An Urysohn-type theorem under a dynamical constraint

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The positive answer to the question is contained in our work

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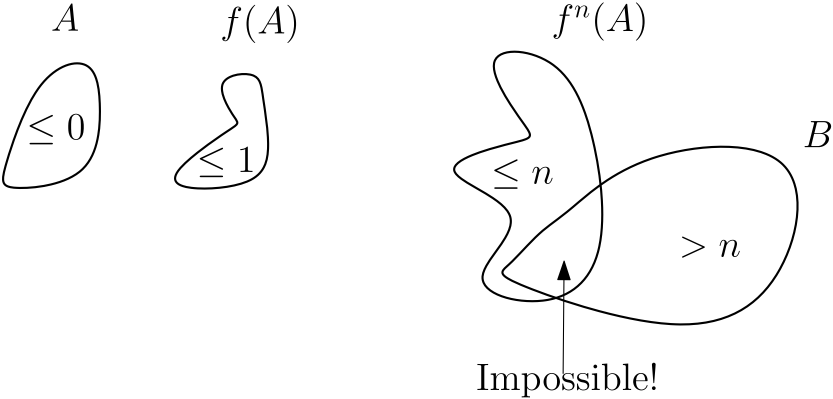

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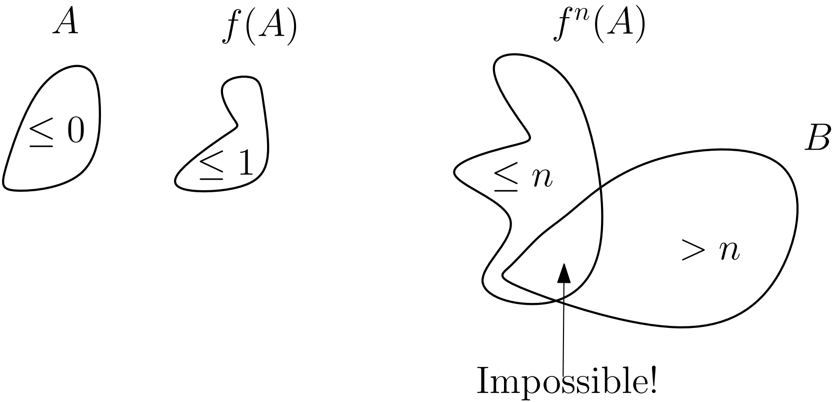

Therefore the condition $B \cap (\cup_{i=0}^{n} f^i(A)) = \emptyset$ is necessary to prove the existence of θ satisfying

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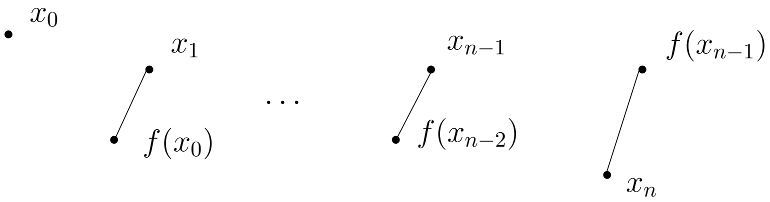
$$A(x_0, \dots, x_n) = \sum_{i=0}^{n-1} d(f(x_i), x_{i+1}) \geq 0.$$



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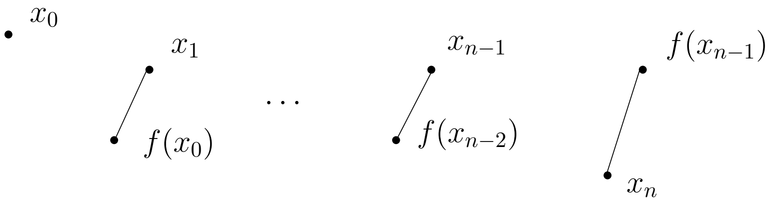

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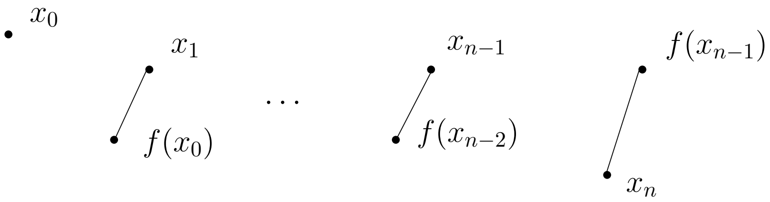

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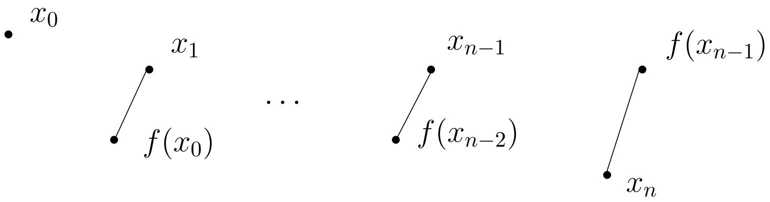

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In fact, as we will see, using $p = 1$, allows to obtain (uniformly) Lipschitz function.

We used the action A to study Lyapunov functions, i.e. functions $\psi : X \to \mathbb{R}$ such that $\psi f \leq \psi$, or equivalently $\psi f - \psi \leq 0$. Since we want instead the condition $\psi f - \psi \leq 1$, we have to modify our action by throwing in the constant potential -1.
For every $k > 0$, we define the cost $c_k : X \times X \to \mathbb{R}$ by

$$c_k(x, y) = kd(f(x), y) + 1.$$

This is to be compared with the Lagrangian associated to the motion of a particule of mass m in a potential field with a potential energy V

$$L(x, v) = \frac{m}{2}\|v\|^2 - V(x).$$

Of course, a discrete speed at the point x is an ordered pair (x, y) $(y = x + v!)$.
If we compare, $k/2$ is therefore the mass. By increasing k, we are making the particle heavier without changing the potential energy $V = -1$

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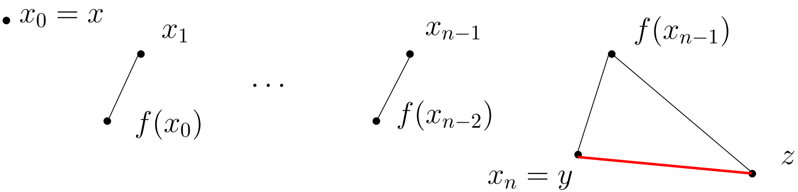
If $(x_0, \ldots, x_n)$ is a chain with $x_0 = x, x_n = y$, we define the chain $(x_0, x_1, \ldots, x_{n-1}, z)$ joining x to z.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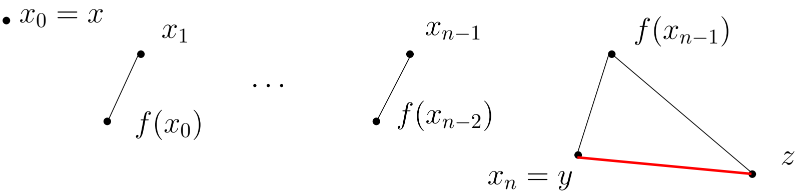

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$$
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Theorem

Assume

- $f : X \to X$ *is a continuous self-map of the metric space* X.
- $A, B \subset X$ *are closed subsets, with* A *compact,*
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Then we can find a Lipschitz function $\theta : X \to [0, +\infty[$ *such that*

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- $B \cap (\cup_{i=0}^{n} f^i(A)) = \emptyset$*, where* $n \geq 0$*.*

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Proof of Theorem

For a subset $S \subset X$, if $\epsilon > 0$, we denote by

$$\bar{V}_\epsilon(S) = \{x \in X \mid d(x, S) \leq \epsilon\}$$

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Note that $\varphi_k | \bar{V}_{1/k}(A) \equiv 0$, and the function φ_k is k-Lipschitz.It is not difficult to estimate from below the values of φ_k on $\bar{V}_{1/k}(B)$ by

$$\varphi_k | \bar{V}_{1/k}(B) \geq kd(A, B) - 2, \tag{0.1}$$

using $d(\bar{V}_{1/k}(A), \bar{V}_{1/k}(B)) \geq d(A, B) - 2/k$.
Since A is compact and B is closed, we have $d(A, B) > 0$. Hence

$$\inf_{\bar{V}_{1/k}(B)} \varphi_k \to +\infty, \text{ as } k \to +\infty.$$

We next define $\theta_k : X \to [0, +\infty[$ by

$$\theta_k(x) = \min[\varphi_k(x), \inf_{y \in X} \varphi_k(y) + \Gamma_k(y, x)].$$

The second part is indeed an "average" in the $(\min, +)$ algebra. In the usual algebra $(+, \times)$, since an infinite (uncountable sum) should be an integral, this "average" would be

$$\int \varphi_k(y) \Gamma_k(y, x) \, dy,$$

which is indeed an average with respect to the measure $\varphi_k(y)dy$! We first observe that θ_k is ≥ 0 everywhere. Moreover, it is k-Lipschitz, since φ_k is k-Lipschitz, and Γ_k is uniformly k-Lipschitz in its second argument.

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Therefore

$$\begin{aligned} \theta_k(f(x)) &\leq \min[\varphi_k(x) + 1, \inf_{y \in X} \varphi_k(y) + \Gamma_k(y, x) + 1] \\ &= \theta_k(x) + 1. \end{aligned}$$

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and $\inf_{\bar{V}_{1/k}(B)} \varphi_k \to +\infty,$ as $k \to +\infty$, without loss of generality, we can assume that

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By the definition of φ_k and C_k, we get

$$k_\ell d(y_0^\ell, \bar{V}_{1/k_\ell}(A)) + \sum_{i=0}^{n_\ell - 1} [k_\ell d(f(y_i^\ell), y_{i+1}^\ell) + 1] < n + 1,$$

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which can be rewritten as

$$k_\ell d(y_0^\ell, \bar{V}_{1/k_\ell}(A)) + n_\ell + \sum_{i=0}^{n_\ell - 1} k_\ell d(f(y_i^\ell), y_{i+1}^\ell) < n + 1.$$

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- $n_\ell \leq n$,
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But $x \in A$ and $m \leq n$. This contradicts the hypothesis $B \cap (\cup_{i=0}^n f^i(A)) = \emptyset$.