The Weyl pseudometric and the Krieger Theorem

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The Weyl Pseudometric for $\mathbb{Z}$-action

(X, ρ)- a compact metric space such that $\mathrm{diam}_\rho(X) \leq 1$,

The Weyl Pseudometric on X^∞

$$D_W(\underline{x}, \underline{z}) = \limsup_{n\to\infty} \frac{1}{n} \sup_k \sum_{i=k}^{k+n-1} \rho(x_i, z_i).$$

The Weyl Pseudometric on X

$T\colon X \to X$ - a homeomorphism

$$D_W(x, z) = D_W(\{T^j(x)\}_{j\in\mathbb{N}}, \{T^j(z)\}_{j\in\mathbb{N}}).$$

- Introduced by Jacobs and Kanae,
- Studied by Downarowicz, Iwanik, Blanchard, Salo, Törmä and others.

More General Setting — Amenable Group Action

A dynamical system (X, G) consists of a compact metric space and an action of G on X by homeomorphisms, where G is a countable discrete amenable group.

A sequence $\{F_n\}_{n\in\mathbb{N}}$ of finite subsets of G is a (left) Følner sequence if

$$\lim_{n\to\infty} \frac{|gF_n \triangle F_n|}{|F_n|} = 0 \quad \text{for every } g \in G.$$

Example: $\mathbb{Z}$, $F_n = \{0, \ldots, n-1\}$.

We say the group G is amenable, if it admits a (left) Følner sequence.

Examples: $\mathbb{Z}^d$, every countable abelian group.

The Weyl Pseudometric for an Amenable Group Actions

The Weyl Pseudometric on X^∞

$$D_W(\underline{x}, \underline{z}) = \limsup_{n\to\infty} \frac{1}{n} \sup_k \sum_{i=k}^{k+n-1} \rho(x_i, z_i).$$

The Weyl Pseudometric on X^G

Fix any Følner sequence $\{H_n\}_{n\in\mathbb{N}}$.

$$D_W(\underline{x}, \underline{z}) = \limsup_{n\to\infty} \frac{1}{|H_n|} \left(\sup_{g\in G} \sum_{f\in H_n g} \rho(x_f, z_f) \right).$$

The Weyl Pseudometric for an Amenable Group Actions

Fix any Følner sequence $\{H_n\}_{n\in\mathbb{N}}$. Then for any $\underline{x}, \underline{z} \in X^G$ one has

$$D_W(\underline{x}, \underline{z}) = \limsup_{n\to\infty} \left(\sup_{g\in G} \frac{1}{|H_n|} \sum_{f\in H_n g} \rho(x_f, z_f) \right) =$$

$$= \sup_{\mathcal{F}} \limsup_{n\to\infty} \frac{1}{|F_n|} \sum_{f\in F_N} \rho(x_f, z_f) = \inf_{F\in Fin(G)} \frac{1}{|F|} \sup_{g\in G} \sum_{f\in F} \rho(x_{fg}, z_{fg}).$$

Moreover, D_W is uniformly equivalent to D'_W given by

$$D'_W\big(\underline{x}, \underline{z}\big) = \inf\left(\left\{\varepsilon > 0 \,:\, \limsup_{N\to\infty} \sup_{g\in G} \frac{1}{|F_N|} |\{f \in F_N g \,:\, \rho(fx, fz) > \varepsilon\}| < \varepsilon\right\}\right).$$

Entropy

For an open cover $\mathcal{U}$ of the space X denote by $\mathcal{N}(\mathcal{U})$ the minimal cardinality of a subcover of $\mathcal{U}$. The join of $\mathcal{U}$ with another open cover $\mathcal{V}$, denoted $\mathcal{U} \vee \mathcal{V}$ is given by

$$\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U},\ V \in \mathcal{V}\}.$$

Let $F = \{f_1, \ldots, f_s\} \subset G$ be a finite set. By $\mathcal{U}^F$ we understand the cover

$$\mathcal{U}^F = \bigvee_{f \in F} f^{-1}\mathcal{U} = (f_1^{-1}\mathcal{U}) \vee \ldots \vee (f_s^{-1}\mathcal{U}).$$

The topological entropy of a system (X, G) with respect to a cover $\mathcal{U}$ is given by

$$h(X, G, \mathcal{U}) := \limsup_{n \to \infty} \frac{\log \mathcal{N}\left(\mathcal{U}^{F_n}\right)}{|F_n|}.$$

The topological entropy of the action of G is defined as

$$h_{top}(X) = \sup\{h(X, G, \mathcal{U}) : \mathcal{U} \text{ is an open cover of } X\}.$$

(Semi)continuity of Entropy

Theorem

The function $(X, D_W) \ni x \to h_{top}(\overline{Gx}) \in (\mathbb{R}_+ \cup \{\infty\}, \tau)$ where τ is the natural topology is lower semicontinuous.

Theorem

Let $x \in \mathcal{A}^G$. The function $x \mapsto h_{top}(\overline{Gx})$ is continuous with respect to D_W-pseudometric on $\mathcal{A}^G$ and usual metric on $[0, \infty)$.

Besicovitch Pseudometric

The Besicovitch pseudometric on X^G

$$D_{B,\mathcal{F}}(\underline{x}_G, \underline{x}'_G) = \limsup_{N\to\infty} \frac{1}{|F_N|} \sum_{g\in F_N} \rho(x_g, x'_g).$$

The Besicovitch pseudometric on (X, G)

$$D_{B,\mathcal{F}}(x, x') = D_{B,\mathcal{F}}(\underline{x}_G, \underline{x'}_G).$$

The Connection Between Weyl and Besicovitch Pseudometric

$$D_W(\underline{x}, \underline{z}) = \sup_{\mathcal{F}} D_{B,\mathcal{F}}(\underline{x}, \underline{z}).$$

Studied by Besicovitch, Aulsander, Fomin, Oxtoby and, more recently, by Blanchard, Downarwicz, Glasner, Garcia-Ramos, Formenti, Kurka, Kwietniak, Oprocha and others...

Empirical and Distribution Measures

Given a set $F \in \mathrm{Fin}(G)$ and a sequence $\underline{x} = \{x_g\}_{g \in G}$ denote by $\mathfrak{m}(\underline{x}, F) \in \mathcal{M}(X)$ the empirical measure of x with respect to F, that is let

$$\mathfrak{m}(\underline{x}, F) = \frac{1}{|F|} \sum_{f \in F} \hat{\delta}_{x_f}.$$

A measure $\mu \in \mathcal{M}(X)$ is a distribution measure for a sequence $\underline{x} \in X^G$ if μ is a weak-* limit of some subsequence of $\{\mathfrak{m}(\underline{x}, F_n)\}_{n=1}^{\infty}$.

The set of all distribution measures of a sequence $\underline{x}$ (with respect to $\mathcal{F}$) is denoted by $\hat{\omega}_{\mathcal{F}}(\underline{x})$.

Properties of $\hat{\omega}_{\mathcal{F}}(x)$ set

- The set $\hat{\omega}_{\mathcal{F}}(x)$ is closed and non-empty.
- If $F_n \subset F_{n+1}$ and $|F_{n+1}|/|F_n| \to 1$ as $n \to \infty$, then $\hat{\omega}_{\mathcal{F}}(x)$ is connected.
- The function

$$(X^G, D_B) \ni \underline{x} \to \hat{\omega}_{\mathcal{F}}(\underline{x}) \in (2^{\mathcal{M}(X)}, H)$$

is uniformly continuous. Moreover, the modulus of continuity does not depend on the choice of the Følner sequence.

Simplices of Invariant Measures

Let $\mathcal{M}_G(\overline{Gx})$ be the simplex of G-invariant probability measures on $\overline{Gx}$.

Theorem

For every $x \in X$ one has

$$\mathcal{M}_G(\overline{Gx}) = \bigcup_{\mathcal{F}} \hat{\omega}_{\mathcal{F}}(x).$$

Corollary

For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $D_W(x, z) < \delta$, then $H(\mathcal{M}_G(\overline{Gx}), \mathcal{M}_G(\overline{Gz})) < \varepsilon$.

Residually Finite Groups

A countable group G is residually finite if there exists a nested sequence $\{H_n\}_{n\in\mathbb{N}}$ of finite index normal subgroups such that

$$\bigcap_{n=0}^{\infty} H_n = \{e\},$$

where $e \in G$ denotes the identity element.

Example: $\mathbb{Z}^d$.

Toeplitz Sequences

Let $\mathcal{A}$ be a finite set.

Toeplitz for $\mathbb{Z}$-action

A sequence $\underline{x} \in \mathcal{A}^{\mathbb{Z}}$ is called a Toeplitz sequence if for very $k \in \mathbb{Z}$ there exists $p \in \mathbb{N}$ such that

$$x_k = x_{k+jp} \text{ for all } j \in \mathbb{Z}.$$

Toeplitz Sequences for residually finite group Actions

An element $x \in \mathcal{A}^G$ is called a Toeplitz sequence if for every $g \in G$ there exists a finite index subgroup $H \subset G$ such that

$$\text{for every } \gamma \in H \text{ one has } x_{\gamma g} = x_g.$$

Toeplitz Sequences

Toeplitz Sequences for $\mathbb{Z}$-action

- Introduced by Jacobs and Kanae.
- Studied by Baake, Downarowicz, Gjerde, Iwaniik, Jaeger, Johansen, Lenz, Markley, Paul, Williams and others.
- Using them one can construct strictly ergodic systems with positive entropy or minimal systems which are not uniquely ergodic, they correspond to some class Bratteli-Vershik systems.

Toeplitz Sequences for Amenable Residually Finite Groups

- Studied by Cortez, Downarowicz, Krieger, Petit and others.
- Every metrizable Choquet simplex can be realized as a simplex of invariant measures of some Toeplitz shift.
- Krieger proved that for any number t in $[0, \log \mathrm{k})$ there exists a Toeplitz shift x over k-letter alphabet such that the entropy of x is equal to t.

Krieger's Theorem

Theorem

The family of Toeplitz sequences is pathwise connected with respect to the Weyl pseudometric.

Krieger's Theorem

Let G be a countable amenable residually finite group and $\mathcal{A}$ be a finite set. Then for every number $h \in [0, \log |\mathcal{A}|)$ there exists a Toeplitz sequence $\eta \in \mathcal{A}^G$ such that $h_{top}(\overline{G\eta}) = h$.