α-deformations of an infinite class of continued fraction transformations

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Outline

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The groups

Fix $n \geq 3$. Let $\nu = \nu_n = 2\cos \pi/n$ and $t = 1 + \nu$.

Let G_n be generated by

$$A = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \; B = \begin{pmatrix} \nu & 1 \\ -1 & 0 \end{pmatrix}, \; C = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \qquad (1)$$

and note that $C = AB$.

The intervals

Fix $\alpha \in [0,1]$ and define

$$\mathbb{I}_\alpha := \mathbb{I}_{n,\alpha} = [\, (\alpha - 1)t, \alpha t \,) \,.$$

Of endpoints

$$\ell_0 := \ell_0(\alpha) = (\alpha - 1)t$$

and

$$r_0 := r_0(\alpha) = \alpha t$$

The maps

Let

$$T_\alpha = T_{n,\alpha} : x \mapsto A^k C^l \cdot x, \tag{2}$$

any 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on reals by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \dfrac{ax+b}{cx+d}$, and

- $l > 0$ is *minimal* such that $C^l \cdot x \notin \mathbb{I}$ Thus, rotate until exit $\mathbb{I}$.
- $k = -\lfloor (C^l \cdot x)/t + 1 - \alpha \rfloor$. Then, translate back into $\mathbb{I}$.

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traditional look

$$x = \cfrac{1}{\mu - \cfrac{1}{\mu - \ddots - \cfrac{1}{\mu - \cfrac{1}{\mu + kt - T_\alpha(x)}}}}.$$

Theorem 1

Theorem

For $n \geq 3$, the set of $\alpha \in (0,1)$ such that there exists $i = i_\alpha, j = j_\alpha$ with

$$T^i_{n,\alpha}(\, r_0(\alpha)\,) = T^j_{n,\alpha}(\, \ell_0(\alpha)\,)$$

is of full Lebesgue measure.

Call the set of these α the synchronization set for n.

Theorem 1, more precision

Theorem

For $n \geq 3$, the synchronization set is the union of intervals, $\mathscr{J}_{k,v} = [\zeta_{k,v}, \eta_{k,v})$ with $k \in \mathbb{Z} \setminus \{0\}$ and $v \in \mathcal{V}$, a tree of words defined below. The complement of the union of the $[\zeta_{k,v}, \eta_{k,v}]$ is a measure zero Cantor set.

2-D set up

- Let $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. For $M \in \mathrm{SL}_2(\mathbb{R})$ and an interval $\mathbb{I}_M$, let

$$\mathcal{T}_M(x,y) := \Big(M \cdot x, RMR^{-1} \cdot y \Big) \quad \text{for} \;\; x \in \mathbb{I}_M, \;\; y \in \mathbb{R}.$$

- Thus, $\mathcal{T}_M(x,y) = (\, M \cdot x, -1/(M \cdot (-1/y))\,)$.
- The measure μ on $\mathbb{R}^2$ given by

$$d\mu = \frac{dx\, dy}{(1+xy)^2}$$

is (locally) $\mathcal{T}_M$-invariant.

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Each T_α is piecewise Möbius — there is a partition into subintervals, $\mathbb{I}_\alpha = \cup_\beta K_\beta$, such that $T_\alpha(x) = M_\beta \cdot x$ for all $x \in K_\beta$.

For $x \in K_\beta$ and $y \in \mathbb{R}$ let

$$\mathcal{T}_\alpha(x, y) = \mathcal{T}_{M_\beta}(x, y) = \left(M_\beta \cdot x, RM_\beta R^{-1} \cdot y \right).$$

Theorem 2

For $k \in \mathbb{N}$ and $v \in \mathcal{V}$ (defined below), let $\mathscr{I}_{k,v} = [\zeta_{k,v}, \eta_{k,v})$.

Theorem

Fix $n \geq 3, k \in \mathbb{N}, v \in \mathcal{V}$ and $\alpha \in (\zeta_{k,v}, \eta_{k,v})$.

There is a connected union of finitely many rectangles $\Omega_{n,\alpha}$ upon which $\mathcal{T}_{n,\alpha}$ is bijective, up to μ-measure zero.

Furthermore, this gives the natural extension of $T_{n,\alpha}$.

Moreover, the collection of heights (top and bottoms) of the rectangles comprising $\Omega_{n,\alpha}$ depends only on (n, k, v).

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One $\Omega_{n,\alpha}$

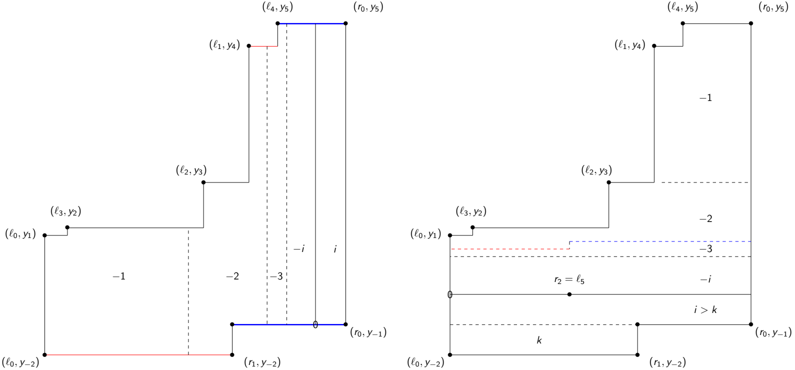

Figure : The domain $\Omega_{3,0.14}$, with blocks $\mathcal{B}_i$ (projecting to cylinders for T_α), and their images, both denoted by i. Here $R_{k,v} = AC$ and $L_{k,v} = A^{-1}CA^{-2}CA^{-2}CA^{-1}CA^{-1}$, and α is an interior point of $\mathscr{J}_{1,1}$.

Cylinders

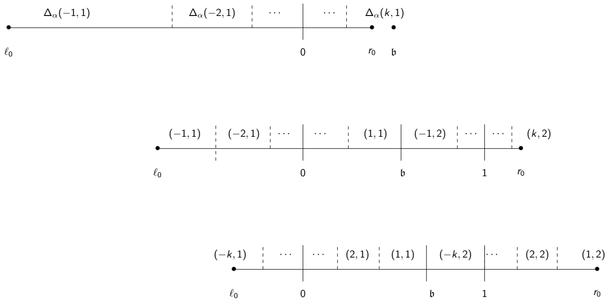

Figure : Schematic representation of cylinders for three values of α (here $n = 3$). For the bottom two, (k, l) denotes $\Delta_\alpha(k, l)$.

A skew product

Fix n. Let $\mathcal{S} = \mathcal{S}_n$ be given as

$$\mathcal{S} : \bigcup_{\alpha \in [0,1]} \{r_0(\alpha)\} \times \mathbb{I}_\alpha \to \bigcup_{\alpha \in [0,1]} \{r_0(\alpha)\} \times \mathbb{I}_\alpha$$

$$(r_0(\alpha), y) \mapsto (r_0(\alpha), T_\alpha y)$$

Recall that $r_0(\alpha) = \alpha t$ and $\ell_0(\alpha) = (\alpha - 1)t = r_0(\alpha) - t$.

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Cylinders, a global perspective

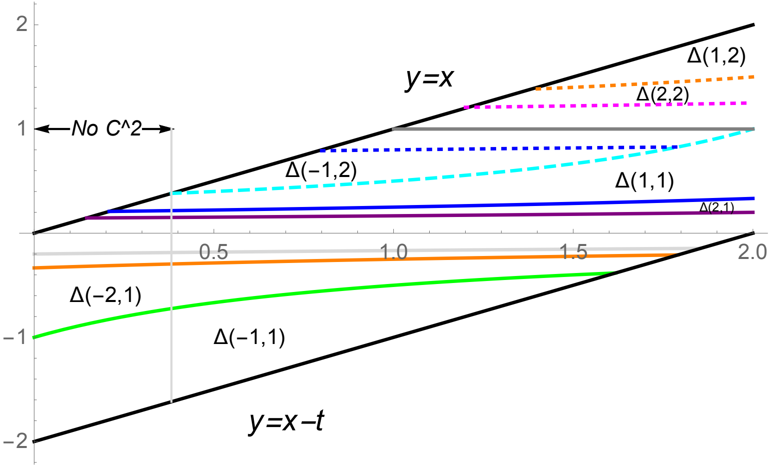

Figure : The unions of the various cylinders for the $T_{n,\alpha}$ form cylinders for $\mathcal{S}_n$. Each $\mathbb{I}_\alpha$ is given as a vertical fiber, with its left endpoint $\ell_0(\alpha) = \alpha t - t$ at the bottom and its right endpoint $r_0(\alpha) = \alpha t$ at the top. Here: $n = 3$.

Discovering synchronization

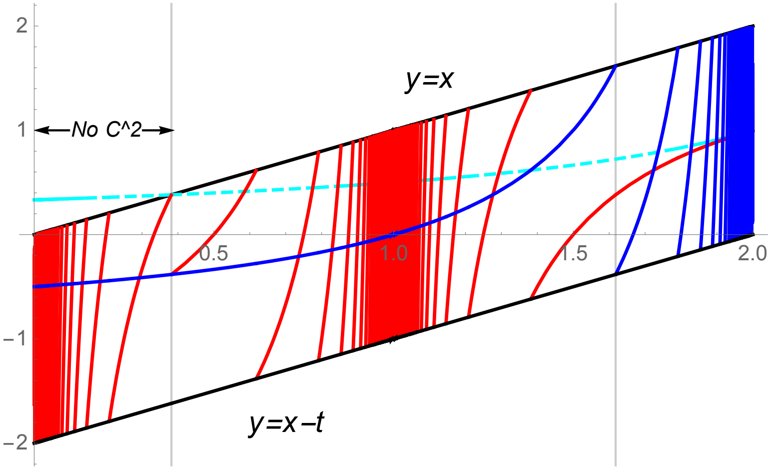

Figure : The graph of $x \mapsto T_{3,\alpha}(x - t)$, with $x = \alpha t$, thus the values of $\ell_1(\alpha)$. In red that of $x \mapsto T_{3,\alpha}(x)$; the red curves give $r_1(\alpha)$. (Here $t = t_3 = 2$.) Gray vertical lines demarcate natural partition; to left of leftmost gray vertical line "C^2 never appears."

Discovering synchronization, 2

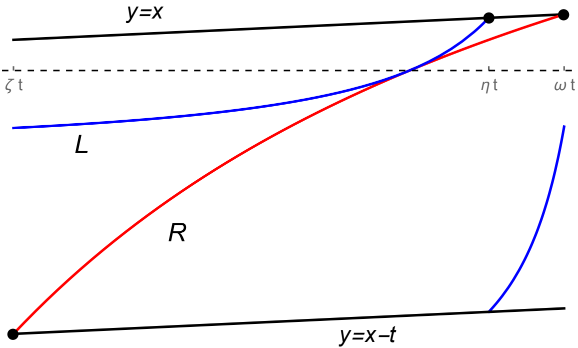

Figure : Zoom in on first red branch in the "no C^2" zone. Red gives the single branch of $y = r_1(\alpha)$ while blue colors the two branches of $y = \ell_4(\alpha)$ for x-range plotted. The x-axis is shown as a dotted line.

A synchronization relation

- In the previous figure, find

$$r_1 = \boxed{r_{j-1} = C^{-1}A^{-1}C \cdot \ell_{i-1}} = C^{-1}A^{-1}C \cdot \ell_4$$

holds for $\alpha \in \mathscr{J}_{1,1}$.

- For some u,

$$r_j = A^u C \cdot r_{j-1} = A^{u-1} C \cdot \ell_{i-1}$$

$$= \ell_i.$$

- This gives ℓ_i because we are in region of "no C^2" and there is a unique translation of $C \cdot \ell_{i-1}$ into $\mathbb{I}_\alpha$.

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Relation reveals further digits of r_0 at right endpoint

At the right endpoint of $\mathscr{I}_{k,v}$, relation gives

- $$\boxed{r_{j-1} = C^{-1}A^{-1}C \cdot r_0} \text{ or } C^{-1}AC \cdot r_{j-1} = r_0$$

- Since $r_1 = A^k C \cdot r_0$,

$$r_j = r_1$$

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Period of $r_0(\eta_{k,v})$ — only $k, k+1$ as digits

- In parameter region where "C^2 does not appear". Use simplified digits, for $\alpha \in \mathscr{J}_{k,v}$,

$$r_0(\alpha) = \underbrace{k^{c_1}, (k+1)^{d_1}, \cdots, (k+1)^{d_{s-1}}, k^{c_s}}_{d(k,v),\; v = c_1 d_1 \cdots c_{s-1} d_{s-1} c_s}, \cdots$$

- At right endpoint $\eta_{k,v}$ find periodic

$$r_0(\alpha) = d(k,v), \overline{k+1, k^{c_1 - 1}, (k+1)^{d_1}, \cdots, (k+1)^{d_{s-1}}, k^{c_s}}$$

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Word v gives v'

For each $s > 1$ and each word $v = c_1 d_1 \cdots c_{s-1} d_{s-1} c_s$, define

$$v' = \begin{cases} 1(c_1 - 1) d_1 c_2 \cdots c_{s-1} d_{s-1} c_s & \text{if} \quad c_1 \neq 1\,, \\ \\ (d_1 + 1) c_2 \cdots c_{s-1} d_{s-1} c_s & \text{otherwise}\,. \end{cases}$$

(When $v = c$ with $c > 1$ then let $v' = 1(c-1)$, and when $v = 1$ then $v' = 1$.)

Operators Θ_q

- Set
  - $\Theta_{-1}(c_1) = c_1 + 1$
  - $\Theta_q(1) = 1q1$ for $q \geq 1$
  - For $c > 1$, set $\Theta_q(c) = c[1(c-1)]^q 1c$ for any $q \geq 0$.
- Recursively ... Suppose $v = \Theta_p(u) = uv''$ for some $p \geq 0$ and some suffix v''. Then define for any $q \geq 0$

$$\Theta_q(v) = v(v')^q v''\,.$$

  This is a palindrome; it is shortest "self-dominant" word extending $v(v')^q$ which is larger than $v(v')^\infty$.
- Let $\mathcal{V}$ be the tree of all words obtained starting from $v = 1$.

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The tree $\mathcal{V}$

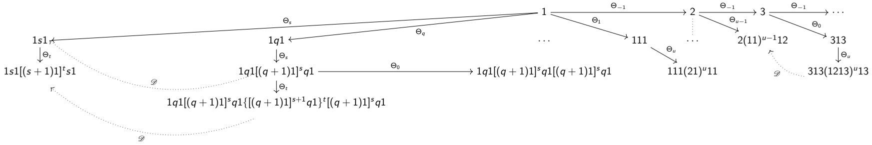

Figure : Each vertex of the directed tree $\mathcal{V}$ has countably infinite valency. A small portion of $\mathcal{V}$ with a hint of the derived words map, $\mathscr{D}$.

Partioning with the $\mathscr{J}_{k,v}$

For $k \in \mathbb{N}, v \in \mathcal{V}$, let

$$\mathscr{I}_{k,v} = \{\alpha \,|\, r_0(\alpha) \text{ has digits } d(k,v)\}$$

This is partitioned

$$\mathscr{I}_{k,v} = \mathscr{J}_{k,v} \cup \bigcup_{q=q'}^{\infty} \mathscr{I}_{k,\Theta_q(v)}\,,$$

where $q' = 0$ unless $v = c_1$, in which case $q' = -1$.

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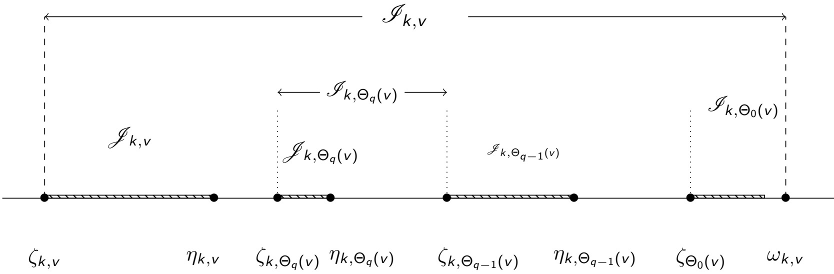

Right endpoint of α-cylinder $\mathscr{I}_{k,v}$

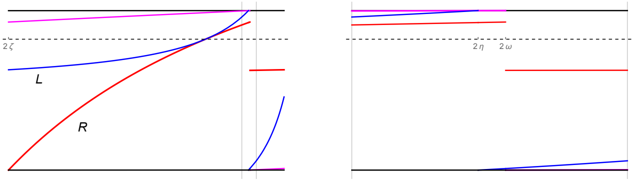

Figure : A non-full branch. Here $n = 3$, $v = 111$ and $k = 1$; we have that $\omega_{1,111}$ is determined by the fixed point of $R_{1,11}$. The labels L, R mark respectively the curves $y = L_{1,111} \cdot r_0(\alpha), y = R_{1,111} \cdot r_0(\alpha)$ where $\alpha = x/2 = x/t_{3,3}$. Red gives of $y = r_3(\alpha)$, while blue gives $y = \ell_9(\alpha)$; Magenta gives the branches of $y = r_2(\alpha)$. The left portion has $.3582 < x < 0.3592$. The right "zooms in" to $0.35910 < x < 0.35915$. (This interval lies between the vertical gray lines in both portions.)

Right endpoint of α-cylinder $\mathscr{I}_{k,v}$, 2

Order on cylinders is $k \succ k+1$, gives order on (shifts of) words: any c_j greater than any d_i, usual order of integers for c_j, reverse for d_i

Define full branch prefix $\mathfrak{f}(v)$ as longest prefix u of v such that u^∞ is maximal among all prefixes.

Find right endpoint of $\mathscr{I}_{k,v}$ has $r_0(\alpha)$ of digits $d(k, \mathfrak{f}(v))^\infty$.

One shows

$$\mathfrak{f}(\Theta_q(v)) = \overleftarrow{(\Theta_{q-1}(v))}'.$$

Can then prove partition result.

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Synchronization relation implies ℓ_0 digits $-1, -2$

Let $W = A^{-2}C\,(A^{-1}C)^{n-3}\,A^{-2}C(A^{-1}C)^{n-2}$.

Lemma (one step)

For $c, k \geq 1$,

$$(A^kC)^c = \underbrace{C^{-1}A^{-1}C}_{synchr.\ rel.}(A^{-1}C)^{n-2}\,[W^{k-1}A^{-2}C(A^{-1}C)^{n-3}]^{c-1}W^kA^{-1}\,.$$

Lemma (glueing)

For $k \geq 1$,

$$WA^{-1}\cdot A^kCA^{-1}C = A^{-2}C\,(A^{-1}C\,)^{n-3}W^k\,A^{-2}C.$$

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Synchronization relation implies ℓ_0 digits $-1, -2$

Let $W = A^{-2}C\,(A^{-1}C)^{n-3}\,A^{-2}C(A^{-1}C)^{n-2}$.

Lemma (one step)

For $c, k \geq 1$,

$$(A^kC)^c = \underbrace{C^{-1}A^{-1}C}_{\text{synchr. rel.}}(A^{-1}C)^{n-2}\,[W^{k-1}A^{-2}C(A^{-1}C)^{n-3}]^{c-1}W^kA^{-1}\,.$$

Lemma (glueing)

For $k \geq 1$,

$$WA^{-1}\cdot A^kCA^{-1}C = A^{-2}C\,(A^{-1}C\,)^{n-3}W^k\,A^{-2}C.$$

Outline of proof of Theorem 1

- Partition result holds, due to descriptions of $\zeta_{k,v}, \eta_{k,v}, \omega_{k,v}$
- Lemmas on previous slide give necessary ℓ_0 digits for synchronization on $\mathscr{J}_{k,v}$.
- Induction shows admissibility of these ℓ_0 digits. Of course, not admissible to right, but relation helps.
- Since only $-1, -2$ can use $\alpha = 0$ maps (actually with acceleration for finite measure from Calta-S), get complement of measure zero.
- Easily show no other α have synchronization (thus exact description of the complement ... follow branch of tree).

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One $\Omega_{n,\alpha}$, again

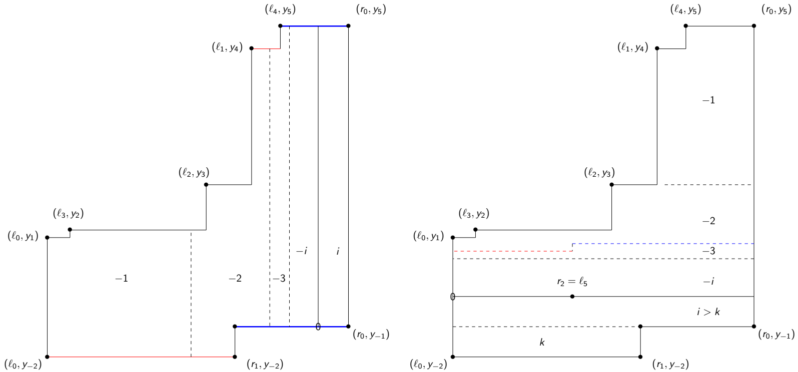

Figure : The domain $\Omega_{3,0.14}$, with blocks $\mathcal{B}_i$ (projecting to cylinders for T_α), and their images, both denoted by i. Here $R_{k,v} = AC$ and $L_{k,v} = A^{-1}CA^{-2}CA^{-2}CA^{-1}CA^{-1}$, and α is an interior point of $\mathscr{J}_{1,1}$.

A second $\Omega_{k,v}$

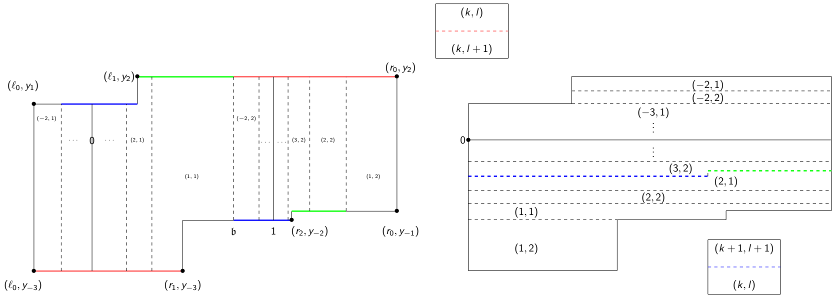

Figure : The domain $\Omega_{3,0.86}$. Blocks $\mathcal{B}_{i,j}$ and their images, both denoted by (i,j). Here $L_{-k,v} = A^{-2}CA^{-1}$ and $R_{-k,v} = AC\,AC^2$, and α is an interior point of $\mathscr{J}_{-2,1}$. Also, hints as to the lamination ordering.

Connectedness of $\Omega_{k,v}$ requires relations on heights

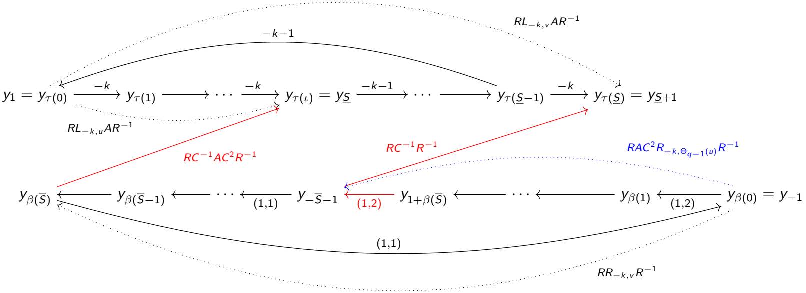

Figure : Relations on the heights of rectangles for general $-k, v$ and $\alpha \in (\eta_{-k,v}, \delta_{-k,v})$. The red paths are used to prove that lamination occurs. Horizontal arrows used to show that boundaries are sent to boundaries.

Thanks

THANK YOU!