On the geometry of Nikulin K3 surfaces

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Polarized Nikulin Surfaces of genus g

$$\mathcal{F}_g^N := \{\text{polarized Nikulin surfaces of genus } g\}/\text{iso}$$

parametrizing pairs $(X, \mathfrak{c})$, where

- X is a K3 surface, $\mathfrak{c}$ big and nef line bundle, $\mathfrak{c}^2 = 2g - 2$.
- X contains 8 disjoint copies of $\mathbb{P}^1$, say $R_1, \ldots, R_8$, satisfying

$$\mathfrak{c} \cdot R_i = 0.$$

These 8 curves form an even set:

$$\mathfrak{e} := \frac{1}{2}(R_1 + \cdots + R_8) \in \text{Pic}(X).$$

Fact

$\mathcal{F}_g^N$ is irreducible and 11-dimensional. The general point corresponds to a K3 surface with Picard number 9.

Motivation

Nikulin surfaces have been studied in relation to:

- Automorphisms: Nikulin (1980s), Garbagnati–Sarti (2008)
- Moduli spaces: Morrison (1984), van Geemen–Sarti (2007)
- Prym curves: Farkas–Kemeny (2016) and the birational geometry of their moduli spaces: Farkas–Verra (2012), Verra (2016).

Sources of interesting geometry: for general $(X, \mathfrak{c}) \in \mathcal{F}_g^N$,

$$|\mathfrak{c}|: X \to S_0 \subset \mathbb{P}^g \quad \text{surface with 8 nodes } (g \geq 3)$$

$$|\mathfrak{c} \otimes \mathfrak{e}^{-1}|: X \hookrightarrow S_1 \subset \mathbb{P}^{g-2} \quad \text{surface with 8 lines } (g \geq 5)$$

Q: positivity and Brill-Noether behaviour of $\boxed{\mathfrak{h}_m := \mathfrak{c} \otimes \mathfrak{e}^{-m}}$ $(m \geq 1)$?

Theorem

For a general $(X, \mathfrak{c}) \in \mathcal{F}_g^N$, *write* $g = 2k^2 + \gamma$, *where* $k \geq 1$ *and* $0 \leq \gamma < 4k + 2$ *and let* $\mathfrak{h}_m := \mathfrak{c} \otimes \mathfrak{e}^{-m}$.

(i) *Assume* $k \geq 2$. *For any* $1 \leq m \leq k - 1$ *the general member of the linear system* $|\mathfrak{h}_m|$ *is a smooth irreducible curve of genus*

$$g_m = 2(k^2 - m^2) + \gamma \geq 6.$$

In fact, $\mathfrak{h}_m$ *is very ample and defines an embedding of* X *in* $\mathbb{P}^{g_m}$.

(ii) *In the extremal case* $m = k$, *the linear system* $|\mathfrak{h}_k|$ *contains a smooth irreducible curve of genus* γ *and* $\mathfrak{h}_k$ *is ample for* $\gamma \geq 2$, *very ample for* $\gamma \geq 3$ *and it defines an elliptic fibration* $X \to \mathbb{P}^1$ *for* $\gamma = 1$.

(iii) *For any* $0 \leq m \leq k$ *(assuming* $\gamma \geq 2$ *when* $m = k$*) all smooth curves in* $|\mathfrak{h}_m|$ *are Brill-Noether general, i.e. have maximal Clifford index* $\lfloor \frac{g_m - 1}{2} \rfloor$.

Proposition

For a general $(X, \mathfrak{c}) \in \mathcal{F}_g^N$*, write* $g = 2k^2 + \gamma$*. For any* $1 \leq m \leq k-1$*, let* $D \in |\mathfrak{h}_{m+1}|$ *be a smooth curve. Then* D *is embedded non-specially by* $\mathfrak{h}_m$ *as a non-degenerate curve of degree* $2g_m - 2 - 4m$ *in* $\mathbb{P}^{g_m}$ *and*

$$\psi_m \colon H^0(X, \mathfrak{h}_m) \xrightarrow{\simeq} H^0(D, \mathfrak{h}_m|_D).$$

Example ($m = k - 1$)

Then D *is a curve of genus* γ *in* $\mathbb{P}^{N+\gamma}$*. For example*
For $\gamma = 0$*,* D *is the rational normal curve of degree* $4k-2$ *in* $\mathbb{P}^{4k-2}$*.*
For $\gamma = 1$*,* D *is an elliptic normal curve in* $\mathbb{P}^{4k-1}$*, etc.*

Corollary ($m = k - 1$, $\gamma = 0$)

For a polarized Nikulin surface $(X, \mathfrak{c})$ *of genus* $g = 2k^2$ $(k \geq 2)$ *let* $D \in |\mathfrak{h}_k|$*. The following locus defines an effective divisor in* $\mathcal{F}_g^N$

$$\mathfrak{D}_k = \left\{ (X, \mathfrak{c}) \in \mathcal{F}_g^N \ : \ \ker\left(H^0(X, \mathfrak{h}_{k-1}) \to H^0(D, \mathfrak{h}_{k-1}|_D)\right) \neq 0 \right\}.$$