Transversely projective foliations

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Codimension 1 foliations: smooth case

A **smooth** codimension 1 **holomorphic foliation** $\mathcal{F}$ on X is defined by

$$(f_i : U_i \to \mathbb{C})_i \quad \text{submersion}$$

satisfying compatibility condition

$$f_i = \varphi_{ij} \circ f_j, \quad \varphi_{ij} \text{ holomorphic diffeomorphisms}$$

The **tangent bundle** $T\mathcal{F} \subset TX$ is locally defined by $df_i = 0$.

Smooth transversely projective foliation:

$$f_i : U_i \to \mathbb{P}^1 \quad \text{and} \quad \varphi_{ij} \in \mathrm{Aut}(\mathbb{P}^1).$$

Smooth codimension 1 foliations: alternate definition

Define $\mathcal{F}$ by its tangent bundle $T\mathcal{F}$:

$$(\omega_i = 0)_i, \quad \omega_i \in \Omega^1(U_i) \text{ non vanishing}$$

satisfying **Frobenius integrability condition**

$$(*) \quad \omega_i \wedge d\omega_i = 0$$

and compatibility condition

$$\omega_i = h_{ij}\omega_j, \quad h_{ij} \in \mathcal{O}^*(U_i)$$

The **normal bundle** $N\mathcal{F} := TX/T\mathcal{F}$ is defined by (h_{ij}).

Frobenius Theorem: $(*) \ \Rightarrow \omega_i = g_i df_i$ where $f_i : U_i \to \mathbb{C}$ submersion, $g_i \in \mathcal{O}^*(U_i)$.

Codimension 1 foliations: the singular case

Define $\mathcal{F}$ by its tangent bundle $T\mathcal{F}$:

$$(\omega_i = 0)_i, \quad \omega_i \in \Omega^1(U_i),\ \omega_i \text{ vanishing in codimension } \geq 2$$

satisfying **Frobenius integrability condition**

$$(*) \quad \omega_i \wedge d\omega_i = 0$$

and compatibility condition

$$\omega_i = h_{ij}\omega_j, \quad h_{ij} \in \mathcal{O}^*(U_i)$$

The **normal bundle** $N\mathcal{F} := TX/T\mathcal{F}$ is defined by (h_{ij}).

Get a smooth holomorphic foliation $\mathcal{F}$ on $X \setminus Z$ where

$$Z := \cup_i \{\text{zero set of } \omega_i\} \subset X$$

is the (closed codimension ≥ 2) **singular set** of $\mathcal{F}$.

Codimension 1 foliations: the singular case

Assuming X projective, we can define $\mathcal{F}$ on a Zariski open subset by

$$\omega = 0, \qquad \text{a global rational 1-form}$$

satisfying Frobenius integrability condition

$$(*) \quad \omega \wedge d\omega = 0.$$

Inconvenient: we introduce fake singular points for $\mathcal{F}$ (non triviality of $N\mathcal{F}$)

Example 1: $\omega = df$ where $f : X \dashrightarrow \mathbb{P}^1$ (pencil of hypersurfaces)

Example 2: $d\omega = 0$ have local first integrals $f := \int \omega$ with additive monodromy outside of poles $(\omega)_\infty$. In general, leaves are dense !

Singular transversely projective foliations

A sing. **transversely projective foliation** is the data of $(P, \mathcal{H}, \sigma)$ where

- $\pi : P \to X$ is a $\mathbb{P}^1$-bundle,

- $\mathcal{H}$ is a sing. codimension 1 foliation on P transversal to a generic $\mathbb{P}^1$-fiber,

- $\sigma = X \dashrightarrow P$ a rational section generically transversal to $\mathcal{H}$.

Then $\mathcal{F} := \sigma^*\mathcal{H}$ on X is transversely projective in the smooth sense in restriction to the Zariski open set $U \subset X$ over which everything is smooth, transversal:

Indeed, outside $S := \pi\,(\mathrm{tang}(\mathcal{H}, \pi) \cup \mathrm{tang}(\mathcal{H}, \sigma) \cup \mathrm{indet}(\sigma))$, we have locally:

$$(P, \mathcal{H}, \sigma)|_{U_i} \xrightarrow{\sim} (U_i \times \mathbb{P}^1_z, dz = 0, f_i : U_i \dashrightarrow \mathbb{P}^1)$$

and $f_i = \varphi_{ij} \circ f_j$ with $\varphi_{ij} \in \mathrm{Aut}(\mathbb{P}^1)$.

Transversely projective foliations: lack of unicity

Another triple $(P', \mathcal{H}', \sigma')$ defines the same transversely projective foliation (same $\mathcal{F}$ with same first special integrals $f_i : U_i \to \mathbb{P}^1$) iff

$$\exists\ \phi : P' \dashrightarrow P \quad \text{birational bundle transformation}$$

such that $\mathcal{H}' = \phi^*\mathcal{H}$ and $\sigma = \phi \circ \sigma'$.

For instance, if X is projective, there exists birationally equivalent triple

$$(P, \mathcal{H}, \sigma) \overset{\sim}{\dashrightarrow} (X \times \mathbb{P}^1_z, \mathcal{H}, z = \infty).$$

Transversely projective foliations: alternate definition

Proposition If $P = X \times \mathbb{P}^1_z$ then $\mathcal{H}$ is defined by a unique equation

$$\underbrace{dz + \alpha z^2 + \beta z + \gamma}_{\omega} = 0 \quad \text{(Riccati foliation)}$$

where α, β, γ rational 1-forms on X. Integrability condition writes

$$\omega \wedge d\omega = 0 \;\Leftrightarrow\; (**) \begin{cases} d\alpha = \alpha \wedge \beta \\ d\beta = 2\alpha \wedge \gamma \\ d\gamma = \beta \wedge \gamma \end{cases}$$

If $\sigma : \{z = \infty\}$, then $\mathcal{F} : \{\alpha = 0\}$. A birational bundle automorphism $z \mapsto az+b$, $a, b \in \mathbb{C}(X)$, $a \not\equiv 0$, gives new equation

$$dz + \tilde{\alpha} z^2 + \tilde{\beta} z + \tilde{\gamma} = 0 \quad (***) \begin{cases} \tilde{\alpha} = a\alpha \\ \tilde{\beta} = \beta + 2b\alpha + \frac{da}{a} \\ \tilde{\gamma} = \frac{1}{a}\left(db + \alpha b^2 + \beta b + \gamma\right) \end{cases}$$

New definition: (Scardua thesis '95) A foliation $\mathcal{F} : \{\alpha = 0\}$ on X is transversely projective if there are β, γ satisfying (**).
Remark: (**) $\Rightarrow$ (*) $\alpha \wedge d\alpha = 0$.

Group reduction

$$(**)\begin{cases} d\alpha = \ \ \alpha\wedge\beta \\ d\beta = 2\alpha\wedge\gamma \\ d\gamma = \ \ \beta\wedge\gamma \end{cases} \quad (***)\begin{cases} \tilde{\alpha} = a\alpha \\ \tilde{\beta} = \beta + 2b\alpha + \frac{da}{a} \\ \tilde{\gamma} = \frac{1}{a}\left(db + \alpha b^2 + \beta b + \gamma\right) \end{cases}$$

A triple $(\alpha, 0, 0)$ satisfies (*) iff $d\alpha = 0$
$\Rightarrow$ **transversely euclidean:** $f := \int \alpha$ has additive monodromy

A triple $(\alpha, \beta, 0)$ satisfies (*) iff $d\alpha = \alpha\wedge\beta$ and $d\beta = 0$
$\Rightarrow$ **transversely euclidean:** $f := \int \frac{\alpha}{\exp(\int\beta)}$ has affine monodromy

Proposition If $\mathcal{F}$ admits two non equivalent projective structures, then $\exists\ p : X' \xrightarrow{2:1} X$ dominant, generically finite (degree 2) map such that $p^*\mathcal{F}$ is transversely euclidean.

Proof Can assume $\tilde{\alpha} = \alpha$ and $\tilde{\beta} = \beta$ up to (***). Then (*) implies that $\tilde{\gamma} = \gamma + h\alpha$, $h \in \mathbb{C}(X)$, and $(\alpha, \frac{1}{2}\frac{dh}{h}, 0)$ satisfies (*). □

Degree d codimension 1 foliations on $\mathbb{P}^n$

Defined $\mathcal{F}_\omega$ on $\mathbb{P}^n$ by $\omega = 0$ on $\mathbb{C}^{n+1}$:

$$\omega = \sum_{i=0}^{n} H_i(X_0, \dots, X_n) dX_i$$

where H_i = homogeneous polynomials of degree $d+1$ without common factor, satisfying

$$(*) \quad \omega \wedge d\omega = 0$$

and the conic structure

$$\omega\left(\sum_{i=0}^{n} X_i \frac{\partial}{\partial X_i}\right) = 0 \quad \Leftrightarrow \quad \sum_{i=0}^{n} X_i H_i = 0.$$

We have $\mathcal{F}_\omega = \mathcal{F}_{\omega'}$ iff $\omega' = c \cdot \omega$, with $c \in \mathbb{C}^*$.

We have $\mathrm{Fol}^d(\mathbb{P}^n) \subset \mathbb{P}^N = \mathbb{P}\{$conic homogeneous 1-forms $\omega\}$ (we omit (*))

In dimension $n = 2$, conic structure $\Rightarrow$ (*) and $\mathrm{Fol}^d(\mathbb{P}^2)$ forms a Zariski open subset of $\mathbb{P}^N$ (have to delete $\omega = H \cdot \tilde{\omega}$).

Foliations on $\mathbb{P}^n$: classification

When dimension $n > 2$, Frobenius condition (*) becomes non trivial and

$$\overline{\mathrm{Fol}^d(\mathbb{P}^n)} \subset \mathbb{P}^N$$

forms a strict subset. Irreducible components are known for degree $d \leq 2$ (Darboux, Jouanolou, Cerveau-LinsNeto).

All known examples are of the form:
- pencils of hypersurfaces, - closed rational 1-forms, - pull-back of a foliation in $\mathbb{P}^2$.

Proposition For any $d \geq 2$, there exist $\mathcal{F} \in \mathrm{Fol}^d(\mathbb{P}^2)$ that are not transversely projective.

($\Rightarrow$ the same holds true for generic $\mathcal{F} \in \mathrm{Fol}^d(\mathbb{P}^2)$ w.r.t. Lebesgue measure)

Conjectures

Conjecture 1: Transversely projective foliations form a closed subset of $\mathrm{Fol}^d(\mathbb{P}^2)$.

Conjecture 2: Given a codimension 1 foliation on a compact complex manifold M. Then:

- $\mathcal{F}$ is the pull-back of a foliation $\mathcal{F}_0$ on a projective surface X:

$$\exists\ \phi : M \dashrightarrow X \quad \text{such that} \quad \mathcal{F} = \phi^*\mathcal{F}_0;$$

- or $\mathcal{F}$ is transversely projective.

A result arXiv:1107.1538v1

Theorem (L.-Pereira-Touzet) Let $\mathcal{F} \in \mathrm{Fol}^d(\mathbb{P}^n)$ with $d < 2n - 2$. Then

- $\mathcal{F}$ is the pull-back of a foliation $\mathcal{F}_0$ on a projective manifold X of dimension

$$\dim(X) \leq \frac{d}{2} + 1 < n;$$

- or $\mathcal{F}$ is transversely euclidean.

Lie Theorem

Theorem (Lie) Let $\mathcal{L} \subset \{v = f(z)\partial_z \ ; \ f \in \mathbb{C}\{x\}\}$ be a finite dimensional Lie sub-algebra w.r.t. the Lie bracket $[f(z)\partial_z, g(z)\partial_z] = (f'g - fg')\partial_z$.
If $\mathcal{L}$ is transitive ($\exists\ v_0 \in \mathcal{L}$ such that $v_0(0) \neq 0$), then $\dim(\mathcal{L}) \leq 3$ and $\exists\ \phi : (\mathbb{C}_z, 0) \to (\mathbb{C}_w, 0)$ such that

$$\mathcal{L} = \phi^* \mathcal{L}_0 \quad \text{with} \quad \mathcal{L}_0 \subset \mathfrak{sl}_2(\mathbb{C}) = \mathbb{C}\langle \partial_w, w\partial_w, w^2\partial_w \rangle.$$

Algebraic version Let $\mathcal{L} \subset \{v = f(z)\partial_z \ ; \ f \in \mathbb{C}(x)\}$ be a finite dimensional Lie sub-algebra w.r.t. the Lie bracket $[f(z)\partial_z, g(z)\partial_z] = (f'g - fg')\partial_z$.
Then $\dim(\mathcal{L}) \leq 3$ and $\exists\ \phi : \mathbb{P}^1_z \to \mathbb{P}^1_w$ such that

$$\mathcal{L} = \phi^* \mathcal{L}_0 \quad \text{with} \quad \mathcal{L}_0 \subset \mathfrak{sl}_2(\mathbb{C}) = \mathbb{C}\langle \partial_w, w\partial_w, w^2\partial_w \rangle.$$