{nk}(\|v\|_\infty \|v\|_1)^{1/2},$$

for all $v \in \mathrm{BV}(Y)$ and all $n \geq 1$.

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