On the partitions with Sturmian-like refinements/On factors of Sturmian subshifts

Michal Kupsa (join work with Štěpán Starosta, published in 2015)

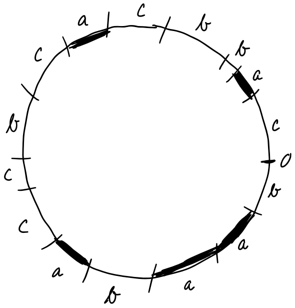
Institute of Information Theory and Automation, Czech Academy of Science, Prague, Czech Republic

Faculty of Information Technology, Czech Technical University, Prague, Czech Republic

New Advances in Symbolic Dynamics, CIRM, February 2nd, 2017

Evolution of a partition under the rotation

fixed irrational angle $\alpha \in [0,1]$, φ $(\mathbb{T}, T)$ - rotation of the unit circle by an irrational angle
$T(x) = (x + \alpha) \mod 1$



$Q = \{R_a, R_b, R_c\}$ - disconnected sets.
Q^n ... ?
disconnected ?

Evolution of a partition - notations

$\mathcal{P}$, $\mathcal{R}$ - finite partitions of $\mathbb{T}$, empty set is not allowed

$$\mathcal{P} \vee \mathcal{R} := \{P \cap R \mid P \in \mathcal{P}, R \in \mathcal{R}\} \setminus \{\emptyset\}$$
$$T^{-1}\mathcal{P} := \{T^{-1}P \mid P \in \mathcal{P}\}$$
$$\mathcal{P}^n := \mathcal{P} \vee T^{-1}\mathcal{P} \vee \ \dots \vee T^{-(n-1)}\mathcal{P}$$

If $\mathcal{P} = \{P_a \mid a \in A\}$, then

$$\mathcal{P}^n = \{P_u := \bigcap_{0 \le i < n} T^{-i} P_{u_i} \mid u = u_0 u_1 \dots u_{n-1} \in A^n\} \setminus \{\emptyset\}$$

$x \in P_u$ iff and only iff $T^i x \in P_{u_i}$ for every $i < n$.

Sturmian partition

From now on, $\mathcal{P}$ is the "Sturmian" partition -

$$\mathcal{P} = \{P_0, P_1\}, \quad \text{where } P_0 = [0, 1-\alpha), P_1 = [1-\alpha, 1).$$

The evolution of this partition is well described by the three lengths theorem.

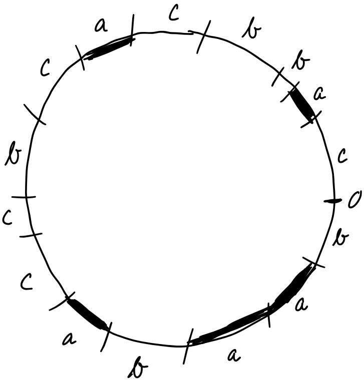
Theorem (Three lengths/Three distances theorem (Sós 1958))

For every n, $\mathcal{P}^n$ consists of $n+1$ intervals into which the points $T^{-i}0$, $0 \leq i \leq n$, divide the interval $[0, 1)$.

These intervals are of two or three lengths, where the third is the sum of other two. They goes to zero in diameter, when n goes to infinity.

The evolution of a stranger partition

How does it look like the refinements of this partition?

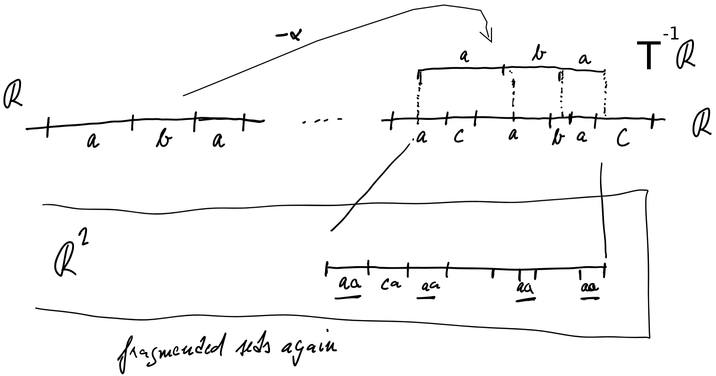


$\mathcal{R} = \{R_a, R_b, R_c\}$ - disconnected sets.

$\mathcal{R}^n$ ... ?

disconnected ?

The evolution of a stranger partition

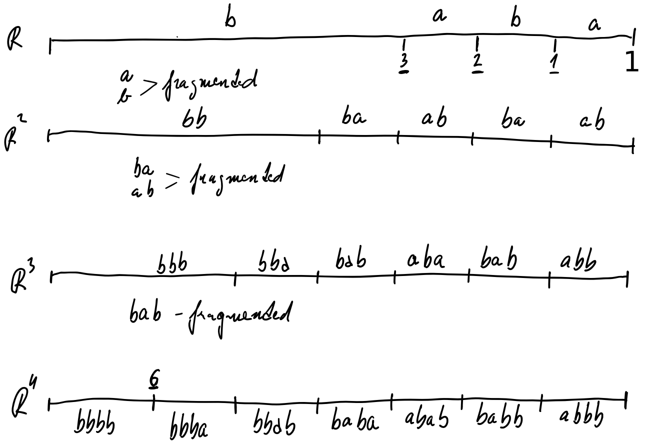


Related questions

Question

- *How do the sets in* $\mathcal{R}^n$ *looks like?*
- *Are they fragmented?*
- *At least, does their diameter go to zero?*

The evolution of a nice partition



Our result, special case

$T(x) = \{x + \alpha\}$, $P_0 = [0, 1 - \alpha)$,$P_1 = [1 - \alpha, 1)$

Theorem (K. - Starosta 2015 - special case - $\ell = 0$)

Let $\mathcal{R}$ be a non-trivial partition whose elements are unions of intervals from some $\mathcal{P}^n$ (we call these partitions Sturmian-measurable).
If 0 belongs to a boundary of a set from $\mathcal{R}$, then there exist natural numbers k and m such that

$$\mathcal{R}^k = \mathcal{P}^m$$

Our result, special case

$T(x) = \{x + \alpha\}$, $P_0 = [0, 1 - \alpha)$,$P_1 = [1 - \alpha, 1)$

Theorem (K. - Starosta 2015 - special case - $\ell = 0$)

Let $\mathcal{R}$ be a non-trivial partition whose elements are unions of intervals from some $\mathcal{P}^n$ (we call these partitions Sturmian-measurable). If 0 belongs to a boundary of a set from $\mathcal{R}$, then there exist natural numbers k and m such that

$$\mathcal{R}^{k+i} = \mathcal{P}^{m+i}, \qquad i \in \mathbb{N}.$$

Our result

$$T(x) = \{x + \alpha\},\ P_0 = [0, 1 - \alpha)\ , P_1 = [1 - \alpha, 1)$$

Theorem (K.- Starosta 2015)

Let $\mathcal{R}$ be a non-trivial partition whose elements are unions of intervals from some $\mathcal{P}^n$. Then there exist natural numbers k, ℓ and m such that $\ell < n$ and

$$\mathcal{R}^k = T^{-\ell}(\mathcal{P}^m) = (T^{-\ell}\mathcal{P})^m$$

In other words, $\mathcal{R}^k$ is the partition of the **unit circle** *that consists of the right-closed left-open intervals whose endpoints are the preimages of zero $T^{-j}(0)$, $\ell \leq j \leq \ell + m$.*

Our result

$T(x) = \{x + \alpha\}$, $P_0 = [0, 1 - \alpha)$,$P_1 = [1 - \alpha, 1)$

Theorem (K.- Starosta 2015)

Let $\mathcal{R}$ be a non-trivial partition whose elements are unions of intervals from some $\mathcal{P}^n$. Then there exist natural numbers k, ℓ and m such that $\ell < n$ and

$$\mathcal{R}^k \quad = T^{-\ell}\left(\mathcal{P}^m \quad\right) = \left(T^{-\ell}\mathcal{P}\right)^m$$

In other words, $\mathcal{R}^k$ is the partition of the **unit circle** *that consists of the right-closed left-open intervals whose endpoints are the preimages of zero $T^{-j}(0)$, $\ell \leq j \leq \ell + m$.*

$$\mathcal{R}^{k+i} = (\mathcal{R}^k)^{i+1} = ((T^{-\ell}\mathcal{P})^m)^{i+1} = (T^{-\ell}\mathcal{P})^{m+i}.$$

Our result

$T(x) = \{x + \alpha\}$, $P_0 = [0, 1 - \alpha)$,$P_1 = [1 - \alpha, 1)$

Theorem (K.- Starosta 2015)

Let $\mathcal{R}$ be a non-trivial partition whose elements are unions of intervals from some $\mathcal{P}^n$. Then there exist natural numbers k, ℓ and m such that $\ell < n$ and

$$\mathcal{R}^{k+i} = T^{-\ell}\left(\mathcal{P}^{m+i}\right) = \left(T^{-\ell}\mathcal{P}\right)^{m+i}, \qquad i \in \mathbb{N}.$$

In other words, $\mathcal{R}^k$ is the partition of the **unit circle** *that consists of the right-closed left-open intervals whose endpoints are the preimages of zero $T^{-j}(0)$, $\ell \leq j \leq \ell + m + i$.*

$$\mathcal{R}^{k+i} = (\mathcal{R}^k)^{i+1} = ((T^{-\ell}\mathcal{P})^m)^{i+1} = (T^{-\ell}\mathcal{P})^{m+i}.$$

Essential remark

Even if we started with a partition $\mathcal{R}$ that can

- have large cardinality (as big as you wish),
- consist of sets that are extremely fragmented (many interval components in each set) and spreaded around all the circle,

after a while we obtained a refinement that consists only of intervals, that are well understood by the classical theory of dynamics on the circle. In particular, these intervals are at most of three different lengths and these lengths go to zero, when m goes to infinity. We say that the partitions have *Sturmian-like* refinements.

Motivation - return times statistics

Extend some results on asymptotic behavior of evolution of the standard Sturmian partition to larger class of partitions. If possible, onto a dense class of partitions.

Sturmian-measurable partitions are dense in the space of all measurable partitions endowed by the weak* topology.

Theorem (K. - work in progress)

Let $(X, \mathcal{A}, T, \mu)$ *be the rotation of the unit circle by the angle* $\alpha = (\sqrt{5}+1)/2$. *Then*

- *for a dense set of partitions and*
- *for almost every point* $x \in X$,

the distribution functions $\tilde{F}_{n,x}$ *of rescaled return times to the cylinder* $\mathcal{R}^n(x)$ *weakly converges to a distribution function with two points of discontinuity.*

Let (X, σ) is a shift on some finite alphabet
Given a mapping $\varphi : \mathcal{L}^m(X, \sigma) \to B$, define the mappings

- $\varphi^{*n} : \mathcal{L}^{m+n-1}(X, \sigma) \to B^n$,

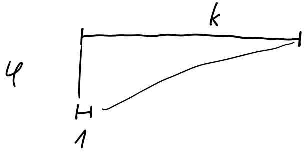
$$(\varphi^{*n}(u))_i = \varphi(u[i, i+m)), \qquad 0 \leq i < n.$$

- $\varphi^* : \mathcal{L}^\infty(X, \sigma) \to B^{\mathbb{N}}$,

$$(\varphi^*(x))_i = \varphi(x[i, i+m)), \qquad i \in \mathbb{N}.$$

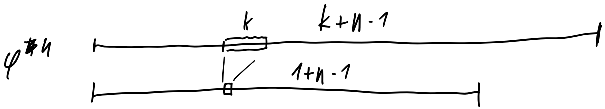
φ - local rule of width m, φ^{*n} - sliding block code of length n
φ^* - a factor mapping from (X, σ) onto $(\varphi^*(X), \sigma)$
Every factor mapping to a shift space can be constructed in this way.

Let (X, σ) is a shift on some finite alphabet



$$\varphi : \mathcal{L}^{k}(X) \to B^{1}$$

$$\varphi^{\# n} : \mathcal{L}^{n+k-1}(X) \to B^{n}$$



Factor mappings in symbolic spaces

Let (X, σ) is a shift on some finite alhpabet

Definition

A local rule $\varphi : \mathcal{L}^m(X, \sigma) \to B$ *ignores*

- *the first letter if for every* $x, y \in \mathcal{L}^m(X, \sigma)$*:*

$$x[1, m) = y[1, m) \qquad \Longrightarrow \qquad \varphi(x) = \varphi(y),$$

- *the last letter if for every* $x, y \in \mathcal{L}^m(X, \sigma)$*:*

$$x[0, m-1) = y[0, m-1) \qquad \Longrightarrow \qquad \varphi(x) = \varphi(y).$$

Theorem (K. - Starosta 2015 - sliding block code version)

If (X,σ) is a Sturmian shift and $\varphi : \mathcal{L}^m(X,\sigma) \to B$ is a non-constant local rule that does not ignore neither the first, nor the last letter, then φ^{*n} is injective for some finite n.

Corollary

If (X,σ) is a Sturmian shift and $\varphi : \mathcal{L}^m(X,\sigma) \to B$ is a local rule that ignore neither the first nor the last letter the following conditions are equivalent:

1. φ^* *is injective,*
2. φ^{*n} *is injective for some finite n.*

$2 \Longrightarrow 1$ — easy

Open problems

For which subshifts is the injectivity on the infinite level equivalent to the injectivity on the finite level?

- It is not true for fullshift
- It is not true for a Toeplitz shift with linear complexity, namely for shift that is defined by the recursive formula $u_0 = 00$, $u_{n+1} = u_n 10 u_n 11 u_n$
- It might be true for coding of rotations with respect to general two-interval partition. We do not know.

Open problems

Which other partitions of the circle do have the property that their refinements with respect to the dynamics of irrational rotation consists of connected sets?

Even for the case, when the starting partition consists of finite unions of intervals whose end-points belong to **two orbits, we have not an answer**, except for some simple situations.

How the proof should looks like:

1. Use some "elementary" facts about Sturmian shifts:

How the proof should looks like:

1. Use some "elementary" facts about Sturmian shifts:
   1. $Y := \psi^*(X)$,
      $p_n(Y) := \# \mathcal{L}^n(Y) \leq \# \mathcal{L}^{n+m-1}(X) = p_{n+m-1}(X) = n + m$
   2. If ψ^* is not constant, then (Y, σ) can not be periodic. (Spectral theory, or - Durand proved that Sturmian shifts are Cantor prime). In particular, $p_{n+1}(Y) - p_n(Y) \geq 1$.
   3. The facts above imply that there is n_0 and $c \leq m$, that for every $n \geq n_0$,
      $$p_n(Y) = n + c.$$

.

How the proof should looks like:

1. Use some "elementary" facts about Sturmian shifts:
   1. $Y := \psi^*(X)$,
      $p_n(Y) := \# \mathcal{L}^n(Y) \leq \# \mathcal{L}^{n+m-1}(X) = p_{n+m-1}(X) = n + m$
   2. If ψ^* is not constant, then (Y, σ) can not be periodic. (Spectral theory, or - Durand proved that Sturmian shifts are Cantor prime). In particular, $p_{n+1}(Y) - p_n(Y) \geq 1$.
   3. The facts above imply that there is n_0 and $c \leq m$, that for every $n \geq n_0$,
      $$p_n(Y) = n + c.$$
      .
2. It is enough to prove, that $c = m$. (Then $\psi^{*n} : \mathcal{L}^{n+m-1}(X) \to \mathcal{L}^n(Y)$ has to be injective).

How the proof should looks like:

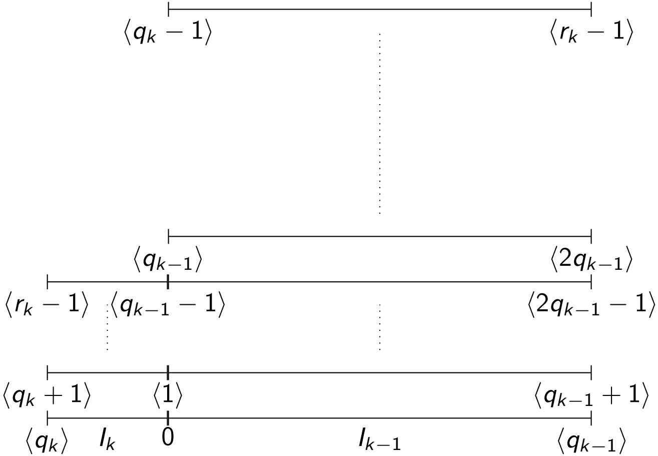
1. Use some "elementary" facts about Sturmian shifts:
   1. $Y := \psi^*(X)$,
      $p_n(Y) := \# \mathcal{L}^n(Y) \leq \# \mathcal{L}^{n+m-1}(X) = p_{n+m-1}(X) = n + m$
   2. If ψ^* is not constant, then (Y, σ) can not be periodic. (Spectral theory, or - Durand proved that Sturmian shifts are Cantor prime). In particular, $p_{n+1}(Y) - p_n(Y) \geq 1$.
   3. The facts above imply that there is n_0 and $c \leq m$, that for every $n \geq n_0$,
      $$p_n(Y) = n + c.$$
      .
2. It is enough to prove, that $c = m$. (Then $\psi^{*n} : \mathcal{L}^{n+m-1}(X) \to \mathcal{L}^n(Y)$ has to be injective).
   WE DO NOT KNOW HOW!

Inspiration

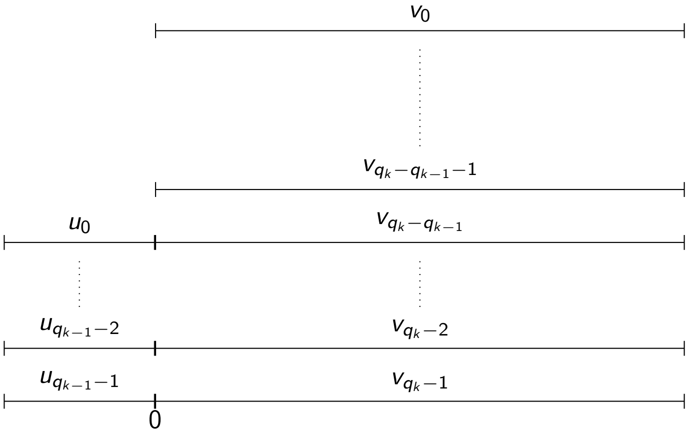
We have exchanged some few e-mails with experts from the field and they advised us where to look in, express their personnal opinion, which was always the same - IT SHOULD BE NOT DIFFICULT TO PROVE IT - using a combinatorial argument. Our shame is we have not succeded even inspired by this communication.

- Berthe, Durand, Didier, Arnoux, Downarowicz ...
- Didier - On the sliding block codes on Sturmian shifts, Evolution of inerval partitions under the rotation of the circle

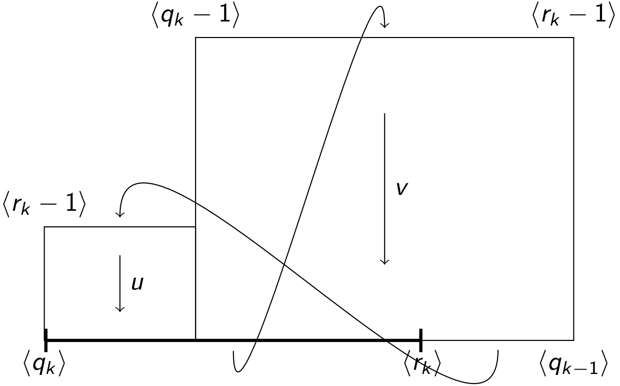
"Two-tower" $\mathcal{P}^{r_k-1}$, endpoints



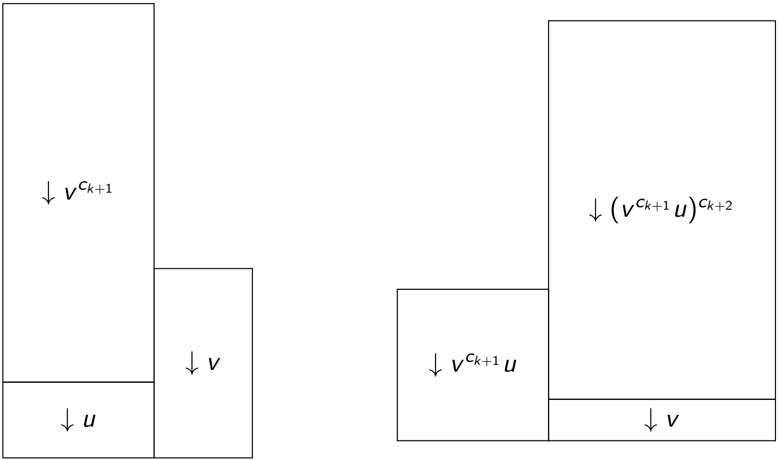
"Two-tower" $\mathcal{P}^{r_k-1}$, $\mathcal{R}$-code



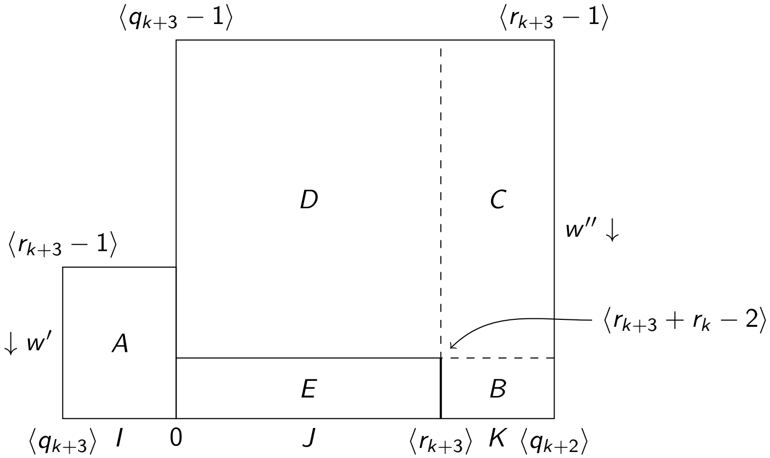
"Two-tower" $\mathcal{P}^{r_k-1}$, dynamics of T



Rokhlin towers, iterates



Important parts of $\mathcal{P}^{r_{k+3}+r_k-2}$

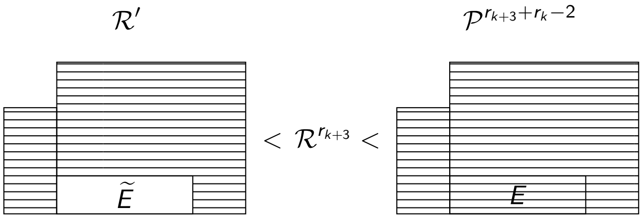


Important parts of $\mathcal{P}^{r_{k+3}+r_k-2}$

$$\begin{aligned} A =&\Lambda(I,0,q_{k+2}), & B =&\Lambda(K,0,r_k-1), & C =&\Lambda(K,r_k-1,q_{k+3}),\\ D =&\Lambda(J,r_k-1,q_{k+3}), & E =&\Lambda(J,0,r_k-1), \end{aligned}$$

where $I = I_{k+3}$, $J = I_{k+2} \setminus T^{-q_{k+2}} I_{k+3}$ and $K = T^{-q_{k+2}} I_{k+3}$. The intervals I, J and K are the bases of the towers A, E and B, respectively. For any of the towers A, B, C, D and E, denote its union using the tilde over the letter, e.g., $\widetilde{A} = \bigcup A$.

Sandwich for $\mathcal{R}'$



Lemma

Partition $\mathcal{R}^{r_{k+3}}$ is rougher than $\mathcal{P}^{r_{k+3}+r_k-2}$, but finer than $\mathcal{R}'$, where

$$\mathcal{R}' = \{\widetilde{E}\} \cup A \cup B \cup \Lambda(J \cup K, r_k - 1, q_{k+3}).$$

The upper bound comes from the inequality $\mathcal{R} < \mathcal{P}^{r_k-1}$.

It remains to prove the lower bound; points from distinct sets from $\mathcal{R}'$ have distinct $\mathcal{R}$-names of the length r_{k+3}.

Yet another r_k steps

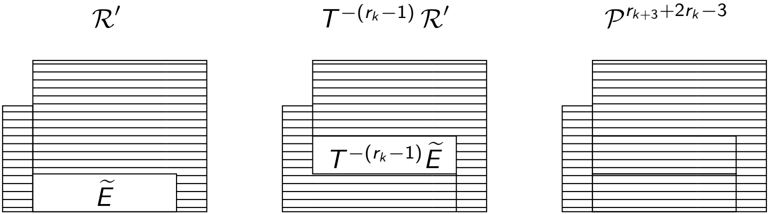


Figure: Partitions $\mathcal{R}'$, $T^{-(r_k-1)}\mathcal{R}'$ and $\mathcal{P}^{r_{k+3}+2r_k-2}$