Quantization from higher genus curves

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Deformation quantization

- R: finite type $\mathbb{C}$-algebra, "base ring".
- A: finite type R-algebra.
- Poisson bracket on A over R: $\{-,-\}: A \otimes_R A \to A$, R-linear Lie bracket, derivation with respect to both entries.
- Example: (Z, Ω_Z) (Ω_Z: non-degenerate algebraic closed 2-form) algebraic symplectic over $\mathbb{C}$, then $A := H^0(Z, \mathcal{O}_Z)$ has a natural structure of Poisson algebra over $\mathbb{C}$:

$$\{f, g\} := \Omega_X^{-1}(df, dg)\,.$$

Definition

A deformation quantization of $(A, \{,\})$ over R is a structure of (non-commutative) associative $R[[\hbar]]$-algebra on $A_\hbar := A \otimes_R R[[\hbar]]$ such that $A_\hbar \mod \hbar = A$ and for every $f, g \in A$

$$[f, g]_{A_\hbar} = \{f, g\}_A \hbar + \mathcal{O}(\hbar^2)\,.$$

Deformation quantization

- Question: given a Poisson algebra $(A, \{-,-\})$ over R, is there a deformation quantization of $(A, \{-,-\})$ over R?
- No in complete generality (Mathieu).
- Yes (Yekutieli) if $R = \mathbb{C}$ and A is smooth over R (follows Kontsevich's result on deformation quantization of the algebra of smooth functions on C^∞ Poisson manifolds).
- Question (open in general): under which conditions on some possibly non-smooth A does it still exist some deformation quantization?
- Question (open in general): in some cases, can we construct a deformation quantization "better" than formal in $\hbar$, e.g. convergent in $\hbar$.
- Purely algebraic questions which seem to have nothing to do with algebraic curves . . .

Mirror symmetry

- Something which has to do with genus zero curves: mirror symmetry.
- Unexpected way to construct algebraic varieties: start with an algebraic variety, do some enumerative geometry (count rational curves), get infinitely numbers, use these numbers to produce a "mirror variety".
- The fact that infinitely numbers, coming from solving infinitely many enumerative problems, contain exactly the information necessary to reconstruct an algebraic variety is absolutely non-trivial: these numbers have to satisfy some "miraculous properties".
- A concrete mirror symmetry construction for log Calabi-Yau surfaces (Gross-Hacking-Keel).

Mirror symmetry

- U: smooth affine surface, $U = Y - D$, Y smooth projective surface, D singular reduced normal crossing divisor. In particular, U is a non-compact Calabi-Yau (algebraic symplectic) surface.
- Example: Y smooth cubic surface in $\mathbb{P}^3$, D triangle of lines (see Simpson's minicourse), more character varieties ...
- Enumerative geometry: counts rational curves in Y touching D in a unique point.
- GHK construction: get a flat "mirror" family $\pi: X \to \operatorname{Spec} R$, $R = \mathbb{C}[NE(Y)]$, affine $A := H^0(X, \mathcal{O}_X)$. Generic fiber: smooth affine algebraic symplectic surfaces. In general, π has some singular fibers. A always comes with a natural Poisson algebra structure over R.

Mirror symmetry

- More precisely, A is obtained by gluing together families of algebraic tori $(\mathbb{C}^*)^2 \times \operatorname{Spec} R$ by gluing functions determined by generating series of te counts of rational curves.
- "Miraculous properties" of the counts of rational curves to make the construction work? Use tropical geometry (Mikhalkin, Nishinou-Siebert correspondence theorem).
- U: complement of a triangle of lines in a cubic surface. Get for X the universal family of cubic surfaces with a triangle of lines (including singular cubics). (U: "symplectic" side of the mirror, only the deformation equivalence class matters, X: "complex" side of the mirror, precise complex structure matters).

Cubic with a triangle of lines

See Simpson's minicourse.

$$xyz+x^2+y^2+z^2-4-x(ab+cd)-y(ad+bc)-z(ac+bd)+abcd+a^2+b^2+c^2+d^2=0$$

Non-trivial Poisson brackets:

$$\{x,y\} = xy + 2z - (ac+bd)\,,$$
$$\{y,z\} = yz + 2x - (ab+cd)\,,$$
$$\{z,x\} = zx + 2y - (ad+bc)\,.$$

Quantization from higher genus curves

Question: is there a deformation quantization of $(A := H^0(X, \mathcal{O}_X), \{-,-\})$ over R?

Theorem (Bousseau, 2017)

There exists a deformation quantization $A_\hbar$ of A over R which is algebraic in $q = e^\hbar$ and constructed from the enumerative geometry of higher genus curves in U.

- Key idea: deform the GHK construction by counting higher genus curves in Y touching D in a unique point ("counting" is much more delicate than in genus zero, non-trivial thing to do to extract numbers from a non-trivial moduli space, log Gromov-Witten theory of the non-compact Calabi-Yau 3-fold $Y \times \mathbb{C}^*$).
- Glue together non-commutative tori using generating series as gluing functions, where $\hbar$ is the parameter keeping track of the genus $(\sum_g N_g \hbar^{2g-1})$.

Quantization from higher genus curves

- Theorem known in some special cases, e.g. for the family of cubics (Oblomkov) using representation theory (Cherednik double affine Hecke algebras) or the realization as character variety. Our construction is more general and independent of some representation theoretic/modular meaning of the geometry. (Alternative viewpoint: new unexpected connection between enumerative geometry of higher genus curves and highly non-trivial representation theoretic structures).
- "Miraculous properties" of the higher genus counts to make the construction work? (and to get the algebraic dependence in $q = e^{\hbar}$). Tropical geometry, use a new correspondence theorem (Bousseau, 1706.07762) (of independent interest), combined with a work of Filippini-Stoppa.

Quantum cubic with a triangle of lines

Non-commutative algebra:

$$q^{-\frac{1}{2}}xy - q^{\frac{1}{2}}yx = (q - q^{-1})z - (q^{\frac{1}{2}} - q^{-\frac{1}{2}})(ac + bd)\,,$$

$$q^{-\frac{1}{2}}yz - q^{\frac{1}{2}}zy = (q - q^{-1})x - (q^{\frac{1}{2}} - q^{-\frac{1}{2}})(ab + cd)\,,$$

$$q^{-\frac{1}{2}}zx - q^{\frac{1}{2}}xz = (q - q^{-1})y - (q^{\frac{1}{2}} - q^{-\frac{1}{2}})(ad + bc)\,,$$

$$q^{-\frac{1}{2}}xyz + q^{-1}(x^2+y^2+z^2) - (q^{\frac{1}{2}}+q^{-\frac{1}{2}})^2 - q^{-\frac{1}{2}}(x(ab+cd)+y(ad+bc)+z(ac+bd))$$
$$+abcd + a^2 + b^2 + c^2 + d^2 = 0\,.$$

Conclusion

- Surprising connection between quantization and enumerative geometry of higher genus curves: a purely algebraic question is solved by counting higher genus curves in the mirror geometry.
- After translation: one of the few known examples of mirror symmetry “at all genus”.
- After translation: non-trivial mathematical check of the “string theory interpretation of Chern-Simons theory” predicted by Witten.
- Expectation (not completely obvious): similar story working in much greater generalities (K3 surfaces, holomorphic symplectic varieties of higher dimensions ...). Part of a much bigger story involving DT theory of 3-Calabi-Yau categories (Kontsevich-Soibelman) ...

Thank you for your attention!