A factorization theorem for rank-two irregular flat connections

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Transversely projective foliations

Let X be a projective manifold. A codimension 1 foliation $\mathcal{F} : \{\alpha = 0\}$ on X is transversely projective if there exists a **flat meromorphic connection**

$$\nabla : Z \mapsto dZ + AZ \quad \text{with} \quad Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2 \quad \text{and} \quad A = \begin{pmatrix} \frac{\beta}{2} & \gamma \\ -\alpha & -\frac{\beta}{2} \end{pmatrix}$$

or a **Riccati 1-form**

$$\omega = dz + \alpha z^2 + \beta z + \gamma, \quad z = z_1/z_2$$

with **integrability condition**

$$\nabla \cdot \nabla = 0 \;\Leftrightarrow\; dA + A \wedge A = 0 \;\Leftrightarrow\; \omega \wedge d\omega = 0 \;\Leftrightarrow\; \begin{cases} d\alpha = \alpha \wedge \beta \\ d\beta = 2\alpha \wedge \gamma \\ d\gamma = \beta \wedge \gamma \end{cases}$$

Let D be the **polar divisor** of the structure (i.e. of coefficients α, β, γ).

Examples

Closed 1-forms: $\mathcal{F} : \{\omega = 0\}$, with ω a closed rational 1-form on X, $d\omega = 0$.

Pull-back of Riccati foliations: given $C =$ curve, α, β, γ rational 1-forms on C, and $\phi : X \to C \times \mathbb{P}^1$, define $\mathcal{F} = \phi^* \mathcal{F}_\omega$ with $\omega = dz + \alpha z^2 + \beta z + \gamma$.

Hilbert modular foliations: Let $\mathbb{Q} \to K \subset \mathbb{C}$ a totally real number field of degree d, let $\mathcal{O}_K \subset K$ be the ring of integers. The group $\Gamma = \mathrm{PSL}_2(\mathcal{O}_K) \subset \mathrm{PSL}_2(\mathbb{R})$ acts naturally on the Poincaré upper half-plane $\mathbb{H} \subset \mathbb{P}^1$, and therefore on the polydisc $\mathbb{H}^d$ as follows. Enumerate $\mathrm{Gal}(K/\mathbb{Q}) = \{\sigma_1, \ldots, \sigma_d\}$. Then define

$$(z_1, \ldots, z_d) \overset{M \in \Gamma}{\longrightarrow} \left(M^{\sigma_1} z_1, \ldots, M^{\sigma_d} z_d\right).$$

This action is discrete and the quotient $\mathbb{H}^d/\Gamma$ admits a compactification

$$X = \overline{\mathbb{H}^d/\Gamma} = \mathbb{H}^d/\Gamma \cup D.$$

Quotient foliations $\mathcal{F}_i : \{dz_i = 0\}/\Gamma$ are transversely projective and not better (Zariski dense monodromy group).

Examples

Explicit models have been given for $K = \mathbb{Q}(\sqrt{5})$ (Mendes-Pereira) and $K = \mathbb{Q}(\sqrt{3})$ (Cousin) where X is rational.

Theorem (Touzet): Hilbert modular foliations are not pull-back of Riccati foliations.

Other examples ? $\longrightarrow$ classify Riccati foliations / flat meromorphic connections $\longrightarrow$ classify representations $\pi_1(X \setminus D) \to \mathsf{PGL}_2(\mathbb{C})$ / $\mathsf{SL}_2(\mathbb{C})$.

Corlette-Simpson Theorem

Theorem: Let (E,∇) be a flat $\mathfrak{sl}_2$-connection on a projective manifold X. If ∇ is logarithmic with normal crossing polar divisor D and rational eigenvalues, then one of the following holds

- the monodromy representation degenerates (finite, dihedral or reducible),
- or $\exists\ \phi : X \dashrightarrow C$ a curve such that $(E,\nabla) \overset{\sim}{\dashrightarrow} \phi^* \underbrace{(E_0,\nabla_0)}_{\text{on } C}$,
- or $\exists\ \phi : X \dashrightarrow \mathfrak{H}$ Shimura polydisk and $(E,\nabla) \sim \phi^*(E_i,\nabla_i)$ for one of the tautological connections (E_i,∇_i) on $\mathfrak{H}$.

Classification Theorem

Theorem (Cousin-Pereira + L.-Pereira-Touzet): Let $\mathcal{F}$ be a transversely projective foliation on X projective. Then:

- $\exists\ \phi : X' \to X$ generically finite such that $\phi^*\mathcal{F}$ is transversely euclidean,
- or $\exists\ \phi : X \dashrightarrow C \times \mathbb{P}^1$ and $\mathcal{H}_0$ a Riccati foliation such that $\mathcal{F} = \phi^*\mathcal{H}_0$,
- or $\exists\ \phi : X \dashrightarrow \mathfrak{H}$ Shimura polydisk such that $\mathcal{F} = \phi^*\mathcal{F}_i$ with $\mathcal{F}_i$ one of the tautological foliations.

Irregular version of Corlette-Simpson Theorem

Theorem (L.-Pereira-Touzet): Let (E,∇) be a flat $\mathfrak{sl}_2$-connection on X projective. If ∇ is irregular, or logarithmic with some irrational eigenvalues, then

- $\exists\ \phi : X' \to X$ generically finite such that

$$\phi^*(E,\nabla) \overset{\sim}{\dashrightarrow} \left(\ X \times \mathbb{C}^2\ ,\ d + \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} \text{ or } \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix}\ \right),$$

- or $\exists\ \phi : X \to C$ a curve such that $(E,\nabla) \overset{\sim}{\dashrightarrow} \phi^*(E_0,\nabla_0)$.