COHOMOLOGICAL CLASSIFICATION OF VECTOR BUNDLES OVER SMOOTH AFFINE VARIETIES

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Abstract. These are expanded lecture notes for the mini-course on the classification of vector bundles on smooth affine varieties given in Luminy during the week Dec 11-15, 2017.

Introduction

The goal of these three lectures is to cover the basics of the new methods of classification of vector bundles over smooth affine varieties.

Conventions. All the schemes considered here are supposed to be of finite type and separated over some field k. Usually, we suppose that k is infinite perfect.

1. Lecture 1

In this lecture, we first start with making precise what we mean by classification of vector bundles on smooth affine varieties. As usual, we'll switch between locally free modules over $X = \mathrm{Spec}(R)$ and projective R-modules.

1.1. **Stable versus unstable classification.** Let $X = \mathrm{Spec}(R)$ be an affine scheme. If V is a locally free sheaf over X, we denote by $\{V\}$ its isomorphism class. The set of isomorphism classes of locally free sheaf over X is endowed the structure of an abelian monoid with operation defined by

$$\{V\} + \{W\} = \{V \oplus W\}$$

and neutral element the trivial module. The group $K_0(X)$ is the group completion of $\mathcal{V}(X)$. More precisely, $K_0(X)$ is the free abelian group generated by the isomorphism classes $\{V\}$ quotiented by the subgroup generated by

$$\{V\} + \{W\} - \{V \oplus W\}$$

for any $\{V\}$ and $\{W\}$. We denote by $[V]$ the class of $\{V\}$ in $K_0(X)$.

Proposition 1.1. *Let V and W be locally free sheaves such that $[V] = [W]$. Then there exists $n \in \mathbb{N}$ such that $V \oplus \mathcal{O}_X^n \simeq W \oplus \mathcal{O}_X^n$.*

Suppose that X is connected (otherwise we can make the following study connected component by connected component). Then we obtain a homomorphism

$$\mathrm{rk} : K_0(X) \to \mathbb{Z}$$

induced by the rank. We denote by $\widetilde{K}_0(X)$ the kernel of rk.

For any $r \in \mathbb{N}$, let $\mathcal{V}_r(X)$ be the set of isomorphism classes of rank r locally free sheaves, pointed by the class $\{\mathcal{O}_X^r\}$. We define a map

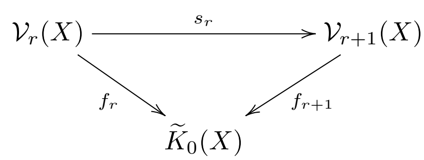
$$s_r : \mathcal{V}_r(X) \to \mathcal{V}_{r+1}(X)$$

by $s_r(\{V\}) = \{V \oplus \mathcal{O}_X\}$ and we observe that s_r is a map of pointed sets. Let $\mathcal{V}(X) := \lim \mathcal{V}_r(X)$. For any $r \in \mathbb{R}$, we denote by

$$\pi_r : \mathcal{V}_r(X) \to \mathcal{V}(X)$$

the "limit" homomorphism. Note that $\mathcal{V}(X)$ is pointed by the class of $\pi_0\{0\}$.

For any $r \in \mathbb{N}$ we define a map $f_r : \mathcal{V}_r(X) \to \widetilde{K}_0(X)$ by $f_r(\{V\}) = [V] - [\mathcal{O}_X^r]$. It is clear that the following diagram commutes for any $r \in \mathbb{N}$



and we then obtain a map (of pointed sets) $f : \mathcal{V}(X) \to \widetilde{K}_0(X)$.

Proposition 1.2. *The map $f : \mathcal{V}(X) \to \widetilde{K}_0(X)$ is bijective.*

Proof. We first prove that f is surjective. Let $\alpha = [V] - [W] \in \widetilde{K}_0(X)$. Then we have $\mathrm{rank}(V) = \mathrm{rank}(W)$. Let W' be such that $W \oplus W' = \mathcal{O}_X^n$ for some $n \in \mathbb{N}$. We have

$$\alpha = [V] - [QW] = [V] + [W'] - [W] - [W'] = [V \oplus W'] - [\mathcal{O}_X^n].$$

Now $\mathrm{rank}(V \oplus W') = n$ and it follows that $\alpha = f_n(\{V \oplus W'\})$. Hence f is surjective.

Let β and γ in $\mathcal{V}(X)$ be such that $f(\beta) = f(\gamma)$. There exists therefore $r, s \in \mathbb{N}$ and $\{V\} \in \mathcal{V}_r(X)$, $\{W\} \in \mathcal{V}_s(X)$ such that $f_r(\{V\}) = f_s(\{W\})$ with $\pi_r(\{V\}) = \beta$ and $\pi_s(\{W\}) = \gamma$.

Since $f_r(\{V\}) = f_s(\{W\})$, we have $[V] - [\mathcal{O}_X^r] = [W] - [\mathcal{O}_X^s]$ and it follows from Proposition 1.1 that $V \oplus \mathcal{O}_X^{s+m} \simeq W \oplus \mathcal{O}_X^{r+m}$ for some $m \in \mathbb{N}$. This yields

$$\beta = \pi_r(\{V\}) = \pi_{r+s+m}(\{V \oplus \mathcal{O}_X^{s+m}\}) = \pi_{r+s+m}(\{W \oplus \mathcal{O}_X^{r+m}\}) = \pi_s(\{W\}) = \gamma.$$

□

Thus we see that K-theory (actually, the reduced K_0 group) studies the isomorphism classes of locally free sheaves "at the limit". In particular, the pointed set $\mathcal{V}(X)$ has the structure of an abelian group *which is part of a perfectly nice cohomology theory*. It can be studied using cohomological methods, and in particular the properties of K-theory (homotopy invariance, localization sequences, Nisnevich excision, etc...). Thus, one has a chance of computing $\mathcal{V}(X)$ if we understand X. or its geometry, well enough. The goal of these lectures is to extend these "cohomological" methods to study the sets $\mathcal{V}_r(X)$ and the maps

$$s_r : \mathcal{V}_r(X) \to \mathcal{V}_{r+1}(X)$$

for any $r \in \mathbb{N}$. We start with a few (well-known) reductions. Note first that $\mathcal{V}_1(X)$ is in fact the Picard group of X. For any $r \geq 1$, there is an obvious map of pointed sets

$$\det_r : \mathcal{V}_r(X) \to \mathcal{V}_1(X)$$

induced by the determinant. At the limit, it yields a group homomorphism $\mathcal{V}(X) = \widetilde{K}_0(X) \to Pic(X) = \mathcal{V}_1(X)$ which is nothing but the classical determinant map.

Corollary 1.3. *For any $r \geq 2$, the map $s_{r-1} \circ \ldots \circ s_1 : \mathcal{V}_1(X) \to \mathcal{V}_r(X)$ is split injective, with section* $\det_r$.

Proof. This follows readily from the fact that $\det(L\oplus\mathcal{O}_X^{r-1})\simeq L$ for any line bundle L over X. □

For this reason, it is customary to study the fibers of the map $\det_r:\mathcal{V}_r(X)\to\mathcal{V}_1(X)$. In particular, let L be a line bundle over X. For any $r\geq 1$, let $\mathcal{V}_r(X,L)$ be the set $\det_r^{-1}(\{L\})$. We obtain maps

$$s_r(L):\mathcal{V}_r(X,L)\to\mathcal{V}_{r+1}(X,L)$$

for any $r\geq 1$ with $\mathcal{V}_1(X,L)=\{L\}$. We may consider these sets as pointed sets (by $L\oplus\mathcal{O}_X^{r-1}$) and the maps as pointed maps. Setting $\mathcal{V}(X,L)=\lim_r\mathcal{V}_r(X,L)$ we observe that $\mathcal{V}(X,L)$ is just the preimage in $\widetilde{K}_0(X)$ of $\{L\}\in Pic(X)$ under the determinant map. It has a structure of an homogeneous space under $\mathcal{V}(X,\mathcal{O}_X)=\ker(\det)$.

Now, we come to a few famous theorems (in the language of projective R-modules).

Theorem 1.4 (Serre)**.** *Let R be a commutative noetherian ring and $X=\mathrm{Max}(R)$. Suppose that X is connected of dimension d. Let P be a projective R-module of rank $r>d$. Then $P\simeq P'\oplus R$.*

Proof. See [5, Part 2, Chapter IV, §2] or [17, Théorème 1]. □

Theorem 1.5 (Bass)**.** *Let R be a commutative noetherian ring and let $X=\mathrm{Max}(R)$. Suppose that X is connected of dimension d. Let P and P' be projective R-modules of rank $r>d$. Suppose that there exists a projective R-module Q such that $P\oplus Q\simeq P'\oplus Q$. Then $P\simeq P'$.*

Proof. [5, Part 2, Chapter IV, Corollary 3.5]. □

As an obvious corollary to these two theorems, we get the following result.

Corollary 1.6. *Let X be a connected affine scheme of dimension d. Then*

$$s_r:\mathcal{V}_r(X)\to\mathcal{V}_{r+1}(X)$$

is surjective if $r\geq d$ and injective if $r\geq d+1$. In particular, $\mathcal{V}(X)=\mathcal{V}_{d+1}(X)$.

Note that this result is a bit better in case X is a curve. Indeed, we know that $\mathcal{V}_1(X)\to\mathcal{V}_2(X)$ is injective and therefore $\mathcal{V}(X)=\mathcal{V}_1(X)$. Thus:

Corollary 1.7. *Let X be an affine curve. Then, $\mathcal{V}(X)=\mathcal{V}_1(X)=Pic(X)$. Moreover, the map*

$$s_r:\mathcal{V}_r(X)\to\mathcal{V}_{r+1}(X)$$

is injective and bijective provided $r\geq 1$.

However, Corollary 1.6 is the best possible in general. Indeed, consider the real algebraic sphere (of dimension 2) S^2. Then, the tangent bundle T is locally free of rank 2, stably isomorphic to $\mathcal{O}_{S^2}^3$ but non isomorphic to $\mathcal{O}_{S^2}^2$. In fact, we may totally describe the sequence

$$\mathcal{V}_1(S^2)\xrightarrow{s_1}\mathcal{V}_2(S^2)\xrightarrow{s_2}\mathcal{V}_3(S^2)=\widetilde{K_0}(S^2)\longrightarrow\dots$$

in that case. First, note that $K_0(S^2)=\mathbb{Z}\oplus\mathbb{Z}/2$ (easy computation) and thus that $\widetilde{K_0}(S^2)=\mathbb{Z}/2$. It follows that $\mathcal{V}_r(S^2)=\mathbb{Z}/2$ if $r\geq 3$ (and obviously that s_r is bijective for such r). Next, $\mathcal{V}_1(S^2)=Pic(S^2)=0$. It remains to describe $\mathcal{V}_2(S^2)$, which is in fact equal to $\mathbb{Z}$ (and has therefore a group structure) by e.g. [7, Theorem 5.9].

However, the situation becomes a bit better if we put more assumptions on the base field.

Theorem 1.8 (Suslin)**.** *Let $k = \overline{k}$ be an algebraically closed field. Let X be an affine k-scheme of dimension d. Then, the map*

$$s_r : \mathcal{V}_r(X) \to \mathcal{V}_{r+1}(X)$$

is injective if $r \geq d$. In particular, $\mathcal{V}_d(X) = \mathcal{V}(X)$.

Proof. [18, Theorem 1]. □

This result was later extended to slightly more general base fields by Suslin himself and Bhatwadekar in respectively [19, Theorem 2.4] and [6, Theorem 4.1]. If X is a surface over an algebraically closed field k, then the above theorem yields $\mathcal{V}_2(X) = \widetilde{K_0}(X)$ and $\mathcal{V}_1(X) = Pic(X)$. The stabilization maps s_i are all injective and it remains to understand the image of $s_1 : Pic(X) \to \mathcal{V}_2(X)$. For this, we introduce the topological filtration on $K_0(X)$. From now on, we assume that the schemes we consider are smooth (even though this is not always necessary). We consider the set of isomorphism classes of coherent sheaves on X (with operation the direct sum) and define $G_0(X)$ to be the free abelian group on this set quotiented by the relation $M + N = P$ if there is an exact sequence $0 \to M \to P \to N \to 0$. There is an obvious group homomorphism $K_0(X) \to G_0(X)$ which is in fact an isomorphism since X is smooth (the inverse map can be defined choosing a locally free resolution of a coherent module). Now, there is an obvious filtration on $G_0(X)$, denoted by $(F^nG_0(X))_{n\in\mathbb{N}}$ where $F^nG_0(X)$ is the subgroup generated by coherent sheaves whose support is of codimension $\geq n$. This filtration extends to a filtration $(F^nK_0(X))_{n\in\mathbb{N}}$ of $K_0(X)$ (via the above isomorphism).

Now, the Chern classes induce maps

$$c_i : K_0(X) \to \mathrm{CH}^i(X)$$

and then maps

$$c_i : F^iK_0(X) \to \mathrm{CH}^i(X)$$

for any $i \in \mathbb{N}$. On the other hand, associating to a point of codimension i the coherent sheaf of its closure yields a map from the free abelian group on the set of codimension i points to $F^iK_0(X)$ and then to $F^iK_0(X)/F^{i+1}K_0(X)$. It is not hard to see that this yields a surjective homomorphism

$$\varphi_i : CH^i(X) \to F^iK_0(X)/F^{i+1}K_0(X)$$

Theorem 1.9 (Grothendieck)**.** *Let X be a quasi-projective smooth scheme over k. For any $i \in \mathbb{N}$, the Chern class c_i induces a homomorphism*

$$c_i : F^iK_0(X)/F^{i+1}K_0(X) \to \mathrm{CH}^i(X)$$

and the composites $c_i \circ \varphi_i$ and $\varphi_i \circ \varphi_i$ are both equal to the multiplication by $(-1)^{i-1}(i-1)!$.

As a corollary of this theorem and Theorem 1.8, we obtain the following result.

Corollary 1.10. *Let X be a smooth affine surface over an algebraically closed field k. Then the Chern classes induce a bijection*

$$(c_1, c_2) : \mathcal{V}_2(X) \to \mathrm{CH}^1(X) \times \mathrm{CH}^2(X).$$

Proof. We know from Theorem 1.8 that $\mathcal{V}_2(X) = \mathcal{V}(X) = \widetilde{K_0}(X)$. By Grothendieck's theorem, we obtain exact sequences

$$0 \to F^1K_0(X) \to K_0(X) \to \mathbb{Z} \to 0$$

and

$$0 \to F^2K_0(X) \to F^1K_0(X) \to \mathrm{CH}^1(X) \to 0,$$

where the right-hand homomorphisms are respectively the rank map and first Chern class. Moreover, $F^2K_0(X) = CH^2(X)$ via the second Chern class since $F^iK_0(X) = 0$ if $i \geq 3$. It follows from Suslin's theorem that $\widetilde{K}_0(X) = \mathcal{V}_2(X)$ and the above exact sequence gives $\widetilde{K}_0(X) = F^1K_0(X)$. The claim now follows from the second exact sequence. □

Remark 1.11. Note that the map

$$(c_1, c_2) : \mathcal{V}_2(X) \to \mathrm{CH}^1(X) \times \mathrm{CH}^2(X).$$

is in general not a group homomorphism. The description of the theorem shows that for any (isomorphism class of) line bundle L over X we have a bijection $\mathcal{V}_2(X, L) \to \mathrm{CH}^2(X)$. In particular, we have an isomorphism $\mathcal{V}_2(X, \mathcal{O}_X) \simeq \mathrm{CH}^2(X)$ and $\mathcal{V}_2(X, L)$ is a homogeneous space under this group. The bijection $\mathcal{V}_2(X, L) \to \mathrm{CH}^2(X)$ is obtained by choosing $\mathcal{O}_X \oplus L$ in $\mathcal{V}_2(X, L)$.

Now, we want to extend this classification to smooth threefolds over an algebraically closed field. The first result we need is the following.

Theorem 1.12. *Let k be an algebraically closed field of characteristic different from 2. Let X be a smooth affine threefold over k. Then $CH^3(X)$ is uniquely 2-divisible.*

We don't give a proof of this result, which is well known. References?

As a final result of this section, let us state (and prove) the following theorem ([12, Theorem 2.1]).

Theorem 1.13 (Mohan Kumar-Murthy)**.** *Let X be a smooth affine threefold over an algebraically closed field k of characteristic different from 2. Then, the Chern classes induce a bijection*

$$(c_1, c_2, c_3) : \mathcal{V}_3(X) \to \mathrm{CH}^1(X) \times \mathrm{CH}^2(X) \times \mathrm{CH}^3(X).$$

Proof. As before, note that $\mathcal{V}_3(X) = \widetilde{K_0}(X)$. We can use the topological filtration $F^iK_0(X)$ and note as before that $F^1K_0(X) = \widetilde{K_0}(X)$, while $F^iK_0(X)/F^{i+1}K_0(X) \simeq \mathrm{CH}^i(X)$ for $i = 1, 2$. Next, the group $CH^3(X)$ is uniquely 2-divisible, showing that $CH^3(X) \simeq F^3K_0(X)$ (as $F^iK_0(X) = 0$ if $i > 3$). This proves that

$$(c_1, c_2, c_3) : \mathcal{V}_3(X) \to \mathrm{CH}^1(X) \times \mathrm{CH}^2(X) \times \mathrm{CH}^3(X).$$

is injective, and we are left with surjectivity. Let then $(a_1, a_2, a_3) \in \mathrm{CH}^1(X) \times \mathrm{CH}^2(X) \times \mathrm{CH}^3(X)$. By definition, there exists a line bundle L over X such that $c_1([L]) = a_1$. Moreover, it follows from Grothendieck's theorem that there exists $\alpha \in F^2K_0(X)$ with $c_1(\alpha) = 0$ and $c_2(\alpha) = a_2$. We can write $\alpha = [P] - [\mathcal{O}_X^3]$ for some P of rank 3 over X. Note that $c_1(P) = 0$ and thus that $c_i(P \oplus L) = a_i$ for $i = 1, 2$. Set $b = c_3(P \oplus L)$ and consider $a_3 - b \in \mathrm{CH}^3(X)$. The third Chern class being surjective, there exists $\beta \in F^3K_0(X)$ such that $c_3(\beta) = a_3 - b$. As above, we write $\beta = [Q] - [\mathcal{O}_X^3]$ with $c_1(Q) = c_2(Q) = 0$Theorem and $c_3(Q) = a_3 - b$.

It follows that the locally free sheaf $Q \oplus P \oplus L$ has (a_1, a_2, a_3) for Chern classes. Using the above theorems, we may write $Q \oplus P \oplus L = V \oplus \mathcal{O}_X^4$ and V is the sheaf we were looking for. □

Remark 1.14. The above theorem is much more precise in [12]. In particular, the authors prove that given c_1, c_2 as above there exists a rank 2 locally free sheaf V with these Chern classes. We'll prove in the third lecture that V is in fact unique with this property.

2. Lecture 2

In this section, we introduce the $\mathbb{A}^1$-homotopy category (after Morel-Voevodsky) which is the main new tool which allows to improve the results of the previous section. A useful reference is [1].

2.1. **Simplicial sets.** Let k be an infinite perfect field and let Sm_k be the category of separated smooth schemes over k. Recall that any $X \in \mathrm{Sm}_k$ defines a functor

$$X : \mathrm{Sm}_k \to \mathrm{Sets}$$

by $Y \mapsto \mathrm{Hom}(Y, X)$. Next, let SSets be the category of simplicial sets (e.g. [8]). Explicitly, consider the category Δ whose objects are $\underline{n} := \{0, 1, \ldots, n\}$ for any $n \in \mathbb{N}$ (seen as a totally ordered set) and whose maps $\varphi : \underline{n} \to \underline{m}$ are order preserving maps of sets, i.e. $\varphi(i) \geq \varphi(j)$ if $i \geq j$. A simplicial set S is a functor

$$S : \Delta^{op} \to \mathrm{Sets}$$

and a morphism of simplicial sets is just a natural transformation. We write S_n for $S(\underline{n})$ and observe that morphisms in the category Δ can be described using basic bricks, called face and degeneracy maps ([8, Definition 3.2]). The basic example of a simplicial set is given by the representable object (called *n-simplex*)

$$\Delta^n : \Delta^{op} \to \mathrm{Sets}$$

given by $\underline{m} \mapsto \mathrm{Hom}_\Delta(\underline{m}, \underline{n})$. Note that there is an obvious functor

$$\mathrm{Sets} \to \mathrm{SSets}$$

associating to E the constant functor $\Delta^{op} \to \mathrm{Sets}$ mapping objects to E and morphisms to the identity of E. Consequently, any smooth scheme X yields a functor

$$X : \mathrm{Sm}_k \to \mathrm{SSets}$$

On the other hand, any simplicial set S also defined a functor

$$S : \mathrm{Sm}_k \to \mathrm{SSets}$$

via $S(X) = S$ and $S(f) = \mathrm{Id}_S$ for any morphism $f : X \to Y$. Thus, we can see both smooth schemes and simplicial sets as presheaves of simplicial sets on Sm_k. Note that a presheaf of simplicial sets is nothing but a simplicial object in the category of presheaves. It is easy to check that the category of simplicial presheaves has all small limits and colimits, as well as internal hom objects.

Definition 2.1. A pointed presheaf of simplicial sets $(\mathcal{X}, x)$ is a presheaf of simplicial sets $\mathcal{X}$, together with a map

$$x : \mathrm{Spec}(k) \to \mathcal{X}$$

We may consider the category of pointed presheaves of simplicial sets, whose morphisms are pointed morphisms of presheaves. This category also has all small limits and colimits, and in particular we can consider the presheaves $(\mathcal{X}, x) \vee (\mathcal{Y}, y)$ and $(\mathcal{X}, x) \wedge (\mathcal{Y}, y)$ for any pointed presheaves $(\mathcal{X}, x)$ and $(\mathcal{Y}, y)$. On the other hand, the obvious functor from the category of pointed presheaves to the category of presheaves has a left adjoint which is defined by $\mathcal{X} \mapsto \mathcal{X}_+$, where the latter denotes the disjoint union of $\mathcal{X}$ with $\mathrm{Spec}(k)$.

2.2. **The Nisnevich topology.** Let X be a smooth scheme. A Nisnevich distinguished square is a Cartesian square

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow[i]{} & X \end{array}$$

where i is an open immersion, $p : V \to X$ is étale and induces an isomorphism $p^{-1}(X \setminus U) \to X \setminus U$. Note that in that case, we have a Cartesian square

$$\begin{array}{ccc} p^{-1}(Z) & \longrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle p} \\ Z & \longrightarrow & X \end{array}$$

where $Z = X \setminus U$ (use the fact that an étale extension of a reduced scheme is indeed reduced). The Nisnevich topology ([15]) is the topology generated by these distinguished squares. More precisely, a cover of X in the Nisnevich topology is a finite family of étale morphisms $U_i \to X$ such that for any $x \in X$ there exists i and $u \in U_i$ such that the field extension $k(u)/k(x)$ is trivial. The Nisnevich topology endows Sm_k with a topology, and in particular we have the notion of Nisnevich sheaf. The points in the Nisnevich topology are the henselizations $\mathcal{O}^h_{X,x}$, where X is a smooth scheme and $x \in X$.

Example 2.2. Any Nisnevich distinguished square yields a covering $V \sqcup U \to X$. In the particular case where $X = \mathrm{Spec}(k)$, we have two possible types of squares. We may first choose $U = \emptyset$, giving the trivial covering of $\mathrm{Spec}(k)$, or we can choose $U = \mathrm{Spec}(k)$ giving a covering $\mathrm{Spec}(k) \sqcup \mathrm{Spec}(A) \to \mathrm{Spec}(k)$ where A is an étale k-algebra. In fact, one can already see the difference of the Zariski, Nisnevich and étale topologies in the coverings of $\mathrm{Spec}(k)$. In the Zariski topology, a cover of $\mathrm{Spec}(k)$ is just $\mathrm{Spec}(k)$ itself. In the Nisnevich topology, a cover is given by an étale algebra $A = L_1 \times \ldots_i$, where L_i/k is a finite separable field extension for each i and $L_1 = k$. For the étale topology, a cover is just given by an étale algebra, without further conditions.

One of the nice features of Nisnevich topology is given by the following theorem.

Theorem 2.3. *Let X be a smooth scheme of dimension d and let A be a Nisnevich sheaf on X. Then $\mathrm{H}^i_{\mathrm{Nis}}(X, A) = 0$ for $i > d$.*

One can associate to a simplicial presheaf a simplicial sheaf following the usual procedure, and it follows that the category of simplicial sheaves also has all small limits and colimits. We can also consider the pointed version of this discussion, and in particular consider the simplicial sheaves $(\mathcal{X}, x) \vee (\mathcal{Y}, y)$ and $(\mathcal{X}, x) \wedge (\mathcal{Y}, y)$ for any pointed sheaves $(\mathcal{X}, x)$ and $(\mathcal{Y}, y)$.

Definition 2.4. A *motivic space* is a simplicial object in the category of Nisnevich sheaves. Equivalently, a motivic space is a simplicial presheaf S whose components S_n are sheaves of sets.

Remark 2.5. We could work with simplicial presheaves instead of simplicial sheaves. In the end, the homotopy categories (constructed below in the case of sheaves) will be equivalent.

2.3. **The model structure on simplicial sheaves.** Now, recall the notion of model category, due to Quillen. The goal is to be able to work "up to homotopy" in an axiomatic way. In particular, model categories provide a nice framework for inverting a class of morphism and getting a grasp at the hom sets in the quotient category.

We start with the category of (pointed) simplicial sets introduced in the previous section. Recall that there is a "topological realization" functor from the category of simplicial sets to the category of topological spaces associating to S the topological space

$$|S| : \sqcup_{n\in\mathbb{N}}(S_n \times |\Delta^n|)/\sim$$

where $|\Delta^n|$ is the usual n-simplex endowed with its usual topology, S_n is the set of n-simplexes endowed with the discrete topology and $\sim$ is the equivalence relation generated by $(s, d_i(p)) = (d_i(s), p)$ for every $s \in S_{n+1}$ and $p \in |\Delta^n|$ (here, $d_i : |\Delta^n| \to |\Delta^{n+1}|$ are the face inclusions) and $(x, s_i(p)) \sim (s_i(x), p)$ for every $x \in S_{n-1}$ and $p \in |\Delta^n|$ (here, $s_i : |\Delta^{n+1}| \to |\Delta^n|$ are the face collapses). Equivalently, one can define the realization functor to be a limit ([10, Chapter I, Lemma 2.1] and the subsequent discussion).

In particular, a direct calculation shows that the topological realization of the n-simplex is just the topological n-simplex. Almost by definition, the n-simplex is weak equivalent to a point. Further, let Δ^n be the simplicial n-simplex for $n \geq 1$ and let $\partial\Delta^n$ be the union of its faces (which is a simplicial set). Then, one can consider the quotient simplicial set $\Delta^n/\partial\Delta^n$ and check that it realizes to the sphere S^n.

Definition 2.6. A morphism of simplicial sets $f : S \to T$ is a *weak-equivalence* if the induced map on topological spaces $|f| : |S| \to |T|$ is a weak-homotopy equivalence (i.e. $|f|$ induces an isomorphism on homotopy groups (for each choice of base point).

The goal is to describe the localization of the category of simplicial sets with respect to the class of weak-equivalences. This can be done as follows. Say that a morphism of simplicial sets $f : S \to T$ is a *cofibration* if it is a monomorphism (i.e. degreewise injective) and say that it is an *acyclic cofibration* if it is further a weak-equivalence. Consider the following diagram

$$\begin{array}{ccc} A & \longrightarrow & E \\ {\scriptstyle i}\downarrow & \overset{f}{\nearrow} & \downarrow{\scriptstyle p} \\ X & \longrightarrow & B \end{array}$$

in the category of simplicial sets. We say that p has the *right lifting property* with respect to acyclic cofibrations if there exists a morphism f making the diagram commutative each time i is an acyclic cofibrations. Define the class of fibrations to be precisely the morphisms satisfying this property. Thus, we have isolated three classes of morphisms: weak-equivalences W, cofibrations C and fibrations F, and this is precisely the data needed to perform our localization process.

Definition 2.7. We say that two maps of simplicial sets $f_i : S \to T$ for $i = 0, 1$ are naively homotopic if there exists a morphism $F : S \times \Delta^1 \to T$ such that $F(i) = f_i$ where $F(i)$ denotes the composite of F with the i-th face.

Note finally that the category of simplicial sets possesses both an initial object $\emptyset$ and a final object $\star$. We say that S is cofibrant if $\emptyset \to S$ is a cofibration (thus any object is cofibrant) and fibrant if $S \to \star$ is a fibration.

Theorem 2.8 (Quillen)**.** *Let S be a cofibrant object and let T be a fibrant object in the category of simplicial sets. Then*

$$\mathrm{Hom}_{\mathrm{SSets}[W^{-1}]}(S, T) = \mathrm{Hom}_{\sim}(S, T)$$

where $\sim$ is the naive homotopy equivalence relation defined above.

Remark 2.9*.* As noted in the theorem, the naive homotopy relation is actually an equivalence relation, provided S is cofibrant and T is fibrant (e.g. [10, Corollary 6.2]).

Now, the good thing with cofibrant and fibrant objects is that they abound in nature. We have already seen that every object is cofibrant. Moreover, given an object S, there exists a trivial cofibration $S \to S_{fib}$ with the latter fibrant. Thus, to compute $\mathrm{Hom}_{\mathrm{SSets}[W^{-1}]}(S, T)$ for any objects S and T, we can choose a fibrant replacement of T and we get

$$\mathrm{Hom}_{\mathrm{SSets}[W^{-1}]}(S, T) = \mathrm{Hom}_{\sim}(S, T_{fib})$$

Of course, the problem with this procedure is that it is hard to explicitly find a fibrant replacement. Note however that there exists another characterization of fibrations, in terms of horns. There are usually called *Kan fibrations* ([10, Chapter I,§3]).

2.4. **The model category structure on simplicial sheaves.** We now endow the category of simplicial sheaves with a model structure using the previous section. Let then S and T be simplicial sheaves of sets and $f : S \to T$ be a morphism of simplicial (pre)sheaves. We say that f is a cofibration if it is a monomorphism and a weak-equivalence if $f(x) : S(x) \to T(x)$ is a weak-equivalence of simplicial sets for any point x in the Nisnevich site (namely, f should be a weak-equivalence at each hensel local essentially smooth scheme). We can define the fibrations to be the class of morphisms having the right lifting property with respect to acyclic cofibrations and obtain as above three classes of morphisms: W, C and F.

Definition 2.10. The simplicial homotopy category $\mathcal{H}_s(k)$ is the localization of the category of simplicial sheaves (of sets) with respect to the class of weak-equivalences defined above. We can also consider its pointed version $\mathcal{H}_{s,\bullet}(k)$ defined in an analogous way.

We may define wedge and smash products as usual, and also an internal hom, induced by the internal hom of simplicial sets.

As before, the way to compute hom sets in this category is to find an explicit fibrant replacement and compute the set of morphisms up to naive homotopies. The problem is again to characterize fibrant objects in a workable way. We'll come back to this point later on, but we now state a few facts that we will use below. First, note that if $S \to T$ is a fibration, then the induced map $S(U) \to T(U)$ is a Kan fibration for each smooth scheme U (this follows from [14, Chapter I, Lemma 1.8 (3)] with $\mathcal{X} = \emptyset$). Next, let S be a fibrant simplicial sheaf. Consider the inclusion $d_0 : \star = \Delta^0 \to \Delta^1$ (given by choosing the vertex 0). It induces a morphism

$$d_0^* : \underline{\mathrm{Hom}}(\Delta^1, S) \to \underline{\mathrm{Hom}}(\star, S) = S$$

which is a fibration ([14, Lemma 1.8 (2)] and [10, Chapter II,Proposition 3.2]). Choosing a base point $s \in S$, we can consider the pull-back square (obviously existing in the category of simplicial sheaves)

$$\begin{array}{ccc} PS & \xrightarrow{j} & \underline{\mathrm{Hom}}(\Delta^1, S) \\ \downarrow & & \downarrow \\ \star & \longrightarrow & S \end{array}$$

Definition 2.11. We define the path space associated to S to be the simplicial sheaf PS fitting in the above diagram.

Now, we have a natural map $d_1 : \Delta^0 \to \Delta^1$ (corresponding to the choice of 1), yielding a map $d_1^* : \underline{\mathrm{Hom}}(\Delta^1, S) \to \underline{\mathrm{Hom}}(\star, S) = S$ as above and a composite

$$\pi : PS \xrightarrow{j} \underline{\mathrm{Hom}}(\Delta^1, S) \xrightarrow{d_1^*} S$$

which is a fibration (use [14, Lemma 1.8 (2)] with $\Delta^0 \sqcup \Delta^0 \to \Delta^1$ and [10, proof of Lemma 7.5]). Moreover, $\pi : PS \to S$ is a weak-equivalence by [10, Chapter I, Lemma 7.5].

We've made use of pull-backs in the category of simplicial sheaves. However, there is a weaker (yet more useful) notion of pull-back in the homotopy category of simplicial sheaves. We follow the exposition of [16, §1] (and say that the model categories we work with are proper in the sense of [16, Definition 1.1.6])

Definition 2.12. A commutative square

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Y & \xrightarrow[g]{} & Z \end{array}$$

of simplicial sheaves is *homotopy cartesian* if for some factorization of $Y \to Z$ as $Y \to V \to Z$, where the first map is an acyclic cofibration and the second is a fibration the induced morphism

$$U \to V \times_Z X$$

is a weak equivalence.

Remark 2.13. We can also require that for some factorization of $X \to Z$ the above property holds, or for some factorization of both $Y \to Z$ and $X \to Z$ ([10, Chapter II, Lemma 9.17]). This is actually equivalent. The same holds if we replace "for some factorization of $Y \to Z$ as $Y \to V \to Z$" by "for any factorization of $Y \to Z$ as $Y \to V \to Z$" ([10, Chapter II, Lemma 9.15]).

The same notion holds for *homotopy cocartesian.* As a particular case, consider the diagram

$$\begin{array}{ccc} & & \star \\ & & \downarrow \\ Y & \xrightarrow[f]{} & Z \end{array}$$

(in the category of pointed sheaves). We've seen that $\pi : PZ \to Z$ is a fibration and that the map $PZ \to \star$ is a weak equivalence. It follows that $\star \to PZ$ (corresponding to the choice of the base point of Z) is also a weak equivalence (and a cofibration). Thus, we may replace $\star \to Z$ in the diagram above with $\pi : PZ \to Z$ and get a homotopy Cartesian square

$$\begin{array}{ccc} F & \longrightarrow & PZ \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow[f]{} & Z. \end{array}$$

We say that F is the *homotopy fiber* of the map $f : Y \to Z$ (we may as well assume that f is pointed). We will consider the sequence

$$F \to Y \xrightarrow{f} Z$$

which is a model of a fiber sequence. The essential property of this type of sequence is a long exact sequence of homotopy sheaves, that we now introduce.

Definition 2.14. Let (T, t) be a pointed simplicial sheaf of sets. We denote by $\pi_i^s(T, t)$ the Nisnevich sheaf associated to the presheaf

$$X \mapsto \mathrm{Hom}_{\mathcal{H}_{s,\bullet}(k)}(S^i \wedge X_+, T).$$

If we have a sequence

$$F \to Y \xrightarrow{f} Z$$

as above, we get a long exact sequence of homotopy sheaves

$$\ldots \to \pi_i^s(F, y) \to \pi_i^s(Y, y) \to \pi_i^s(Z, f(y)) \to \pi_{i-1}^s(F, y) \to \ldots$$

where $\pi_i^s(\_, \_)$ is an abelian group provided $i \geq 2$, a group if $i = 1$ and a pointed set if $i = 0$. This follows from the fact that the diagram

$$\begin{array}{ccc} F & \longrightarrow & PZ \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow[f]{} & Z. \end{array}$$

is homotopy Cartesian, the general properties of model categories and the fact that sheafification is exact. Here, recall that a sequence of pointed sets (X_i, x_i) and groups G_i

$$G_1 \xrightarrow{h_1} G_0 \xrightarrow{h_0} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0$$

is exact if

- any $x \in X_1$ mapping to the base point x_0 of X_0 comes from X_2.
- There is an action of G_0 on X_2 such that we have $h_0(g) = g \cdot x_2$ and for any y, z with $f_2(y) = f_2(z)$ there exists $g \in G_0$ with $g \cdot y = z$.
- any $g \in G_0$ acting trivially on the base point of X_2 comes from G_1.

2.5. **Eilenberg-MacLane spaces.** We will need later the notion of Eilenberg MacLane space. We give a quick recollection, following [14, Chapter I, §1]. To a sheaf of simplicial abelian groups A, one may associate a normalized complex of sheaves of abelian groups, denoted $N(A)$ ([10, Chapter III,§2]). This construction is obviously functorial in A and then one obtains a functor from the category of simplicial sheaves of abelian groups to the category of complexes of sheaves (of abelian groups) lying in positive degrees. This functor admits a right adjoint Γ ([10, Chapter III, Corollary 2.3]), which is actually an equivalence of categories, known as the Dold-Kan correspondence. More generally, one may consider the category of bounded below complexes of sheaves of abelian groups and define Γ in an analogue way (this time, it is not an equivalence). The functor Γ has the very useful consequence to allow to define useful spaces.

Definition 2.15. Let A be a sheaf of abelian groups and let $n \in \mathbb{N}$. The space $\Gamma(A[n])$ is called the Eilenberg-MacLane space associated to the pair (A, n). Here, $A[n]$ denotes the complex whose terms are all zero, except in degree n, where it is equal to A. We denote $\Gamma(A[n])$ by $K(A, n)$.

Now the Dold-Kan correspondence can be derived. More precisely, one endows the category of (bounded below) chain complexes of sheaves of abelian groups with the structure of a model category. The weak-equivalences are just the quasi-isomorphisms (while the other classes are described for instance in [10, Chapter III, §2]). Then, we obtain an adjunction at the level of the homotopy categories and it follows readily that the Eilenberg-MacLane space satisfy the following useful property. For any smooth scheme X, we have

$$\mathrm{Hom}_{\mathcal{H}_s(k)}(X, K(A,n)) = \mathrm{Hom}_{\mathcal{H}_{s,\bullet}(k)}(X_+, K(A,n)) = \mathrm{H}^n_{\mathrm{Nis}}(X, A).$$

One can extend the definition of Eilenberg-MacLane spaces to bounded below complexes, but we won't need this fact here. Arguably, the importance of these spaces lies in the Postnikov tower associated to a space, which allows to compute the set of maps up to homotopy using a (generalized) spectral sequence.

2.6. **Postnikov tower.** Let (X, x) be a pointed space. We assume first that X is simply connected, in the sense that $\pi_i^s(X, x) = *$ for $i = 0, 1$. The Postnikov tower is a sequence of pointed spaces $X^{(i)}$ (for $i \geq 0$), together with pointed morphisms for any $i \geq 0$

$$p_i : X \to X^{(i)}$$

and

$$f_i : X^{(i+1)} \to X^{(i)}$$

satisfying the following properties:

(1) $f_i p_{i+1} = p_i$ for any $i \geq 0$.
(2) $\pi_j^s(X^{(i)}) = 0$ if $j > i$.
(3) The morphism p_i induces an isomorphism $\pi_j^s(X, x) \to \pi_j^s(X^{(i)})$ for $j \leq i$.
(4) The morphism f_i is a fibration with homotopy fiber $K(\pi_{i+1}^s(X, x), i + 1)$.
(5) Further, the fibration f_i is principal in the sense that there is a morphism

$$k_{i+1} : X^{(i)} \to K(\pi_{i+1}^s(X, x), i + 2)$$

whose homotopy fiber is $X^{(i+1)}$.
(6) The morphism from X to the homotopy colimit of the $X^{(i)}$ is a weak-equivalence.

It is certainly not clear a priori that such a tower exists. We'll give a few arguments an references for this construction, but we start with a few easy observations. Since X is simply connected, then we can take $X^{(i)} = *$ for $i = 0, 1$. Second, we may of course assume that X is fibrant (the tower having anyway properties related to the homotopy sheaves of X). Then, the sections $X(U)$ for any smooth scheme U form a fibrant simplicial set. One may apply the procedure of [10, Chapter VI, §2] to get a fibrant simplicial set $X(U)^{(i)}$ and, since the construction is functorial a simplicial presheaf $U \mapsto X(U)^{(i)}$. Taking the associated sheaf and a fibrant replacement of the latter, we find a simplicial sheaf $X^{(i)}$. It remains to check that it satisfies the expected properties. Property (1) is clear from the construction, while the properties on homotopy sheaves follow from the fact that $X^{(i)}$ is locally weak-equivalent to the presheaf defined above, and that the presheaf $X(U)^{(i)}$ satisfies the conditions by construction. For (5), observe that there is a morphism of simplicial sets

$$k_{i+1}(U) : X(U)^{(i)} \to K(\pi_{i+1}(X(U), x), i+2)$$

for any smooth U whose homotopy fiber is actually $X(U)^{(i+1)}$ ([10, Corollary 5.3]). Now, there is a morphism of abelian groups $\pi_{i+1}(X(U), x) \to \pi^s_{i+1}(X, x)(U)$ (corresponding to sheafification) and we obtain a composite

$$X(U)^{(i)} \to K(\pi_{i+1}(X(U), x), i+2) \to K(\pi^s_{i+1}(X, x)(U), i+2)$$

which factors through $X^{(i)}(U)$ by the universal property of the sheafification. We then have a morphism of sheaves

$$k_{i+1} : X^{(i)} \to K(\pi^s_{i+1}(X, x), i+2)$$

which we can suppose to be a fibration between fibrant objects. Its homotopy fiber is locally weak-equivalent to the sheafification of $X(U)^{(i+1)}$ and the claim follows. Surprisingly, the last point is actually the delicate one. Its proof can be found in [14, Chapter I, Theorem 1.37].

One can extend the definition of the Postnikov tower to a space which is not simply connected, but only connected. We start by observing that the construction of [10, Chapter VI, §2] provides spaces $X^{(i)}$ satisfying conditions (1)-(4) above. The same applies for (6) and the only notable difference is (5). Now, $X^{(1)}$ has a single homotopy sheaf in degree 1, namely $\pi^s_i(X^{(1)}) = 0$ if $i \neq 1$ and $\pi^s_1(X^{(1)}) = \pi^s_1(X, x)$. We denote $X^{(1)}$ by $B\pi_1$ and observe that it is the classifying space of a sheaf of groups (see the next sections for more on classifying spaces). In particular, there is an action of $\pi^s_1(X, x)$ on $B\pi_1$ and also on $\pi^s_i(X, x)$ for $i \geq 1$. This allows to construct an action on $K(\pi^s_i(X, x), i+2)$ and one may consider the space

$$K^{\pi_1}(\pi^s_i(X, x), i+2) := E\pi_1 \times_{\pi^s_1(X,x)} K(\pi^s_i(X, x), i+2).$$

The theorem in this situation is the following.

Theorem 2.16. *Let (X, x) be a pointed space which is connected. For any $i \geq 0$, there are pointed spaces $X^{(i)}$, morphisms $p_i : X \to X^{(i)}$ and $f_i : X^{(i+1)} \to X^{(i)}$ such that the following conditions are satisfied:*

(1) $f_i p_{i+1} = p_i$ for any $i \geq 0$.
(2) $\pi^s_j(X^{(i)}) = 0$ if $j > i$.
(3) The morphism p_i induces an isomorphism $\pi^s_j(X, x) \to \pi^s_j(X^{(i)})$ for $j \leq i$.
(4) The morphism f_i is a fibration with homotopy fiber $K(\pi^s_{i+1}(X, x), i+1)$.

(5) The morphism from X to the homotopy colimit of the $X^{(i)}$ is a weak-equivalence.

Further, the fibration f_i is a twisted principal fibration in the sense that there is a morphism

$$k_{i+1} : X^{(i)} \to K^{\pi_1}(\pi_i^s(X,x), i+2)$$

and a homotopy Cartesian square

$$\begin{array}{ccc} X^{(i+1)} & \longrightarrow & B\pi_1 \\ {\scriptstyle f_i}\downarrow & & \downarrow \\ X^{(i)} & \xrightarrow[k_{i+1}]{} & K^{\pi_1}(\pi_i^s(X,x), i+2) \end{array}$$

The proof can be obtained as in the previous situation, using this time the more general situation on simplicial sets as described in [10, Chapter VI, §5].

2.7. **Using Postnikov towers.** The Postnikov tower of a pointed (connected) space (X,x) is precisely designed to be able to construct a (pointed) morphism $f:(Y,y)\to(X,x)$ stage by stage. Suppose first that (X,x) is simply connected. To construct a morphism from (Y,y) to (X,x), we can start by constructing a pointed morphism $Y \to X^{(i)}$ for some I. Under our assumption, $X^{(i)} = *$ for $i=0,1$ and the problem is particularly simple for such i. Suppose then that we have a morphism

$$g^{(i)} : Y \to X^{(i)}.$$

On the other hand, we have a fiber sequence

$$X^{(i+1)} \to X^{(i)} \stackrel{k_{i+1}}{\to} K(\pi_{i+1}(X,x), i+2)$$

and the general theory of fiber sequences ([11, Proposition 6.5.3]) shows that $g^{(i)}$ lifts to a morphism $g^{(i+1)}: Y \to X^{(i+1)}$ if and only if the composite $k_{i+1}g^{(i)}$ is null homotopic. Now, $\mathrm{Hom}_{\mathcal{H}_{s,\bullet}}(Y, K(\pi_{i+1}(X,x), i+2))$ is equal to $\mathrm{H}^{i+2}_{\mathrm{Nis}}(Y, \pi_{i+1}(X,x))$, a group that is supposedly computable. Thus, there is a cohomological way of deciding if $g^{(i)}$ can be lifted to a morphism $g^{(i+1)}$. Moreover, we can as well parametrize the different choices of $g^{(i+1)}$. Indeed, the general properties of fiber sequences say that two different choices of $g^{(i+1)}$ differ from an element of the group $\mathrm{Hom}_{\mathcal{H}_{s,\bullet}}(Y, K(\pi_{i+1}(X,x), i+1))$. In particular, if both of the above cohomology groups are trivial, then there is a unique choice of $g^{(i+1)}$ given a map $g^{(i)}$.

Let's now come back to the more general situation where (X,x) is connected, but not necessarily simply connected. Suppose that we are given a map $g^{(i)} : Y \to X^{(i)}$. Then the homotopy Cartesian square

$$\begin{array}{ccc} X^{(i+1)} & \longrightarrow & B\pi_1 \\ {\scriptstyle f_i}\downarrow & & \downarrow \\ X^{(i)} & \xrightarrow[k_{i+1}]{} & K^{\pi_1}(\pi_i^s(X,x), i+2) \end{array}$$

shows that there exists a lift $g^{(i+1)} : Y \to X^{(i+1)}$ if and only if $k_{i+1}g^{(i)}$ lifts to a map $Y \to B\pi_1$. Note that there is a projection morphism

$$K^{\pi_1}(\pi_i^s(X,x), i+2) \to B\pi_1$$

and that the indicated morphism $B\pi_1 \to K^{\pi_1}(\pi_i^s(X,x), i+2)$ is a section of this projection. Thus, we exactly know what the potential lift should be. In some sense, the lifting problem is the same as the previous one, but in the category of spaces above $B\pi_1$. We may compare this to the discussion after Corollary 1.3 for motivation.

2.8. **The motivic homotopy category.** We finally introduce the motivic homotopy category, which will be our playground in the next section. This category (introduced by Morel and Voevodsky) is a very nice place for the general study of homotopy invariant cohomology theories over smooth schemes. The general idea is start from the simplicial homotopy category studied above and force the affine line to be contractible. This can be done using the general process of Bousfield localization and we follow here the exposition of [14, §3.2].

Definition 2.17. A sheaf of simplicial sets X is $\mathbb{A}^1$-local if for any simplicial sheaf Y, the map

$$\mathrm{Hom}_{\mathcal{H}_s(k)}(Y,X) \to \mathrm{Hom}_{\mathcal{H}_s(k)}(Y\times\mathbb{A}^1,X)$$

induced by the projection $Y\times\mathbb{A}^1 \to Y$ is a bijection. Further, a morphism a simplicial sheaves $Y \to Z$ is an $\mathbb{A}^1$-weak equivalence if for any $\mathbb{A}^1$-local object X, the map

$$\mathrm{Hom}_{\mathcal{H}_s(k)}(Z,X) \to \mathrm{Hom}_{\mathcal{H}_s(k)}(Y,X)$$

is a bijection.

Example 2.18. For any smooth scheme X, the projection $X\times\mathbb{A}^1 \to X$ is an $\mathbb{A}^1$-weak equivalence. More generally, suppose that we have morphisms of smooth schemes $f : X \to Y$ and $g : Y \to X$ such that both $g\circ f$ and $f\circ g$ are naively homotopic to identity, i.e. there exists (for $g\circ f$, and similarly for $f\circ g$) a morphism $F : X\times\mathbb{A}^1 \to X$ such that $F(0) = \mathrm{Id}$ and $F(1) = g\circ f$. Then, both f and g are $\mathbb{A}^1$-weak equivalences.

One can endow the category of simplicial sheaves with a proper model structure in which the weak equivalences are the $\mathbb{A}^1$-weak equivalences, the cofibrations are monomorphisms and the fibrations have the right lifting property with respect to trivial cofibrations (i.e. cofibrations that are also $\mathbb{A}^1$-weak equivalences ([14, Theorem 3.2])). These fibrations are called $\mathbb{A}^1$-fibrations and we note that they are in particular fibrations in the simplicial model category as weak equivalences are particular cases of $\mathbb{A}^1$-weak equivalences. We denote by $\mathcal{H}_{\mathbb{A}^1}(k)$ the homotopy category of this model category, and call it the *motivic homotopy category.* The category $\mathcal{H}_{\mathbb{A}^1}(k)$ has the following useful property. Let $\mathcal{H}_{s,\mathbb{A}^1}(k) \subset \mathcal{H}_s(k)$ be the full subcategory of $\mathbb{A}^1$-local objects. This functor has a left adjoint $L_{\mathbb{A}^1}$ identifying $\mathcal{H}_{s,\mathbb{A}^1}(k)$ with $\mathcal{H}_{\mathbb{A}^1}(k)$. As in any model category, we have the notion of fibre sequence, homotopy pull-backs, etc... Of particular importance for us is the following definition.

Definition 2.19. Let S be pointed a simplicial sheaf. For any $i \geq 0$, we define

$$\pi_i^{\mathbb{A}^1}(S) := \pi_i^s(L_{\mathbb{A}^1}(S)).$$

Equivalently, $\pi_i^{\mathbb{A}^1}(S)$ is the Nisnevich sheaf associated to the presheaf

$$U \mapsto \mathrm{Hom}_{\mathcal{H}_{\mathbb{A}^1,\bullet}(k)}(S^i \wedge U_+, S).$$

Remark 2.20. A version of the Whitehead theorem holds in $\mathcal{H}_{\mathbb{A}^1}(k)$ ([14, §3, Proposition 2.14]). More precisely, let $f : S \to T$ be a morphism of (pointed) simplicial sheaves. Suppose that $\pi_0^{\mathbb{A}^1}(S) = \pi_0^{\mathbb{A}^1}(T) = *$. Then f is an $\mathbb{A}^1$-weak equivalence if and only if it induces an isomorphism $\pi_i^{\mathbb{A}^1}(S) \to \pi_i^{\mathbb{A}^1}(T)$ for any $i \geq 0$ (actually $i > 0$).

Having these definitions at hand, we can now repeat the constructions of the previous sections (for $\mathcal{H}_s(k)$) in the category $\mathcal{H}_{\mathbb{A}^1}(k)$. In particular, we have the notion of homotopy Cartesian square and of fibre sequences. More precisely, a sequence

$$F \to E \to B$$

is an $\mathbb{A}^1$-fiber sequence if it is homotopy cartesian, i.e. if F is $\mathbb{A}^1$-weak equivalent to the actual fiber of a morphism $E' \to B$ where $E \to E' \to B$ is a factorization of the original morphism into a trivial cofibration followed by an $\mathbb{A}^1$-fibration. As before, a fibre sequence induces a long exact sequence of the associated homotopy sheaves.

2.9. **Convenient $\mathbb{A}^1$-fibrant replacements.** As usual, the problem with the motivic homotopy category is to get a grasp to $\mathbb{A}^1$-fibrations, and in particular to $\mathbb{A}^1$-fibrant objects. By [14, §2, Proposition 2.28], a fibrant simplicial sheaf S is $\mathbb{A}^1$-local if and only if it is $\mathbb{A}^1$-fibrant. This suggests then the following procedure for finding an $\mathbb{A}^1$-fibrant replacement of a simplicial sheaf X. First, take a fibrant replacement in the category of simplicial sheaves, then force this suitable replacement to be $\mathbb{A}^1$-local.

As a first approximation to the second step, we may consider the following construction, called the $\mathbb{A}^1$-singular construction. For any $n \in \mathbb{N}$, let $\Delta(k)_n := \operatorname{Spec}(k[t_0, \ldots, t_n]/\sum t_i = 1)$. We can form a cosimplicial object $\Delta(k)_\bullet$ by looking at the face maps induced by $\Delta(k)_n \to \Delta(k)_{n+1}$ sending t_i to 0 (and reindexing the others) and the degeneracies induced by $\Delta(k)_{n+1} \to \Delta(k)_n$ sending t_i to $t_i + t_{i+1}$ (and reindexing the others).

For any simplicial sheaf S, let $\operatorname{Sing}_\bullet^{\mathbb{A}^1}(S)$ be the simplicial sheaf defined by

$$\operatorname{Sing}_\bullet^{\mathbb{A}^1}(S)(U)_n = S_n(U \times \Delta(k)_n)$$

and obvious face and degeneracy maps. Note that this construction is obviously functorial in S and therefore defines an endofunctor of the category of simplicial sheaves.

There is a natural transformation $\operatorname{Id} \to \operatorname{Sing}_\bullet^{\mathbb{A}^1}(\_)$ (induced by the choice of a base point of $\Delta(k)_\bullet$) which is obviously a cofibration when evaluated to any simplicial sheaf S (it is a monomorphism term by term). It is moreover an $\mathbb{A}^1$-weak equivalence ([14, §3, Corollary 3.8]) (additionally this construction preserves $\mathbb{A}^1$-fibrations). The problem is that it is not clear (and not true in general) that $\operatorname{Sing}_\bullet^{\mathbb{A}^1}(S)$ is $\mathbb{A}^1$-local! In fact, one has to iterate the construction above infinitely many times to get a suitable $\mathbb{A}^1$-fibrant replacement ([14, §2, Lemma 3.12])

Definition 2.21. A simplicial sheaf S is said to satisfy Nisnevich excision (or the B.G. property) if the diagram

$$\begin{array}{ccc} S(X) & \longrightarrow & S(V) \\ \downarrow & & \downarrow \\ S(U) & \longrightarrow & S(W) \end{array}$$

is homotopy cartesian for any elementary Nisnevich square

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow[i]{} & X. \end{array}$$

Remark 2.22. Note that the diagram

$$\begin{array}{ccc} S(X) & \longrightarrow & S(V) \\ \downarrow & & \downarrow \\ S(U) & \longrightarrow & S(W) \end{array}$$

is Cartesian (since S is a sheaf). However, it is not homotopy Cartesian in general. However, suppose that S is fibrant. Then, it follows from [14, Lemma 1.8(2)] that if $U \subset X$ is an open subscheme, then $S(X) \to S(U)$ is a fibration of simplicial sets. Consequently, fibrant simplicial sheaves satisfy Nisnevich excision.

The main use of Nisnevich excision as above is the fact that if a simplicial sheaf S satisfies Nisnevich excision, and $S \to S_{fib}$ is a fibrant replacement of S then we have that $S(X) \to S_{fib}(X)$ is a weak-equivalence for any smooth X ([14, Proposition 1.16]). Next, we turn to homotopy invariance properties.

Definition 2.23. A simplicial (pre)sheaf S is *homotopy invariant* if for any smooth scheme X, the map

$$S(X) \to S(X \times \mathbb{A}^1)$$

is a weak equivalence (of simplicial sets).

For instance, $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(S)$ is $\mathbb{A}^1$-invariant for any simplicial sheaf S ([14, §3, Corollary 3.5]). The following theorem (which is basically due to Schlichting and formalized by Asok-Hoyois-Wendt) provides a convenient way to approximate an $\mathbb{A}^1$-fibrant replacement.

Theorem 2.24. *Let S be a simplicial sheaf satisfying Nisnevich (or Zariski) excision. Suppose that $\pi_0(S)$ is homotopy invariant. Then $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(S)$ also satisfies Nisnevich (Zariski) excision.*

Proof. See [4, Theorem 4.2.3] □

Remark 2.25. Here, one has to understand $\pi_0(S)$ in the naive sense, i.e. $\pi_0(S)(U) = \pi_0(S(U))$.

Corollary 2.26. *Let S be a simplicial presheaf satisfying the conditions of the above theorem. Then,*

$$\mathrm{Hom}_{\mathcal{H}_{\mathbb{A}^1}(k)}(X, S) = \mathrm{Hom}_{\mathcal{H}_{\mathbb{A}^1}(k)}(X, \mathrm{Sing}_\bullet^{\mathbb{A}^1}(S)) = \mathrm{Hom}_{\mathcal{H}_s(k)}(X, \mathrm{Sing}_\bullet^{\mathbb{A}^1}(S)) = \pi_0(\mathrm{Sing}_\bullet^{\mathbb{A}^1}(S)(X)).$$

Proof. Consider the (simplicially) fibrant replacement

$$\mathrm{Sing}_\bullet^{\mathbb{A}^1}(S) \to \mathrm{Sing}_\bullet^{\mathbb{A}^1}(S)_{fib}$$

If X is any smooth scheme, the above theorem shows that we have a weak-equivalence of simplicial sets

$$\mathrm{Sing}_\bullet^{\mathbb{A}^1}(S)(X) \to \mathrm{Sing}_\bullet^{\mathbb{A}^1}(S)_{fib}(X)$$

On the other hand, we know that the map $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(S)(X) \to \mathrm{Sing}_\bullet^{\mathbb{A}^1}(S)(X \times \mathbb{A}^1)$ is a weak-equivalence, and it follows that the map

$$\mathrm{Sing}_\bullet^{\mathbb{A}^1}(S)_{fib}(X) \to \mathrm{Sing}_\bullet^{\mathbb{A}^1}(S)_{fib}(X \times \mathbb{A}^1)$$

is a weak-equivalence for any smooth X. It follows then from [14, §2, Proposition 3.19] that $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(S)_{fib}$ is $\mathbb{A}^1$-local. Finally, we have already seen that the morphism

$$S \to \mathrm{Sing}_\bullet^{\mathbb{A}^1}(S)$$

is an $\mathbb{A}^1$-weak equivalence. □

3. Lecture 3

In this section, we put everything together to actually construct a method to "compute" $\mathcal{V}_n(X)$ for any $n \in \mathbb{N}$ and any X smooth. We start with the construction of the relevant classifying space.

3.1. **Classifying spaces.** For a smooth scheme X, we consider the category $\mathrm{Vect}_n(X)$ whose objects are vector bundles of rank n over X, and whose morphisms are isomorphisms of vector bundles. This form an essentially small category, and we may consider its classifying space $\mathbf{B}\mathrm{Vect}_n(X)$, which is the simplicial set

$$\Delta \to \mathrm{SSets}$$

with $\underline{i} \mapsto \mathrm{Hom}(\underline{i}, \mathrm{Vect}_n(X))$ and obvious morphisms. If $f : Y \to X$, we have $f^* : \mathrm{Vect}_n(X) \to \mathrm{Vect}_n(Y)$. This assignment is not strictly functorial, but one can slightly modify the definition to obtain a functor ([4, §5]), i.e we get a presheaf $\mathbf{B}\mathrm{Vect}_n$ of simplicial sets on Sm_k defined by

$$X \mapsto \mathbf{B}\mathrm{Vect}_n(X)$$

Using Quillen's theorem B (e.g. [20]) and gluing, we see that $\mathbf{B}\mathrm{Vect}_n$ satisfies Nisnevich excision. On the other hand, this sheaf is *not* homotopy invariant for a general smooth scheme X (think to $X = \mathbb{P}^1$ for instance). So, we can't conclude that $\mathbf{B}\mathrm{Vect}_n$ is globally weak-equivalent to its $\mathbb{A}^1$-fibrant replacement. However, the same kind of arguments work (using [4, Theorem 5.1.3]) to prove that

$$\mathrm{Hom}_{\mathcal{H}_{\mathbb{A}^1}(k)}(X, \mathbf{B}\mathrm{Vect}_n) = \pi_0(\mathbf{B}\mathrm{Vect}_n(X)) = \mathcal{V}_n(X)$$

for any smooth *affine* scheme X. So, the functor $X \mapsto \mathcal{V}_n(X)$ is representable in $\mathcal{H}_{\mathbb{A}^1}(k)$ by $\mathbf{B}\mathrm{Vect}_n$.

Remark 3.1. Observe that we have constructed a presheaf that represent the functor we are interested with. However, we defined the spaces to be simplicial sheaves of sets, and not presheaves. Let $\mathcal{X}$ be the simplicial sheaf associated to the presheaf $\mathbf{B}\mathrm{Vect}_n$ and let $\mathcal{X}_{fib}$ be a simplicially fibrant resolution of $\mathcal{X}$. Then the maps $\mathbf{B}\mathrm{Vect}_n \to \mathcal{X} \to \mathcal{X}_{fib}$ are all weak-equivalences. Now, $\mathbf{B}\mathrm{Vect}_n$ satisfies Nisnevich excision, and it turns out that the above composite is a weak-equivalence on sections. Thus, we also get a simplicial sheaf representing $\mathcal{V}_n$.

We can find equivalent formulations as follows. Let $GL_n(X)$ be the usual group. It can be seen as a category with one object and morphisms the elements of the group. Thus, we have an obvious map of simplicial sets $\mathbf{B}GL_n(X) \to \mathbf{B}\mathrm{Vect}_n(X)$ that can be turned into a functor, and thus to a morphism of simplicial presheaves

$$\mathbf{B}GL_n \to \mathbf{B}\mathrm{Vect}_n.$$

This is obviously a local weak-equivalence (as the underlying categories are equivalent when evaluated at local rings), and it follows that the simplicial sheaf BGL_n also represents the functor $\mathcal{V}_n$. One can also use the infinite Grassmannian as a representative ([4, Theorem 5.2.3]).

Note as a variant that we can also consider the category $\mathrm{SVect}_n(X)$ of *oriented* vector bundles of rank n, i.e. vector bundles V of rank n over X equipped with an isomorphism $\varphi : \det V \to \mathcal{O}_X$. The morphisms are just given by isomorphisms of vector bundles respecting the respective trivializations of the determinant. The same discussion as above leads to a representability result for the functor $X \mapsto S\mathcal{V}_n(X)$, the set of isomorphism classes of oriented vector bundles of rank n. Indeed, we have

$$\mathrm{Hom}_{\mathcal{H}_{\mathbb{A}^1}(k)}(X, \mathbf{B}SL_n) = S\mathcal{V}_n(X).$$

If we want to classify vector bundles of rank n that are oriented, we may use the Postnikov tower as described in Section 2.6. More precisely, we use the following theorem.

Theorem 3.2. *Let (X, x) be a pointed space which is $\mathbb{A}^1$-simply connected. For any $i \geq 0$, there are pointed spaces $X^{(i)}$, morphisms $p_i : X \to X^{(i)}$ and $f_i : X^{(i+1)} \to X^{(i)}$ such that the following conditions are satisfied:*

1. *$f_i p_{i+1} = p_i$ for any $i \geq 0$.*
2. *$\pi_j^{\mathbb{A}^1}(X^{(i)}) = 0$ if $j > i$.*
3. *The morphism p_i induces an isomorphism $\pi_j^{\mathbb{A}^1}(X, x) \to \pi_j^{\mathbb{A}^1}(X^{(i)})$ for $j \leq i$.*
4. *The morphism f_i is a fibration with homotopy fiber $K(\pi_{i+1}^{\mathbb{A}^1}(X, x), i + 1)$.*
5. *The morphism from X to the homotopy colimit of the $X^{(i)}$ is a weak-equivalence.*

Further, the fibration f_i is a principal fibration in the sense that there is a morphism

$$k_{i+1} : X^{(i)} \to K(\pi_{i+1}^{\mathbb{A}^1}(X, x), i + 2)$$

and an $\mathbb{A}^1$-fiber sequence

$$X^{(i+1)} \xrightarrow{f_i} X^{(i)} \xrightarrow{k_{i+1}} K(\pi_{i+1}^{\mathbb{A}^1}(X, x), i + 2)$$

The construction of the Postnikov tower is the same as the one we described in Section 2.6. Indeed, we start with an $\mathbb{A}^1$-fibrant replacement of (X, x) and work our way using the same constructions. However, one has to use that the spaces $K(\pi_{i+1}^{\mathbb{A}^1}(X, x), n)$ are $\mathbb{A}^1$-local for any $i \geq 0$. This amounts to saying that the cohomology groups of X with coefficients in $\pi_{i+1}^{\mathbb{A}^1}(X, x)$ are all homotopy invariant. This is a hard theorem due to Morel ([13, §6]). In any case, we see (repeating the process described in Section 2.7) that we need to compute the homotopy sheaves of BSL_2 if we want to classify rank 2 bundles. If X is a threefold, note however that we don't have to compute too much of the relevant homotopy sheaves, due to Theorem 2.3. Indeed, as $\mathrm{H}^i_{\mathrm{Nis}}(X, A) = 0$ for any Nisnevich sheaf A and $i > 3$, we are left with $\pi_i^{\mathbb{A}^1}(\mathbf{B}SL_2)$ for $i = 0, 1, 2, 3$. Note additionally that we have to prove that $\pi_i^{\mathbb{A}^1}(\mathbf{B}SL_2)$ is trivial for $i=0,1$ in order even to use the Postnikov tower as above.

3.2. **The relevant homotopy sheaves.** To compute homotopy sheaves, one usually uses fiber sequences. In general, it is not easy to find such sequences. However, there is now a well understood procedure to get a few of them ([3, Theorem 2.2.5]). In particular, we get a fiber sequence

$$SL_2 \to * \to \mathbf{B}SL_2$$

Showing that $\pi_i^{\mathbb{A}^1}(\mathbf{B}SL_2) = \pi_{i-1}^{\mathbb{A}^1}(SL_2)$ for any $i \geq 1$. By the above result, we know that

$$\mathrm{Hom}_{\mathcal{H}_{\mathbb{A}^1}(k)}(X, \mathbf{B}SL_n) = S\mathcal{V}_n(X).$$

for any smooth scheme X, and in particular this set is trivial for local X. Thus $\pi_0^{\mathbb{A}^1}(\mathbf{B}SL_2) = *$. Next, we can use [3, Theorem 2.2.4] to compute $\mathrm{Hom}_{\mathcal{H}_{\mathbb{A}^1}(k)}(X, SL_2)$ as the quotient of $SL_2(X)$ by the subgroup $H_2(X)$ of matrices which are homotopic to identity. The subgroup $E_2(X)$ generated by elementary matrices is contained in $H_2(X)$ and for R local we have $SL_2(R)/E_2(R) = 1$. It follows that the sheaf associated to

$$X \mapsto \mathrm{Hom}_{\mathcal{H}_{\mathbb{A}^1}(k)}(X, SL_2)$$

is trivial i.e. that $\pi_0^{\mathbb{A}^1}(SL_2) = \pi_1^{\mathbb{A}^1}(\mathbf{B}SL_2) = *$. All in all, we are then in position to use the theorem on the existence of the Postnikov tower of $\mathbf{B}SL_2$.

We now pass to $\pi_2^{\mathbb{A}^1}(\mathbf{B}SL_2) = \pi_1^{\mathbb{A}^1}(SL_2)$. The projection to the first row map $SL_2 \to \mathbb{A}^2\backslash 0$ is easily seen to be Zariski locally of the form $U\times\mathbb{A}^1 \to U$ and it follows that this projection is actually an $\mathbb{A}^1$-weak equivalence. Therefore $\pi_1^{\mathbb{A}^1}(SL_2) = \pi_1^{\mathbb{A}^1}(\mathbb{A}^2 \setminus 0)$ and we are reduced to computing some of the homotopy sheaves of the sphere $\mathbb{A}^2 \setminus 0$. Let us introduce a sheaf (whose definition is due to Morel) that we will use in the computation.

Let R be a ring, and let $R^\times$ be the set of invertible elements in R. Let $\mathrm{K}_*^{\mathrm{MW}}(R)$ be the quotient of the free associative (unital) $\mathbb{Z}$-graded ($\mathbb{Z}$-)algebra generated by symbols $[a]$ in degree 1 and a symbol η in degree -1 by the ideal generated by the following relations:

(1) $[a][1-a] = 0$ for any $a \in R^\times \setminus \{1\}$.
(2) $[ab] = [a] + [b] + \eta[a][b]$ for any $a, b \in R^\times$.
(3) $\eta[a] = [a]\eta$ for any $a \in R^\times$.
(4) $\eta(\eta[-1] + 2) = 0$.

We get a presheaf on Sm_k defined by

$$X \mapsto \mathrm{K}_*^{\mathrm{MW}}(\mathcal{O}_X(X)).$$

The associated Nisnevich sheaf is denoted by $\mathbf{K}_*^{\mathrm{MW}}$ and is called the Milnor-Witt K-theory sheaf. Note that it is a sheaf of graded rings. The following theorem describes the homotopy sheaves of spheres of low degrees ([13, Theorems 5.38 and 5.40]).

Theorem 3.3. *For any $n \geq 2$, we have*

$$\pi_i^{\mathbb{A}^1}(\mathbb{A}^n \setminus 0) = \begin{cases} 0 & \text{if } i \leq n-2. \\ \mathbf{K}_n^{\mathrm{MW}} & \text{if } i = n-1. \end{cases}$$

Remark 3.4. In fact, the definition of the sheaf $\mathbf{K}_n^{\mathrm{MW}}$ given by Morel is different from the one we gave above. The fact that the two definitions coincide follows from [9, Theorem 6.3].

Remark 3.5. We can also compute the homotopy sheaves of $\mathbb{A}^n \setminus 0$ if $n = 1$. We get $\pi_i^{\mathbb{A}^1}(\mathbb{A}^n \setminus 0) = 0$ if $i > 0$ and $\pi_0^{\mathbb{A}^1}(\mathbb{A}^1 \setminus 0) = \mathbb{G}_m$.

Given the above theorem, we are left with the computation of $\pi_2^{\mathbb{A}^1}(\mathbb{A}^2 \setminus 0)$. The computation is the main object of [2], and requires the introduction of yet another sheaf. For any smooth scheme X, one may define (in analogy with K-theory) the groups $K_i^{\mathrm{Sp}}(X)$ which (roughly speaking) classify symplectic vector bundles over X. This construction is functorial in X and we obtain a presheaf $X \mapsto K_3^{\mathrm{Sp}}(X)$ whose associated Nisnevich sheaf is denotes by $\mathbf{K}_3^{\mathrm{Sp}}$. The following theorem is a simplified (yet sufficient for our purpose) form of [2, Theorem 3.1].

Theorem 3.6. *Let k be an infinite perfect field of characteristic different from 2. Then, we have an exact sequence of sheaves*

$$\mathbf{K}_4^{\mathrm{MW}}/6(\eta[-1]+2) \to \pi_2^{\mathbb{A}^1}(\mathbb{A}^2 \setminus 0) \to \mathbf{K}_3^{\mathrm{Sp}} \to 0.$$

We are now in position to prove the main theorem of this course.

Theorem 3.7. *Let X be a smooth affine threefold over an algebraically closed field of characteristic different from 2. Then the second Chern class induces a bijection*

$$c_2 : S\mathcal{V}_2(X) \to \mathrm{CH}^2(X)$$

Proof. The general process of Postnikov tower shows that we have to prove two statements. The first one is

$$\mathrm{H}^2(X, \mathbf{K}_2^{\mathrm{MW}}) = \mathrm{CH}^2(X)$$

and the map $\mathbf{B}SL_2 \to K(\mathbf{K}_2^{\mathrm{MW}}, 2)$ is the second Chern class. The second one is $\mathrm{H}^3(\pi_2^{\mathbb{A}^1}(X, \mathbb{A}^2 \setminus 0)) = 0$. This is obtained in [2, Theorem 6.6]. □

Remark 3.8. In fact, the above theorem has a stronger form (developed in [2, Theorem 6.6]), showing that for such a threefold X, the first and second Chern class induce a bijection

$$(c_1, c_2) : \mathcal{V}_2(X) \to \mathrm{CH}^1(X) \times \mathrm{CH}^2(X)$$

The proof requires the refined notion of the Postnikov tower developed in Section 2.6, as well as a computation of the action of $\pi_1^{\mathbb{A}^1}(\mathbf{B}GL_2) = \mathbb{G}_m$ on the sheaves $\pi_i^{\mathbb{A}^1}(\mathbf{B}GL_2)$ for $i = 2, 3$.

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