Combinatorics on Words, Calculability, Automata
Marseille, January 30 - February 3, 2017

Ergodic invariant measures for finite rank Bratteli diagrams

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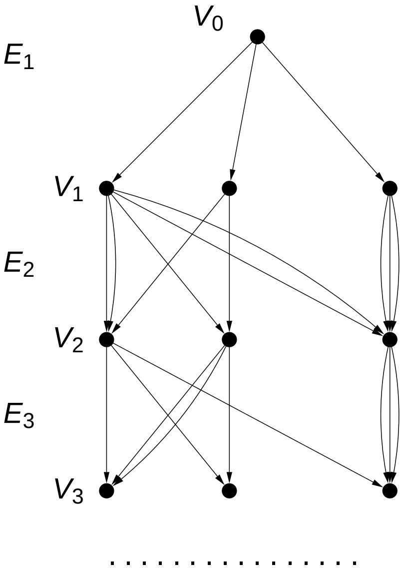
Bratteli diagrams



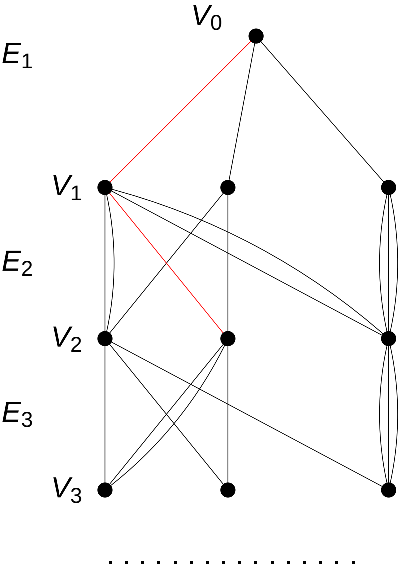
A Bratteli diagram is an infinite directed graph $B = (V, E)$:

- vertex set $V = \bigsqcup_{i \geq 0} V_i$,
- edge set $E = \bigsqcup_{i \geq 1} E_i$,
- $V_0 = \{v_0\}$ is a single point,
- V_i and E_i are finite sets,
- edges E_i connect V_{i-1} to V_i
- every $v \in V$ has an outcoming edge and every $v \in V \setminus V_0$ has and incoming edge.

V_i is called the i-th level of the diagram.

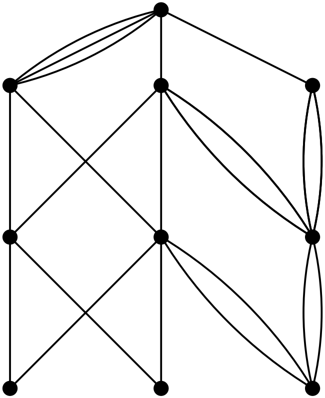
X_B is the set of all infinite paths that start at v_0.

Bratteli diagrams

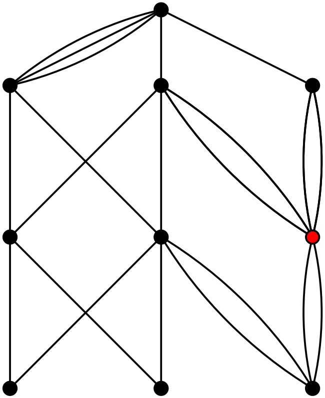


- Topology on X_B is generated by cylinder sets $[\overline{e}] \subset X_B$, where $\overline{e} = (e_1, \ldots, e_n)$ is a finite path that starts at v_0.
- Two infinite paths are close if they agree on a large initial segment.
- X_B is a zero-dimensional compact metric space. If X_B has no isolated points then X_B is a Cantor set.

Ordered Bratteli diagrams

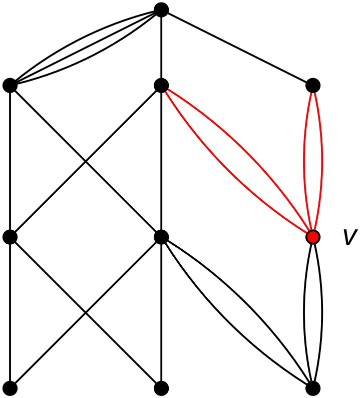


Ordered Bratteli diagrams



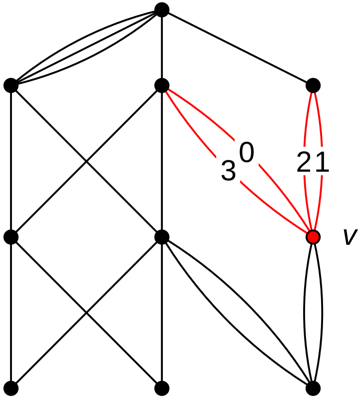
- Take a vertex $v \in V \setminus V_0$.

Ordered Bratteli diagrams



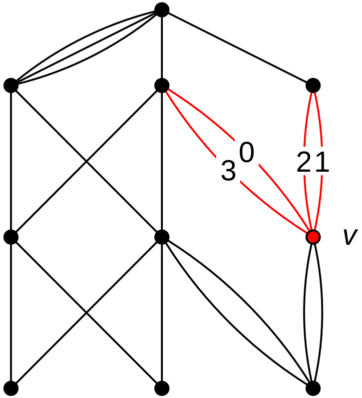
- Take a vertex $v \in V \setminus V_0$.
- Consider the set of all edges that end in v.

Ordered Bratteli diagrams



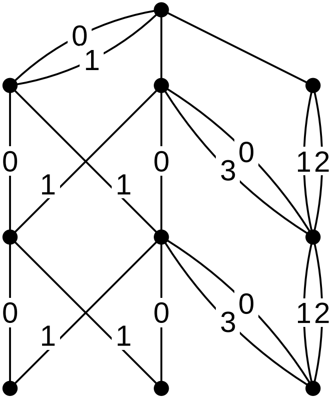
- Take a vertex $v \in V \setminus V_0$.
- Consider the set of all edges that end in v.
- Enumerate edges from this set.

Ordered Bratteli diagrams



- Take a vertex $v \in V \setminus V_0$.
- Consider the set of all edges that end in v.
- Enumerate edges from this set.
- Do the same for every vertex.

Ordered Bratteli diagrams



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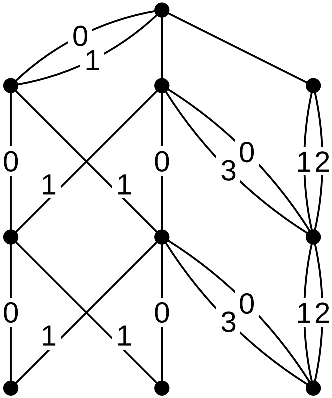
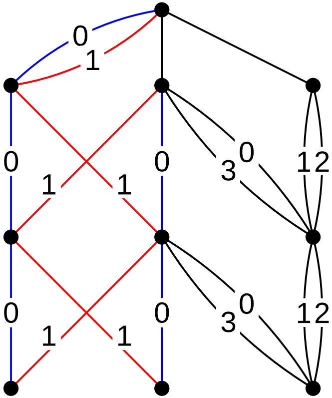


- An infinite path $x = (x_n)$ is called maximal if for every n, x_n is *maximal* among all edges that end in the same vertex as x_n.

Ordered Bratteli diagrams



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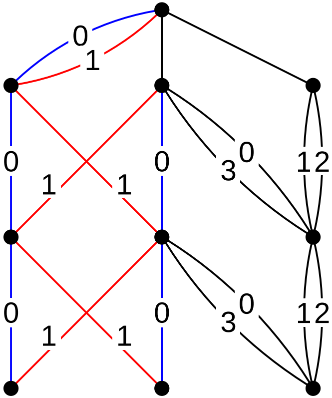
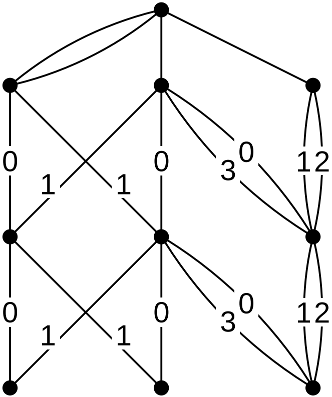
Ordered Bratteli diagrams

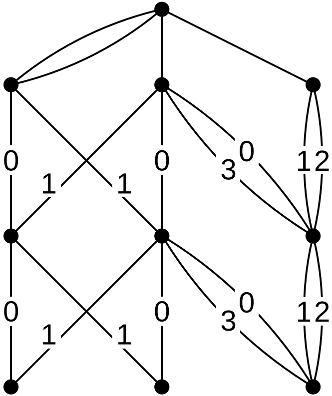


- An infinite path $x = (x_n)$ is called maximal if for every n, x_n is *maximal* among all edges that end in the same vertex as x_n.
- The sets $X_{\max}$ and $X_{\min}$ of all maximal and minimal paths are non-empty and closed.

Vershik map



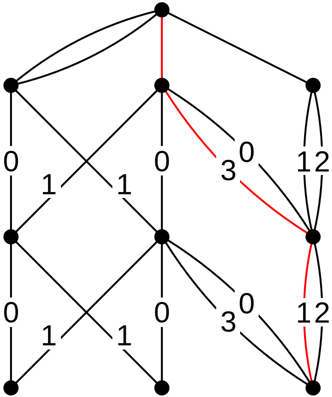
Vershik map



Define the Vershik map

$\varphi_B : X_B \setminus X_{\max} \to X_B \setminus X_{\min}$:

Vershik map

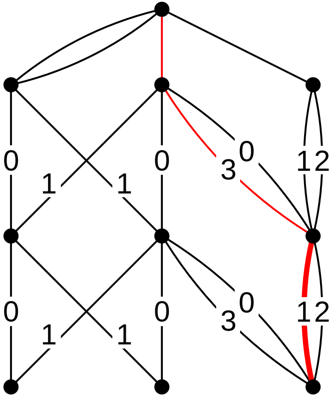


Define the Vershik map

$\varphi_B : X_B \setminus X_{\max} \to X_B \setminus X_{\min}$:

Fix $x \in X_B \setminus X_{\max}$.

Vershik map



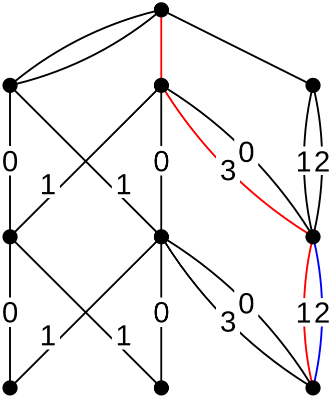
Define the Vershik map

$\varphi_B : X_B \setminus X_{\max} \to X_B \setminus X_{\min}$:

Fix $x \in X_B \setminus X_{\max}$.

Find the first k with x_k non-maximal.

Vershik map



Define the Vershik map

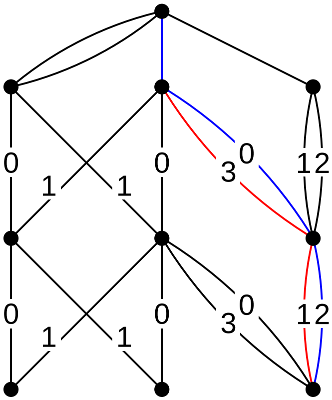
$\varphi_B : X_B \setminus X_{\max} \to X_B \setminus X_{\min}$:

Fix $x \in X_B \setminus X_{\max}$.

Find the first k with x_k non-maximal.

Take the successor $\overline{x}_k$ of x_k.

Vershik map



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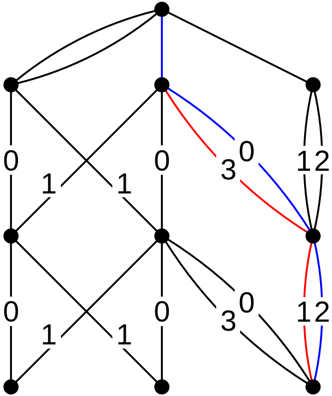
Connect $s(\overline{x}_k)$ to the top vertex V_0 by the minimal path.

Vershik map



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Connect $s(\overline{x}_k)$ to the top vertex V_0 by the minimal path.

φ_B is defined everywhere on $X_B \setminus X_{\max}$,
$\varphi_B(X_B \setminus X_{\max}) = X_B \setminus X_{\min}$

If the map φ_B can be extended to a homeomorphism of X_B such that $\varphi_B(X_{\max}) = X_{\min}$, then (X_B, φ_B) is called a Bratteli-Vershik system and φ_B is called the Vershik map.

Bratteli-Vershik models for homeomorphisms of a Cantor set

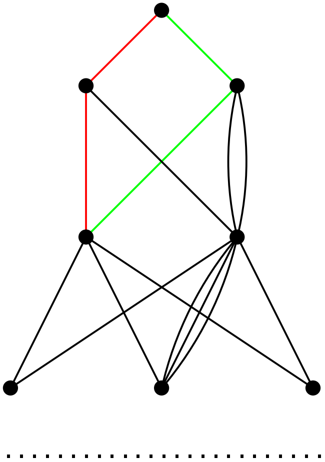
Theorem (Herman-Putnam-Skau, 1992)

Every minimal homeomorphism of a Cantor space can be represented as a Vershik map acting on a path space of an ordered Bratteli diagram, which has a unique minimal and a unique maximal paths.

Theorem (Downarowicz-K, 2017)

A (compact, invertible) zero-dimensional system (X, T) is "Bratteli-Vershikizable" (i.e. φ_B can be prolonged uniquely to X_{max}) if and only if the set of aperiodic points is dense, or its closure misses one periodic orbit.

Invariant measures on Bratteli diagrams



Two infinite paths are called tail (cofinal) equivalent if they coincide starting from some level.

We will assume that the tail equivalence relation is aperiodic.

A measure μ on X_B is called invariant if $\mu([\overline{e}]) = \mu([\overline{e}'])$ for any two cylinders $[\overline{e}]$ and $[\overline{e}']$, such that the finite paths $\overline{e}$ and $\overline{e}'$ have the same range.

An invariant measure μ is ergodic for B if it is ergodic wrt tail equivalence relation (i.e. if A is a Borel subset of X_B, $[A]$ is the set of all paths equivalent to some path in A and $A = [A]$ then $\mu(A) = 0$ or $\mu(X_B \setminus A) = 0$).

Invariant measures on Bratteli diagrams

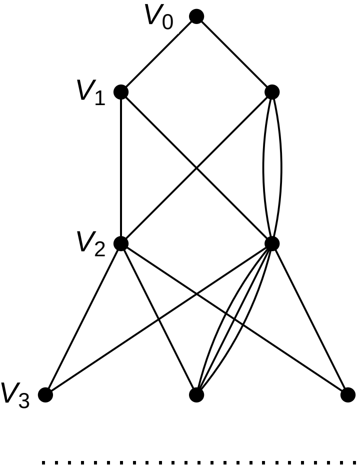
Theorem (Bezuglyi-Kwiatkowski-Medynets-Solomyak, 2010)

Let B be an ordered Bratteli diagram which admits an aperiodic Vershik map φ_B, and suppose that the tail equivalence relation $\mathcal{R}$ is aperiodic. Then the set $\mathcal{M}_1(\mathcal{R})$ of all probability $\mathcal{R}$-invariant measures coincides with the set $\mathcal{M}_1(\varphi_B)$ of all probability φ_B-invariant measures.

Theorem (Medynets, 2006)

Every Cantor aperiodic system is homeomorphic to a Vershik map acting on an ordered Bratteli diagram with aperiodic tail equivalence relation.

Incidence matrices



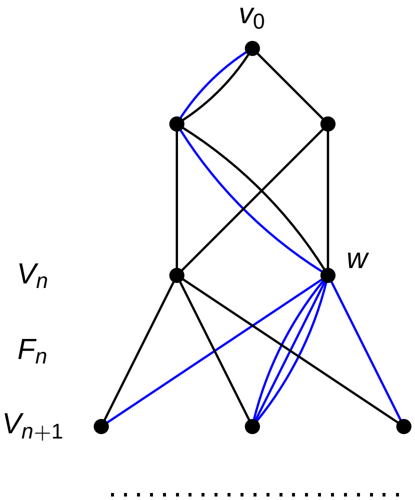
The *n*-th incidence matrix $F_n = (f^{(n)}_{v,w})$, $n \geq 0$, is a $|V_{n+1}| \times |V_n|$ matrix such that $f^{(n)}_{v,w}$ is the number of edges between $v \in V_{n+1}$ and $w \in V_n$.

$$F_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$F_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

$$F_2 = \begin{pmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 1 \end{pmatrix}.$$

Invariant measures on Bratteli diagrams



Let μ be a Borel non-atomic probability invariant measure on X_B.

Let $p_w^{(n)}$ be a measure of a cylinder set corresponding to a finite path between v_0 and $w \in V_n$.

Then

$$p_w^{(n)} = \sum_{v \in V_{n+1}} f_{v,w}^{(n)} p_v^{(n+1)}.$$

Let $p^{(n)} = (p_w^{(n)} : w \in V_n)$ be a column-vector. Then

$$p^{(n)} = F_n^T p^{(n+1)}.$$

Invariant measures on Bratteli diagrams

For any $n, m \in \mathbb{N}$

$$p^{(n)} = F_n^T F_{n+1}^T \cdots F_{n+m}^T \, p^{(n+m+1)}.$$

Let

$$C_m^{(n)} = F_n^T \cdots F_{n+m}^T(\mathbb{R}_+^{|V_{n+m+1}|})$$

Then $p^{(n)} \in C_m^{(n)}$ and

$$\mathbb{R}_+^{|V_n|} \supset C_1^{(n)} \supset C_2^{(n)} \supset \ldots$$

and

$$C_\infty^{(n)} = \cap_{m=1}^{\infty} C_m^{(n)}$$

is a closed nonempty convex subcone of $\mathbb{R}_+^{|V_n|}$ and

$$F_n^T C_\infty^{(n+1)} = C_\infty^{(n)}.$$

Invariant measures on Bratteli diagrams

Theorem (Bezuglyi-Kwiatkowski-Medynets-Solomyak, 2010)

Let B be a Bratteli diagram with incidence matrices F_n such that the tail equivalence relation is aperiodic.

Let μ be a probability invariant measure on B. Then $p^{(n)} \in C_\infty^{(n)}$ and $p^{(n)} = F_n^T p^{(n+1)}$ for all n.

Conversely, let $q^{(n)} \in \mathbb{R}_+^{|V_n|}$ be a sequence of vectors such that $q^{(n)} = F_n^T q^{(n+1)}$ for all n. Then there exists a probability invariant measure μ on B with $p^{(n)} = q^{(n)}$.

Corollary

The tail equivalence relation on a Bratteli diagram is uniquely ergodic if and only if the cone $C_\infty^{(n)}$ reduces to a single ray for all $n \geq 1$.

Stationary Bratteli diagrams

A Bratteli diagram is called stationary if $F_n = F$ for all $n \geq 1$.

Let $F^T(x) = \lambda x$, where x is a non-negative probability vector. Then the corresponding measure μ satisfies the relation:

$$p_w^{(n)} = \frac{x_i}{\lambda^{n-1}}.$$

(Recall that $p_w^{(n)}$ is a measure of a cylinder set corresponding to a finite path between v_0 and $w \in V_n$.)

Theorem (Forrest, 1997; Durand-Host-Skau, 1999)

The family of Bratteli-Vershik systems associated with stationary, properly ordered Bratteli diagrams is (up to isomorphism) the disjoint union of the family of substitution minimal systems and the family of stationary odometer systems.

Bratteli diagrams of finite rank

A Bratteli diagram $B = (V, E)$ is called of finite rank if there exists $k \in \mathbb{N}$ such that $|V_n| \leq k$ for every n.

Let B have finite rank. Then B has rank k if k is the smallest integer such that $|V_n| = k$ infinitely often.

Let B be a Bratteli diagram, and $n_0 = 0 < n_1 < n_2 < \ldots$ be a strictly increasing sequence of integers. The telescoping of B with respect to (n_j) is the Bratteli diagram B', whose incidence matrices (F'_j) are defined by $F'_j = F_{n_{j+1}-1} \circ \ldots \circ F_{n_j}$, where (F_j) are the incidence matrices for B.

After an appropriate telescoping, we can assume that the diagram B of rank k has exactly k vertices at each level (hence all incidence matrices are square $k \times k$ matrices).

Bratteli diagrams of finite rank

A Cantor dynamical system (X, S) has the topological finite rank $k > 0$ if (X, S) can be represented by a Bratteli diagram of rank k and k is the smallest such integer.

The following Cantor dynamical systems have topological finite rank:

- substitution dynamical systems (can be represented by stationary Bratteli diagrams);
- Cantor version of interval exchange transformations;
- linearly recurrent subshifts.

Open question: Which exactly classes of Cantor dynamical systems can be represented by Bratteli diagrams of finite rank?

Measures on finite rank Bratteli diagrams

Theorem ("Folklore")

A Bratteli diagram of rank k has no more than k invariant ergodic probability measures.

(Bressaud - Durand - Maass, On the eigenvalues of finite rank Bratteli-Vershik dynamical systems)

Question: Suppose B is a Bratteli diagram of rank k and $1 \leq l \leq k$. When does B have exactly l ergodic probability invariant measures? Under what conditions on the incidence matrices of B there exist exactly k ergodic measures?

Bratteli diagrams of rank 2

Theorem (Adamska-Bezuglyi-K.-Kwiatkowski, 2016)

Let B be a Bratteli diagram with 2×2 incidence matrices F_n such that

$$F_n = \begin{pmatrix} a_n & c_n \\ d_n & b_n \end{pmatrix},$$

where $a_n + c_n = d_n + b_n = r_n$ for every n (i.e. $F_n \in \mathrm{ERS}(r_n)$). Then there are two finite ergodic invariant measures if and only if

$$\sum_{n=1}^{\infty} \left(1 - \frac{|a_n - d_n|}{r_n}\right) < \infty.$$

There is a unique invariant measure μ on B if and only if

$$\sum_{n=1}^{\infty} \left(1 - \frac{|a_n - d_n|}{r_n}\right) = \infty.$$

Sketch of proof

Let μ be any probability invariant measure on B. Let $p_0^{(n)}$ and $p_1^{(n)}$ be the measures of cylinder sets of length n that end in the vertices $v_0, v_1 \in V_n$. Then for any $n \geq 1$ we have

$$\begin{cases} p_0^{(n)} = a_n p_0^{(n+1)} + d_n p_1^{(n+1)}, \\ p_1^{(n)} = c_n p_0^{(n+1)} + b_n p_1^{(n+1)}. \end{cases}$$

We have

$$p_0^{(n)} + p_1^{(n)} = \frac{1}{r_0 \ldots r_{n-1}}.$$

The measure μ is completely defined by the sequence of numbers $\{p_0^{(n)}\}$ such that $0 \leq p_0^{(n)} \leq \frac{1}{r_0 \ldots r_{n-1}}$ and

$$p_0^{(n)} = (a_n - d_n) p_0^{(n+1)} + \frac{d_n}{r_0 \ldots r_n}.$$

Sketch of proof

Denote

$$\Delta^{(n)} = \left[0, \frac{1}{r_0 \dots r_{n-1}}\right].$$

Let

$$G_n \colon \Delta^{(n+1)} \to \Delta^{(n)}$$

such that

$$G_n(y) = (a_n - d_n)y + \frac{d_n}{r_0 \dots r_n}.$$

We have

$$\Delta^{(1)} \xleftarrow{G_1} \Delta^{(2)} \xleftarrow{G_2} \Delta^{(3)} \xleftarrow{G_3} \dots$$

There is a unique invariant measure iff for infinitely many n

$$\lim_{m\to\infty} |G_n \circ \dots \circ G_{n+m}(\Delta^{(n+m+1)})| = 0.$$

Sketch of proof

Since

$$|G_n \circ \ldots \circ G_{n+m}(\Delta^{(n+m+1)})| = \frac{1}{r_0 \ldots r_{n-1}} \prod_{k=0}^{m} \frac{|a_{n+k} - d_{n+k}|}{r_{n+k}},$$

there is a unique measure μ on B if and only if

$$\prod_{k=0}^{\infty} \frac{|a_k - d_k|}{r_k} = 0.$$

Or, equivalently,

$$\sum_{k=1}^{\infty} \left(1 - \frac{|a_k - d_k|}{r_k}\right) = \infty.$$

Examples

In case when

$$F_n = \begin{pmatrix} n^2 & 1 \\ 1 & n^2 \end{pmatrix},$$

there are two finite ergodic invariant measures.

In case when

$$F_n = \begin{pmatrix} n & 1 \\ 1 & n \end{pmatrix},$$

there is a unique invariant measure which is not an extension from any odometer.

New results

Theorem (Adamska-Bezuglyi-K.-Kwiatkowski, 2016)

Let $B = (V, E)$ be a Bratteli diagram of rank k and $B \in ERS(r_n)$ such that $r_n \geq 2$ for every n. Let $F_n = (f_{v,w}^{(n)})$ be incidence matrices for B and $\det F_n \neq 0$ for every n. Let

$$z^{(n)} = \det \begin{pmatrix} \frac{f_{1,1}^{(n)}}{r_n} & \cdots & \frac{f_{1,k-1}^{(n)}}{r_n} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ \frac{f_{k,1}^{(n)}}{r_n} & \cdots & \frac{f_{k,k-1}^{(n)}}{r_n} & 1 \end{pmatrix}.$$

Then there exist exactly k ergodic invariant measures on B if and only if

$$\prod_{n=1}^{\infty} |z^{(n)}| > 0.$$

Bratteli diagrams of finite rank

Let $X_v^{(n)}$ be the set of all infinite paths in X_B that go through the vertex $v \in V_n$ (a "tower").

Theorem (Bezuglyi-Kwiatkowski-Medynets-Solomyak, 2013)

For any ergodic probability measure μ on a finite rank diagram $B = (V, E)$, there exists a subdiagram $\overline{B}$ of B defined by a sequence $W = (W_n)$, where $W_n \subset V_n$, such that $\mu(X_w^{(n)})$ is bounded from zero for all $w \in W_n$ and n.
The measure μ can be obtained as an extension of an ergodic measure on the subdiagram $\overline{B}$ (in other words, $\overline{B}$ supports μ).

Stochastic incidence matrices

Let $h_w^{(n)}$ be the number of all finite paths between v_0 and $w \in V_n$ ("height"). Then

$$h_v^{(n+1)} = \sum_{w \in V_n} f_{v,w}^{(n)} h_w^{(n)}.$$

Let $h^{(n)} = (h_w^{(n)} : w \in V_n)$ be a column-vector. Then

$$h^{(n+1)} = F_n h^{(n)}.$$

Define incidence stochastic matrices $\{\widetilde{F}_n = (\widetilde{f}_{v,w}^{(n)})\}$ of a Bratteli diagram:

$$\widetilde{f}_{w,v}^{(n)} = f_{w,v}^{(n)} \frac{h_v^{(n)}}{h_w^{(n+1)}}.$$

Theorem (Bezuglyi-K.-Kwiatkowski, 2017)

Let $B = (V, E)$ be a Bratteli diagram of rank k and $1 \leq l \leq k$. Let $\widetilde{F}_n$ be nonsingular for every n. Then B has exactly l ergodic invariant probability measures if and only if there is a telescoping of B such that for every n there exists a partition $\{V_{n,1}, \ldots, V_{n,l}, V_{n,0}\}$ of V_n and:

(a) $V_{n,i} \neq \emptyset$ *for* $i = 1, \ldots, l$;

(b) $|V_{n,i}| = |V_i|$ *for* $i = 0, 1, \ldots, l$ *and* $n \geq 1$;

(c) $\sum_{n=1}^{\infty} \left(1 - \min_{v \in V_{n+1,j}} \sum_{w \in V_{n,j}} \widetilde{f}_{vw}^{(n)}\right) < \infty$ *for* $j = 1, \ldots, l$;

(d)

$\max_{v,v' \in V_{n+1,j}} \sum_{w \in V_n} |\widetilde{f}_{vw}^{(n)} - \widetilde{f}_{v'w}^{(n)}| \to 0$ *as* $n \to \infty$ *for* $j = 1, \ldots, l$;

(e) $vol_l S(\overline{q}_1^{(n)}, \ldots, \overline{q}_l^{(n)}, \overline{f}_{v'}^{(n)}) \to 0$ *as* $n \to \infty$ *for any* $v' \in V_{n+1,0}$, *where* $\overline{q}_j^{(n)} = \frac{1}{|V_{n+1,j}|} \sum_{v \in V_{n+1,j}} \overline{f}_v^{(n)}$; *and* $\overline{f}_v^{(n)} = (\widetilde{f}_{v,w}^{(n)})_{w \in V_n}$ *and* $S(\overline{a}_1, \ldots, \overline{a}_{l+1})$ *denotes a simplex with vertices* $(\overline{a}_i)_{i=1}^{l+1}$;