Subvector Inference in Partially Identified Models with Many Moment Inequalities

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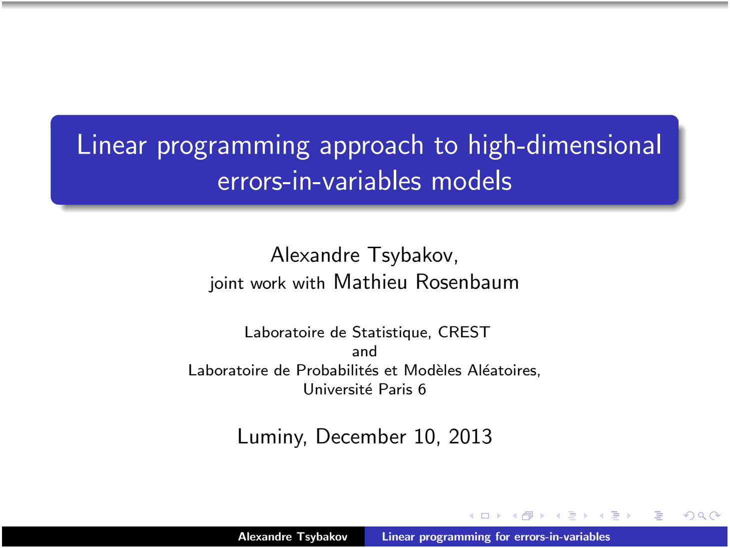
joint work with Federico Bugni (Duke) and Victor Chernozhukov (MIT)

Meeting in Mathematical Statistics
CIRM, December 20th, 2017

Happy Birthday

Luminy, December 12, 2013

Happy Birthday



(Mathieu: "Did you get the slides I sent? ... [send] my best to Sacha, I am feeling bad not having been able to make it to Luminy!")

Introduction

- There is a large literature on inference in partially identified (PI) models defined by moment (in)equalities.

- We consider a model characterized by (θ^*, F)
  - $\theta^* \in \mathbb{R}^{d_\theta}$ is a finite dimensional parameter of interest,
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- The *main prediction* of the model is that the true parameter θ^* satisfies

$$\begin{aligned} \mathbb{E}[m_j(W,\theta)] &\leqslant 0, \quad \text{for } j = 1, \ldots, p_I, \\ \mathbb{E}[m_j(W,\theta)] &= 0, \quad \text{for } j = p_I + 1, \ldots, p_I + p_E. \end{aligned} \tag{1}$$

- key issue: θ^* is *not* assumed to be point identified, i.e., given F, there might be a **set** of θ that satisfy (1).

$$\Theta_I \equiv \left\{ \theta \in \Theta \text{ s.t. } \begin{array}{l} \mathbb{E}[m_j(W,\theta)] \leqslant 0 \text{ for } j = 1, \ldots, p_I \\ \mathbb{E}[m_j(W,\theta)] = 0 \text{ for } j = p_I + 1, \ldots, p_I + p_E \end{array} \right\}.$$

Motivating Examples

Interval-Outcome Linear Regression (e.g., Manski and Tamer 2002)

- let Y_i^* denote a latent dependent variable

$$Y_i^* = X_i'\theta^* + \varepsilon_i, \quad \mathbb{E}[\varepsilon_i \mid X_i] = 0 \text{ a.s.}$$

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- We only observe an interval s.t. $Y_i \in [Y_i^l, Y_i^u]$ which leads to

$$\mathbb{E}[X_i'\theta^* - Y_i^u \mid X_i] \leqslant 0$$
$$\mathbb{E}[Y_i^l - X_i'\theta^* \mid X_i] \leqslant 0$$

- We could use

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$$\mathbb{E}[(Y_i^l - X_i'\theta^*)X_{ij}1\{X_{ij} \geqslant 0\}] \leqslant 0$$
$$\mathbb{E}[-(X_i'\theta^* - Y_i^u)X_{ij}1\{X_{ij} \leqslant 0\}] \leqslant 0$$
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Discrete Choice Model with Multiple Equilibria (Ciliberto and Tamer, 2009)

- m firms play an entry game (Nash Equilibrium) on n independent markets
- On each market, a firm makes an entry decision $d_j \in \{0, 1\}$

$$\text{profit function} \quad \pi_j(d_j, d_{-j}, X, \varepsilon, \theta^*)$$

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- There are set-valued functions R_1, R_2 such that
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- Conclude that

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- If conditional distribution of ε given X is known (up to a subvector of θ_0), we can calculate numerically right-hand sides of both inequalities
- we have 2^{m+1} moment inequalities for each value of $X \in \mathcal{X}$ (a discrete set).

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- ▷ We formally deal with unconditional moments
  but conditional moments can be approximated via

$$\mathbb{E}[m_j(W,\theta^*) \mid z_i] \leqslant 0 \Rightarrow \mathbb{E}[m_j(W,\theta^*)1\{z_i \in [a,b]\}] \leqslant 0 \ \text{ for all } [a,b]$$

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- We are not interested on θ^* but on $h(\theta^*)$ for a *known* fn. $h : \Theta \to \Lambda$. This is the problem addressed in this paper:

  Hypothesis test (HT): For fixed h_0, we want to test:

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- Main application: *Subvector inference*: For $\theta^* \in \Theta \subset \mathbb{R}^{d_\theta}$, $d_\theta > 1$,

$$H_0 \ : \ \theta_1^* = h_0 \quad \text{vs.} \quad H_1 \ : \ \theta_1^* \neq h_0.$$

  $\Rightarrow$ Special case of Eq. (3) with $h(\theta) = \theta_1$ and $h_0 \in \Lambda \subseteq \mathbb{R}$.

Literature review

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- **Testing unconditional moment inequalities:** Chernozhukov, Hong, and Tamer (2007), Romano and Shaikh (2008), Andrews and Guggenberger (2009), Andrews and Soares (2010), Canay (2010), Bugni (2011), Andrews and Jia Barwick (2012), Romano, Shaikh, and Wolf (2012)
- **Testing conditional moment inequalities:** Andrews and Shi (2013), Chernozhukov, Lee, and Rosen (2013), Armstrong (2011), Chetverikov (2011), Armstrong and Chan (2012)

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In both cases, the number of moments p is fixed (explicitly or due to the structure)

Testing unconditional moment inequalities with $p \to \infty$

- Menzel (2014), where $p \ll n$
- Chernozhukov, Chetverikov and Kato (WP 2013), where $p \gg n$

Literature review: subvector

Asymptotically uniformly valid inference for

$$H_0 : h(\theta^*) = h_0 \ \text{ vs. } \ H_1 : h(\theta^*) \neq h_0$$

- **Projections of CS:** Project usual CS for θ onto space of H_0. Considered by Andrews et. al. (2009, 10).
  - *Related work improving projections:*
    Kaido, Molinari & Stoye (WP, 2015), Gafarov (affine models, WP 2017)
- **Subsampling:** Profile the criterion function and approximate critical value with subsampling. Proposed by Romano & Shaikh (2008, 10).
- **Project the Criterion Function**: Bugni, Canay and Shi (2017)

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In all cases, the number of moments inequalities p **is fixed** and asymptotic analysis (i.e., based on the limiting distribution of the process)

Positioning in the literature

	Donsker (e.g. p fixed)	non-Donsker (e.g. p growing)
Vector Inference	Chernozhukov et al (2007) Romano and Shaikh (2008) Andrews and Guggenberger (2009) Andrews and Soares (2010) ...	Menzel (2014, $p \ll n$) Chernozhukov et al (WP 2013, $p \gg n$)
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Starting point:

- ▷ the minimum resampling critical value in Bugni, Canay and Shi (2017); and
- ▷ CLTs for the max of high-dim vectors used in Chernozhukov et al (WP 2013)

Setting and Contributions

Profiled test statistics for $H_0 : h(\theta^*) = h_0$ vs. $H_1 : h(\theta^*) \neq h_0$

$$T_n(h_0) = \inf_{\theta \in \Theta(h_0)} \max_{j \in [p]} \sqrt{n} \bar{m}_{\theta,j} / \widehat{\sigma}_{\theta,j}$$

where $\Theta(h_0) = h^{-1}(h_0) = \{\theta \in \Theta : h(\theta) = h_0\}$ and

$$\bar{m}_{\theta,j} = \frac{1}{n} \sum_{i=1}^{n} m_j(W_i, \theta) \quad \text{and} \quad \widehat{\sigma}^2_{\theta,j} = \frac{1}{n} \sum_{i=1}^{n} \{m_j(W_i, \theta) - \bar{m}_{\theta,j}\}^2$$

The test: reject if $T_n(h_0) > c_n(h_0, 1-\alpha)$ where $c_n(h_0, 1-\alpha)$ is a critical value.

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Our contribution is to construct critical values $c_n(h_0, 1-\alpha)$ that

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- uniformly controls asymptotic size over a large class of dgps ($F \in \mathcal{P}_n$)
- in the presence of many moment inequalities ($p \to \infty$ as $n \to \infty$)
- allow for $p \gg n$ (also $d_\theta \to \infty$ but not clear if empirically relevant)
- finite sample analysis, and rate for size error (e.g. polynomially in n)
- towards data-driven choice of parameters

Overview of Proposals

Profiled test statistics for $H_0 : h(\theta^*) = h_0$ vs. $H_1 : h(\theta^*) \neq h_0$

$$T_n(h_0) = \inf_{\theta \in \Theta(h_0)} \max_{j \in [p]} \sqrt{n} \bar{m}_{\theta,j} / \widehat{\sigma}_{\theta,j}$$

We consider different methods to calculate the critical value $c_n(h_0, 1-\alpha)$:

▷ Self-Normalized method (not covering today)

- fast
- works under very weak conditions
- potentially conservative

▷ Bootstrap-based methods

- slower (requires simulations)
- requires stronger conditions
- but less conservative

▷ Hybrids are possible (not covering today)

- potentially useful to speed up bootstrap-based methods

Proposal via Bootstrap-based methods

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$$T_n(h_0) = \inf_{\theta \in \Theta(h_0)} \max_{j \in [p]} \quad \widehat{v}_{\theta,j} + \sqrt{n}\mathbb{E}[m_j(W,\theta)]/\widehat{\sigma}_{\theta,j}$$

bootstrap: $\widehat{v}^*_{\theta,j} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i \frac{m_j(W_i,\theta) - \bar{m}_{\theta,j}}{\widehat{\sigma}_{\theta,j}}$ where ξ_i's are i.i.d. $N(0,1)$.

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bootstrap: $\widehat{v}^*_{\theta,j} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i \frac{m_j(W_i,\theta) - \bar{m}_{\theta,j}}{\widehat{\sigma}_{\theta,j}}$ where ξ_i's are i.i.d. $N(0,1)$.

Remark: It has been shown that although we can suitably approximate

$$\widehat{v}_{\theta,j} \text{ by } \widehat{v}^*_{\theta,j}$$

Proposal via Bootstrap-based methods

Profiled test statistics for $H_0 : h(\theta^*) = h_0$ vs. $H_1 : h(\theta^*) \neq h_0$

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Andrews and Soares (2010) show it is more delicate to approximate

$$\sqrt{n}\mathbb{E}[m_j(W,\theta)]/\widehat{\sigma}_{\theta,j} \quad \text{by} \quad \sqrt{n}\bar{m}_{\theta,j}/\widehat{\sigma}_{\theta,j}$$

as there is a non-vanishing noise due to the scaling by $\sqrt{n}$.

Standard Bootstrap-based methods via GMS

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A standard way to proceed is to use Generalized Moment Selection (GMS)

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for a tuning parameter $\kappa_n \to \infty$

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for a tuning parameter $\kappa_n \to \infty$ (recommendation $\sim \{\log n\}^{1/2}$ when p is fixed)

Then set

$$T_n^{GMS*}(h_0) := \inf_{\theta \in \Theta(h_0)} \max_{j \in [p]} \quad \widehat{v}^*_{\theta,j} + \varphi_{\theta,j}$$

and compute the critical values based on the quantile of $T_n^{GMS*}(h_0)$.

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Example: $d_\theta = 2$, and $\Theta = [-1, 1]^2$. Let $p = 2$, and consider

$$\begin{aligned} \mathbb{E}[m_1(W_i, \theta)] &= \mathbb{E}[\theta_1 + \theta_2 - W_{i,1}] \leqslant 0 \\ \mathbb{E}[m_2(W_i, \theta)] &= \mathbb{E}[W_{i,2} - \theta_1 - \theta_2] \leqslant 0 \end{aligned}$$

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In turn, for GMS, using $\kappa_n = \sqrt{\log n}$ we select both inequalities whp and

$$T_n^{GMS*}(0) \mid (W_i)_{i=1}^n \approx \inf_{-1 \leqslant \theta_2 \leqslant 1} \max \{-Z_1 + \varphi_{\theta_2,1}, Z_2 + \varphi_{\theta_2,2}\}$$

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Critical values based on $T_n^{GMS*}(0)$ fail to control size.

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Critical values based on $T_n^{GMS*}(0)$ fail to control size. Indeed, for $\alpha = 0.1$

- $c_n^{GMS}(0, 1-\alpha) \approx 0.5$ and $P(T_n(0) > c_n^{GMS}(0, 1-\alpha)) \approx 0.24$.
- $c_n(0, 1-\alpha) \approx 0.86$
- GMS quantiles are "too" small

Bootstrap-based methods for subvector inference

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1) **"Discard Resampling" (DR):**

$$T_n^{DR*}(h_0) \equiv \inf_{\theta \in \widehat{\Theta}_I(h_0)} \max_{j \in [p]} \widehat{v}^*_{\theta,j} + \varphi_{\theta,j}$$

where $\widehat{\Theta}_I(h_0) \subseteq$ "$\arg\min$"$_{\theta \in \Theta(h_0)} \max_{j \in [p]} \sqrt{n}\bar{m}_{\theta,j}/\widehat{\sigma}_{\theta,j}$

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2) **"Penalized Resampling" (PR):**

$$T_n^{PR*}(h_0) \equiv \inf_{\theta \in \Theta(h_0)} \max_{j \in [p]} \quad \widehat{v}^*_{\theta,j} + \kappa_n^{-1}\sqrt{n}\bar{m}_{\theta,j}/\widehat{\sigma}_{\theta,j},$$

where $\kappa_n \geqslant 1$ is a penalty parameter

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3) **"Minimum Resampling" (MR):**

$$T_n^{MR*}(h_0) \equiv \min\{T_n^{DR*}(h_0), T_n^{PR*}(h_0)\}$$

The impact of many moment inequalities, $p \gg n$

- ▷ lack of a Donsker property for the whole process $\{v_{\theta,j} : \theta \in \Theta(h_0), j \in [p]\}$
  - no limiting distributions guaranteed to exist
  - cannot invoke Donsker's functional CLT to establish the convergence in distribution of $T_n(h_0)$

- ▷ restriction on the criterion functions
  - we use $Q(\theta) = \max_{j\in[p]} \sqrt{n}\bar{m}_{\theta,j}/\widehat{\sigma}_{\theta,j}$
  - do not use (MMM): $Q(\theta) = \sum_{j=1}^{p}\{\sqrt{n}\bar{m}_{\theta,j}/\widehat{\sigma}_{\theta,j}\}_+^2$
  - do not use (AQLR): $Q(\theta) = \min_{t\in\mathbb{R}^p}(\sqrt{n}\bar{m}_{\theta,j}/\widehat{\sigma}_{\theta,j} - t)'\widetilde{\Sigma}_\theta^{-1}(\sqrt{n}\bar{m}_{\theta,j}/\widehat{\sigma}_{\theta,j} - t)$

- ▷ tuning parameters need to account for growing entropy

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(iii) Polynomial Minorant condition away from the identified set

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- *with constants* $\bar{A}$ *and* $v \geqslant 1$ *and envelope* F *(i.e.* $F(W) \geqslant |\widetilde{m}_j(W,\theta)|$*)*
- *for some* $b > 0$, $q \geqslant 4$, *we have*

$$Ep[F^q]^{1/q} \leqslant b \ \ \textit{and} \ \ \mathbb{E}[|\widetilde{m}_j(W,\theta)|^k] \leqslant b^{k-2}, \quad k = 3, 4$$

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Define $\gamma = o(1)$, in particular $\gamma \ll \alpha$

- ▷ $\bar{w}_n = (1 - \gamma)$-quantile of $\sup_{\theta \in \Theta(h_0), j \in [p]} |\widehat{v}^*_{\theta, j}|$
- ▷ $K_n = v \log(n\bar{A}b) + d_\theta \log(nb) + \log p$

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Condition M. *The following conditions hold:*
(i) set $\Theta(h_0)$ *is convex and* $\sup_{\theta\in\Theta(h_0)}\|\theta\|_\infty \leqslant C\sqrt{n}$
(ii) $\{\widetilde{m}_j(\cdot,\theta) : \theta \in \Theta(h_0), j \in [p]\}$ *is VC type class of functions*

- *with constants* $\bar{A}$ *and* $v \geqslant 1$ *and envelope* F *(i.e.* $F(W) \geqslant |\widetilde{m}_j(W,\theta)|$*)*
- *for some* $b > 0$*,* $q \geqslant 4$*, we have*

$$Ep[F^q]^{1/q} \leqslant b \;\; \textit{and} \;\; \mathbb{E}[|\widetilde{m}_j(W,\theta)|^k] \leqslant b^{k-2}, \quad k = 3, 4$$

- $\mathbb{E}[\{\widetilde{m}_j(W,\theta) - \widetilde{m}_j(W,\tilde{\theta})\}^2] \leqslant L_C\|\theta - \tilde{\theta}\|^\chi$ *for some* $\chi \geqslant 1$.
- $\max_{j\in[p]}\|\nabla_\theta \mathbb{E}[\widetilde{m}_j(W,\theta)]\| \leqslant L_G$ *for every* $\theta \in \Theta(h_0)$

(iii) For every $\theta \in \Theta(h_0) \setminus \Theta_I$ *we have*

$$\max_{j\in[p]} \mathbb{E}\left[\widetilde{m}_j\left(W,\theta\right)\right] \geqslant \vartheta_n \min\left\{\delta, \inf_{\tilde{\theta}\in\Theta(h_0)\cap\Theta_I} \|\theta - \tilde{\theta}\|\right\}$$

Define $\gamma = o(1)$, in particular $\gamma \ll \alpha$ (e.g., $\gamma = n^{-c}$ for some $c > 0$)

▷ $\bar{w}_n = (1-\gamma)$-quantile of $\sup_{\theta\in\Theta(h_0),j\in[p]}|\widehat{v}^*_{\theta,j}| \lesssim \sqrt{d_\theta \log(pn)}$

▷ $K_n = v\log(n\bar{A}b) + d_\theta \log(nb) + \log p \quad \lesssim d_\theta \log(pn)$

Rates for Size Control

Rates for Size Control for Discard Resampling

$$T_n^{DR*}(h_0) \equiv \inf_{\theta \in \widehat{\Theta}_I(h_0)} \max_{j \in [p]} \widehat{v}^*_{\theta,j} + \varphi_{\theta,j}$$

where $\widehat{\Theta}_I(h_0) \subseteq$ "arg min"$_{\theta \in \Theta(h_0)} \max_{j \in [p]} \sqrt{n}\bar{m}_{\theta,j}/\widehat{\sigma}_{\theta,j}$

$$\varphi_{\theta,j} = \begin{cases} 0, & \text{if } \sqrt{n}\bar{m}_{\theta,j}/\widehat{\sigma}_{\theta,j} \geqslant \max_{\ell \in [p]} \sqrt{n}\bar{m}_{\theta,\ell}/\widehat{\sigma}_{\theta,\ell} - \kappa_n, \\ -\infty, & \text{otherwise} \quad \text{(i.e., inequality will not be used)} \end{cases}$$

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Issues to address:

- no functional min max CLT since $p \to \infty$ (and potentially $d_\theta \to \infty$)
- handle random set $\widehat{\Theta}_I(h_0)$
- handle random selection of inequalities
- penalty parameter κ_n

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Theorem (Simplified)

Suppose Condition M is satisfied with $d_\theta + L_G/\vartheta_n \leqslant C$ and that H_0 holds. Then

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$$\delta'_{n,\gamma} \lesssim \frac{\log^{2/3}(np)}{\gamma^{1/3} n^{1/6}}$$

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for some $0 < c < 1/6$

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Theorem

Assume that Condition M is satisfied and that H_0 holds. Then

$$P(T_n(h_0) \geqslant t) \leqslant P(T_n^{DS*}(h_0) \geqslant t - C\delta'_{n,\gamma}) + C\{\gamma + n^{-1}\}$$

where we have

$$\delta'_{n,\gamma} \quad := \frac{CbK_n}{\gamma^{3/q} n^{1/2}} + \frac{C(bK_n^2)^{1/3}}{\gamma^{1/3} n^{1/6}} + CL_C^{1/2} \left(\frac{CK_n^{1/2}}{\gamma^{1/q} n^{1/2} \vartheta_n} \right)^{\chi/2} \frac{K_n^{1/2}}{\gamma^{1/q}} + \frac{CbK_n}{\gamma^{1/q} n^{1/2 - 1/q}}$$

provided that $\kappa_n \geqslant \bar{w}_n\{6 + 2L_G/\vartheta_n\}$

Rates for Size Control for Penalized Resampling

$$T_n^{PR*}(h_0) = \inf_{\theta \in \Theta(h_0)} \max_{j \in [p]} \widehat{v}^*_{\theta,j} + \kappa_n^{-1} \sqrt{n} \bar{m}_{\theta,j} / \widehat{\sigma}_{\theta,j}$$

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Issues to address:

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Theorem (Simplified)

Suppose Condition M is satisfied with $d_\theta + L_G/\vartheta_n \leqslant C$, $\chi = 2$, and that H_0 holds. Then

$$P(T_n(h_0) \geqslant t) \leqslant P(T_n^{PR*}(h_0) \geqslant t - C\delta''_{n,\gamma}) + C\{\gamma + n^{-1}\}$$

where we have

$$\delta''_{n,\gamma} := \frac{\log^{2/3}(np)}{\gamma^{1/3} n^{1/6}} + \kappa_n \frac{\log^{3/2}(np)}{n^{1/2}} + \frac{\bar{w}_n}{\kappa_n}$$

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$$+ \frac{\bar{w}_n}{\kappa_n} + L_C^{1/2} \left(\frac{\kappa_n K_n^{1/2}}{n^{1/2} \vartheta_n \gamma^{1/q}} \right)^{\chi/2} \frac{K_n^{1/2}}{\gamma^{1/q}}$$

Rates for Size Control for Minimum Resampling

$$T_n^{MR*}(h_0) = \min\{T_n^{DR*}(h_0), T_n^{MR*}(h_0)\}$$

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Issues:

- ▷ note that T_n^{MR*} is also a MinMax statistics
  - as it is the minimum of two MinMax statistics
  - need to handle random set in the minimization
  - need to handle random centering
- ▷ clearly need to couple the statistics (use the same $\xi_i \sim N(0,1)$ for both)
- ▷ no functional CLT for MinMax as $p \to \infty$ (and potentially $d_\theta \to \infty$)

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Theorem

Assume that Condition M is satisfied and that H_0 holds. Then

$$P(T_n(h_0) \geqslant t) \leqslant P(T_n^{MR*}(h_0) \geqslant t - C\delta_{n,\gamma}) + C\{\gamma + n^{-1}\}$$

where we have

$$\delta_{n,\gamma} := \delta'_{n,\gamma} + \delta''_{n,\gamma}$$

Key New Coupling Result

Theorem. Let $X_1, \ldots, X_n$ be independent random matrices in $\mathbb{R}^{N \times p}$ ($Np \geqslant 2$), $Y_1, \ldots, Y_n$ be independent random matrices in $\mathbb{R}^{N \times p}$ with $Y_i \sim N(\mathbb{E}[X_i], \operatorname{Var} X_i)$.

$$\text{Define} \quad T = \min_{k \in [N]} \max_{j \in [p]} \sum_{i=1}^{n} \frac{X_{ikj}}{\sqrt{n}}, \quad \text{and} \quad \widetilde{T} = \min_{k \in [N]} \max_{j \in [p]} \sum_{i=1}^{n} \frac{Y_{ikj}}{\sqrt{n}}$$

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Then for every $\delta > 0$ and every Borel subset A of $\mathbb{R}$ we have

$$P(T \in A) \leqslant P(\widetilde{T} \in A^{C\delta}) + \frac{C \log^2(Np)}{\delta^3 n^{1/2}} \{L_n + M_{n,X}(\delta) + M_{n,Y}(\delta)\}$$

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where C is a universal positive constant, and

$$L_n = \max_{k \in N, j \in [p]} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[|\widetilde{X}_{ikj}|^3],$$

$$M_{n,W}(\delta) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[\max_{k \in N, j \in [p]} |\widetilde{W}_{ikj}|^3 \cdot 1\left\{ \max_{k \in N, j \in [p]} |\widetilde{W}_{ikj}| > \delta\sqrt{n}/\log(Np) \right\} \right],$$

for $\widetilde{W}_i = W_i - \mathbb{E}[W_i]$.

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Proof is based on Stein's method.
(Extends to processes, Empirical Bootstrap as in Deng and Zhang, 2017)
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that satisfies:

$$-\beta^{-1} \log N \leqslant G_\beta(X) - \min_{k \in [N]} \max_{j \in [p]} X_{kj} \leqslant \beta^{-1} \log p$$

$$\|\nabla G_\beta(X)\|_1 \leqslant 1$$

$$\|\nabla^2 G_\beta(X)\|_1 \leqslant 4\beta$$

$$\|\nabla^3 G_\beta(X)\|_1 \leqslant 24\beta^2$$

Key New Coupling Result

LEMMA 11. *Consider* $m(X) = g \circ G_\beta(X)$. *Then we have*

(1) for $(k, j) \in [N] \times [p]$

$$m'_{kj}(X) = g'(G_\beta(X))\pi_k(-F_\beta(X))\pi_j^{\mu_k}(X_{k\cdot})$$

(2) for $(k, j) \in ([N] \times [p])^2$, *we have*

$$\begin{aligned}
m''_{(k,j)}(X) \quad &= g''(G_\beta(X))\pi_{k_2}(-F_\beta(X))\pi_{j_2}^{\mu_{k_2}}(X_{k_2\cdot})\pi_{k_1}(-F_\beta(X))\pi_{j_1}^{\mu_{k_1}}(X_{k_1\cdot}) \\
&-g'(G_\beta(X))\beta w_{k_1k_2}(-F_\beta(X))\pi_{j_2}^{\mu_{k_2}}(X_{k_2\cdot})\pi_{j_1}^{\mu_{k_1}}(X_{k_1\cdot}) \\
&+g'(G_\beta(X))\pi_{k_1}(-F_\beta(X))\delta_{k_1k_2}\beta w_{j_1j_2}^{\mu_{k_1}}(X_{k_1\cdot})
\end{aligned}$$

(3) for $(k, j) \in ([N] \times [p])^3$, *we have*

$$\begin{aligned}
m'''_{(k,j)}(X) \quad &= g'''(G_\beta(X)) \textstyle\prod_{\ell=1}^3 \pi_{k_\ell}(-F_\beta(X))\pi_{j_\ell}^{\bar{\mu}_{k_\ell}}(X_{k_\ell\cdot}) \\
&-g''(G_\beta(X))\beta w_{k_2k_3}(-F_\beta(X))\pi_{k_1}(-F_\beta(X)) \textstyle\prod_{\ell=1}^3 \pi_{j_\ell}^{\bar{\mu}_{k_\ell}}(X_{k_\ell\cdot}) \\
&+g''(G_\beta(X))\pi_{k_2}(-F_\beta(X))\delta_{k_2k_3}\beta w_{j_2j_3}^{\bar{\mu}_{k_2}}(X_{k_2\cdot})\pi_{k_1}(-F_\beta(X))\pi_{j_1}^{\bar{\mu}_{k_1}}(X_{k_1\cdot}) \\
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&+g'(G_\beta(X))\beta^2 q_{k_1k_2k_3}(-F_\beta(X)) \textstyle\prod_{\ell=1}^3 \pi_{j_\ell}^{\bar{\mu}_{k_\ell}}(X_{k_\ell\cdot}) \\
&-g'(G_\beta(X))\beta w_{k_1k_2}(-F_\beta(X))\delta_{k_2k_3}\beta w_{j_2j_3}^{\bar{\mu}_{k_2}}(X_{k_2\cdot})\pi_{j_1}^{\bar{\mu}_{k_1}}(X_{k_1\cdot}) \\
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\end{aligned}$$

Back to the Size Control Bound

We obtained

$$P(T_n(h_0) \geqslant t) \leqslant P(T_n^{MR*}(h_0) \geqslant t - C\delta_{n,\gamma}) + C\{\gamma + n^{-1}\}$$

where we can take $\gamma \to 0$, and $\delta_{n,\gamma} = o(1)$.

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We need to ensure that

$$P(|T_n^{MR*}(h_0) - c_{n,1-\alpha}| \leqslant C\delta_{n,\gamma}) \quad \text{is small}$$

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i.e., that $T_n^{MR*}(h_0)$ does not concentrate too fast around $c_{n,1-\alpha}$ as $p \to \infty$.

Anti-Concentration

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Anti-concentration essentially bounds the probability density function

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Theorem (Chernozhukov, Chetverikov and Kato (2011))

Let $X \in \mathbb{R}^p$ be a vector of Gaussian random variables such that $\mathrm{Var}(X_j) \geqslant 1$. Let $Z = \max_{j \in [p]} X_j$. Then for any $\epsilon > 0$ and $x \in \mathbb{R}$

$$\mathbb{P}\left(|Z - x| \leqslant \epsilon\right) \leqslant C\epsilon\sqrt{\log p}$$

In particular the probability density function of Z satisfies $\max_{t \in \mathbb{R}} f_Z(t) \leqslant C\sqrt{\log p}$

- allows for non-central and arbitrary correlation structure

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That is, $\log^{7/6}(p) = o(n^{1/6})$.

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Lemma

For $X_{kj} \sim N(0,1)$, i.i.d., $k \in [N]$, $j \in [p]$, let

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That is, if $p = N$, for some universal constants $0 < c < C$ we have

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- ▷ suggests anti-concentration of MinMax is quite different from the Max
- ▷ currently only partial results for arbitrary correlation structures

Anti-Concentration

However note that our bounds are

$$P(T_n(h_0) \geqslant t) \leqslant P(T_n^{MR*}(h_0) \geqslant t - C\delta_{n,\gamma}) + C\{\gamma + n^{-1}\}$$

It suffices to control the concentration of the bootstrapped statistics

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We can estimate

$$P(|T_n^{MR*}(h_0) - t| \leqslant 2C\delta_{n,\gamma})$$

via bootstrap for $t = c_n(h_0, 1-\alpha)$ and bound the anti-concentration factor

▷ adaptive to the setting (in contrast to analytical bounds)

▷ can be estimated using the same bootstrap that computed $c_n(h_0, 1-\alpha)$

Let $\mathcal{A}_{1-\alpha}^* := \frac{P(|T_n^{MR*}(h_0)-t| \leqslant 2C\delta_{n,\gamma})}{2C\delta_{n,\gamma}}$ denote the anti-concentration rate.

Examples of Simple Conditions for "Penalized Resampling"

Suppose Condition M, m_j and its derivatives are uniformly bounded, $\sigma_{\theta,j} \geqslant c$, $L_G/\vartheta_n + L_C \leqslant C$. Then, letting $\mathcal{A}^*_{1-\alpha}$ denote the anti-concentration rate, provided

$$\frac{K_n^{2/3}}{n^{1/6}} + \kappa_n \frac{d_\theta^{1/2} K_n}{n^{1/2}} + \frac{\bar{w}}{\kappa_n} = o\left(\frac{1}{\mathcal{A}^*_{1-\alpha}}\right), \tag{4}$$

where $K_n = \log p + d_\theta \log n$, we have $P(T_n(h_0) \geqslant c_n(h_0, 1-\alpha)) \leqslant \alpha + o(1)$

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Remark: we can simulate $\bar{w}$ and bound $\mathcal{A}^*_{1-\alpha}$

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For the traditional setting, e.g., fixed p and d_θ

- ▷ $\mathcal{A}^*_{1-\alpha} \leqslant C$
- ▷ $K_n \leqslant C$
- ▷ $\kappa_n \to \infty$ and $\kappa_n/n^{1/2} \to 0$

Examples of Simple Conditions for "Penalized Resampling"

For non-Donsker cases:

Example (Many inequalities and fixed d_θ)

Let $p = n^C$ for some fixed $C > 1$, $d_\theta \leqslant C$, and the anti-concentration $\mathcal{A}^*_{1-\alpha} \leqslant C \log^{3/2} n$. It suffices $\kappa_n \in [\log^{5/2} n, \quad n^{\frac{1}{2}} \log^{-3} n]$.

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Let $p = n^C$ for some fixed $C > 1$, $d_\theta = n^a$ for some $a < 1/4$, and the anti-concentration $\mathcal{A}^*_{1-\alpha} \leqslant C \log^{3/2} n$. It suffices $\kappa_n \in [n^{a/2} \log^{5/2} n, \quad n^{\frac{1}{2} - \frac{3}{2}a} \log^{-3} n]$.

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Example (Exponentially many inequalities)

Suppose that $d_\theta \leqslant C \log n$, $p \geqslant n^{\log n}$ and the anti-concentration $\mathcal{A}^*_{1-\alpha} \leqslant C \log^{3/2} p$. It suffices $\kappa_n \in [\log^2 p \, \log n, \quad n^{1/2} \log^{-5/2} p \log^{-1} n]$, provided that $n^{-1/6} \log^{13/6} p \, \log n = o(1)$.

Conclusion

- ▷ subvector inference in PI models with many moment restrictions
  - allow for non-Donsker classes
  - finite sample analysis
  - need more than $\kappa_n \to \infty$ and $\kappa_n/\sqrt{n} \to 0$ when $p \to \infty$
  - valid data-driven choice of penalty parameters (via additional bootstrap)
- ▷ new CLTs for $\min_{k\in[N]} \max_{j\in[p]} W_{kj}$
  - results parallel results for $\max_{j\in[p]} W_j$
  - approximation based on composition of smooth maximum (LSE)
- ▷ new anti-concentration pattern
  - does not parallel results for $\max_{j\in[p]} W_j$ (counter example)
  - estimate anti-concentration via bootstrap
- ▷ Future (ongoing) work
  - sharper constants
  - hybrid methods
  - power comparisons
  - analytical bounds for anti-concentration
  - orthogonal moment conditions