Entropies of mixing subshifts

Silvère Gangloff, Benjamin Hellouin de Menibus, Cristóbal Rojas.

ATER, Université Paris 7 Denis Diderot

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Outline

1. Motivations, definitions, examples
2. State of the art: subshifts of finite type
3. Our work: more general subshifts

John Milnor, 2002:

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- cellular automata
- Turing machines
- C^∞ interval maps
- smooth diffeomorphisms ($d > 2$)

Computable: (hypotheses missing)

- positively expansive CA
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Two related questions

For a class of systems,

- How hard is computing the topological entropy?
- What are the possible values of the topological entropy?

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$\mathcal{A}^*$ the (finite) **patterns**;

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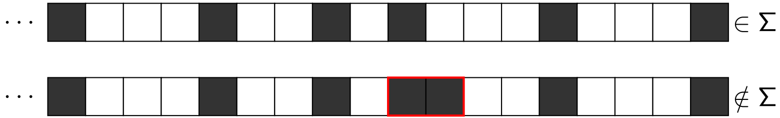
Example.

$d = 1,\ \mathcal{F} = \{11\} = \{\blacksquare\blacksquare\}$



$\mathcal{L}_n(\Sigma)$: square patterns of side length n appearing in Σ.

Entropy

$$\frac{\log \#\mathcal{L}_n(\Sigma)}{n^d} \searrow h_{top}(\Sigma).$$

Some more definitions

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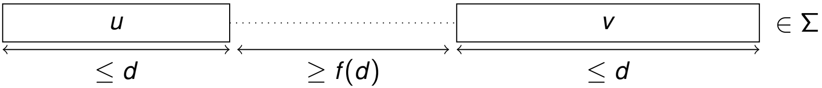

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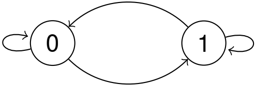

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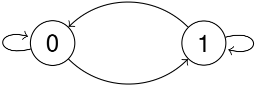

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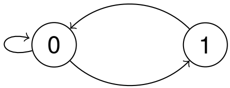

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad \#\mathcal{L}_n(\Sigma) \simeq \sum (M^n)_{i,j} \simeq \varphi^n \qquad h_{top}(\Sigma) = \log \varphi$$

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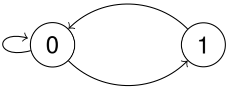

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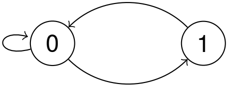

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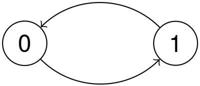

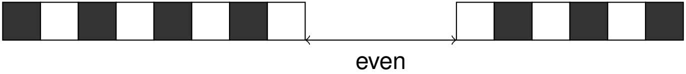

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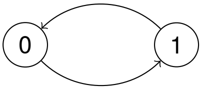

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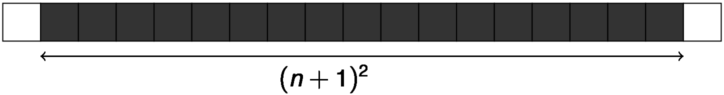

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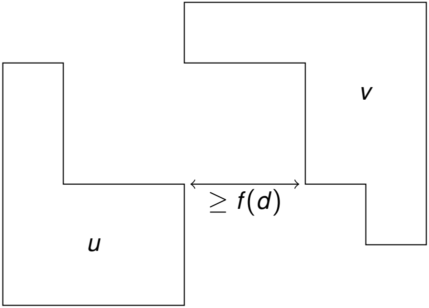
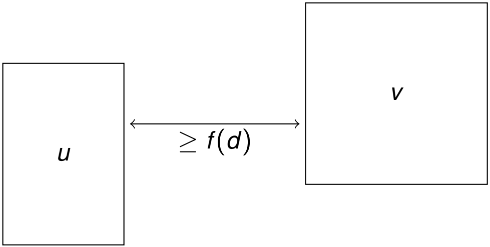


Block gluing



Computability of real numbers

A real number α is **computable** if there is a computable function $n \mapsto \alpha_n$ such that

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α is **upper-semi-computable** if instead:

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Computing the entropy

If we can compute (or approximate from above) the value of $\#\mathcal{L}_n(\Sigma)$, then $h_{top}(\Sigma)$ is **upper-semi-computable**.

Proof:

$$\frac{\log \#\mathcal{L}_n(\Sigma)}{n^d} \searrow h_{top}(\Sigma).$$

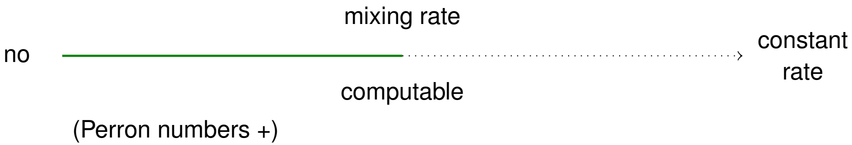
The entropy of **1D SFT** is computable (computing the dominant eigenvalue of a matrix).

One-dimensional subshifts of finite type

The entropy of **1D SFT** is computable (computing the dominant eigenvalue of a matrix).

Lind 1974

Entropies of 1D SFT are exactly reals of the form $q \log p$, where $q \in \mathbb{Q}^+$ and p is a Perron number.



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Entropies of $O(1)$-mixing 1D SFT are exactly reals of the form $q \log p$, where $q = 1$ and p is a Perron number.

mixing rate

no → constant rate

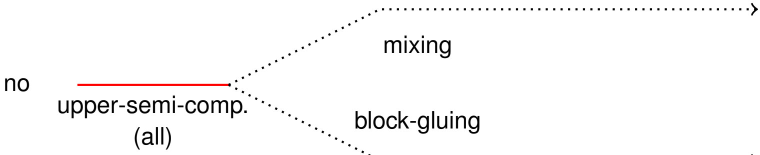
computable

(Perron numbers +) (Perron numbers)

Hochman and Meyerovitch 2007

Entropies of **dD SFT** are the **upper-semi-computable** real numbers.

In particular, entropy of dD SFT is **not computable**.



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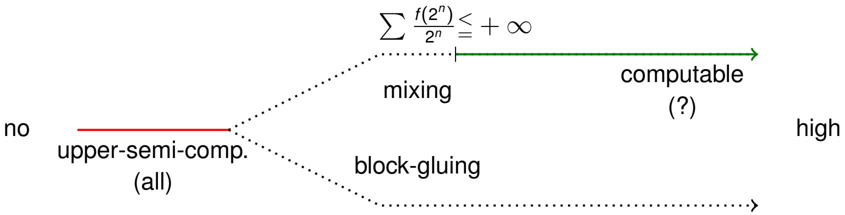

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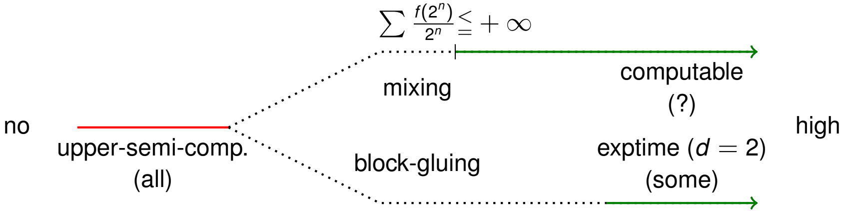
Pavlov and Schraudner, 2015

Entropy of block-gluing 2D SFT is exptime-computable, and there is a partial characterisation.



Gangloff, H., Rojas

Any real number is the entropy of some $O(1)$-mixing subshift.

	mixing / block-gluing	
no	$\longrightarrow$	constant rate
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Compare Grillenberger 73.

General subshifts

Gangloff, H., Rojas

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mixing / block-gluing

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Where did the computability go?

Deciding the language

Input $w \in \mathcal{A}^*$

Output $w \in \mathcal{L}(\Sigma)$?

We consider subshifts with **decidable language**.

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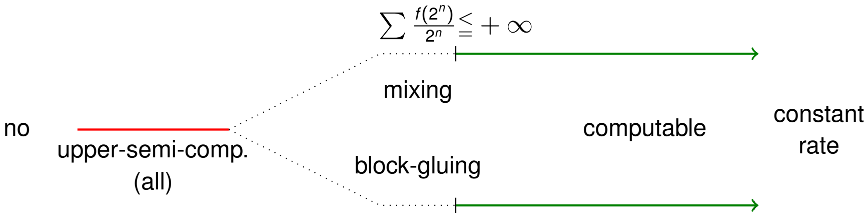
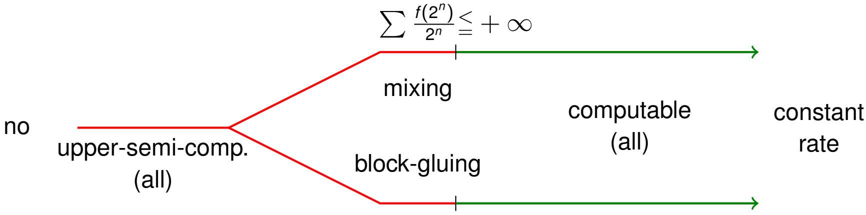


Simonsen 06, Hertling and Spandl 07

A threshold

Gangloff, H., Rojas

Assume that $\sum \frac{f(2^n)}{2^n} \substack{< \\ =} +\infty$. The entropies of $O(f)$-mixing subshifts with decidable language are the computable / upper-semi-computable real numbers.



Conclusion

- Fixing the computational complexity of the language unveils the effect of mixing properties;
- Can we complete the SFT picture ($d \geq 2$) ?