Old and new geometric isoperimetric inequalities, Monge–Ampère equation with drifts. Shortcut to applied math: sharp Beckner–Sobolev inequality on Hamming cube

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What follows is a joint work with Paata Ivanisvili.

Theorem

If a real valued function $M(x, y)$ *is such that* $M(x, \sqrt{y}) \in C^2(\Omega \times \mathbb{R}_+)$ *and it satisfies the differential inequalities*

$$\begin{bmatrix} M_{xx} + \frac{M_y}{y} & M_{xy} \\ M_{xy} & M_{yy} \end{bmatrix} \leq 0 \quad \text{and} \quad M_y \leq 0, \tag{1}$$

then for any $f \in C_0^\infty(\mathbb{R}^n;\ \Omega)$ *we have*

$$\int_{\mathbb{R}^n} M(f, \|\nabla f\|) d\gamma \leq M\left(\int_{\mathbb{R}^n} f d\gamma, 0\right). \tag{2}$$

2. Log-Sobolev inequality

$$M(x,y) = x\ln x - \frac{y^2}{2x}, \quad x>0 \quad \text{and} \quad y \ge 0. \tag{3}$$

Notice that $M(x,y)$ satisfies (1). Indeed, $M_y = -\frac{y}{x} \le 0$ and

$$\begin{bmatrix} M_{xx} + \frac{M_y}{y} & M_{xy} \\ M_{xy} & M_{yy} \end{bmatrix} = \begin{bmatrix} -\frac{y^2}{x^3} & \frac{y}{x^2} \\ \frac{y}{x^2} & -\frac{1}{x} \end{bmatrix} \le 0. \tag{4}$$

Log-Sobolev inequality of Gross states that

$$\int_{\mathbb{R}^n} |f|^2 \ln |f|^2 d\gamma - \left(\int_{\mathbb{R}^n} |f|^2 d\gamma\right) \ln \left(\int_{\mathbb{R}^n} |f|^2 d\gamma\right) \le 2\int_{\mathbb{R}^n} \|\nabla f\|^2 d\gamma \tag{5}$$

whenever the right hand side of (5) is well-defined and finite for complex-valued f.

3. Beckner–Sobolev and spectral gap inequality

Beckner:
For $f \in L^2(d\gamma)$ and $1 \leq p \leq 2$ we have

$$\int |f|^p d\gamma - \left(\int |f| d\gamma\right)^p \leq \frac{p(p-1)}{2} \int_{\mathbb{R}^n} f^{p-2} \|\nabla f\|^2 d\gamma \qquad (6)$$

For $p = 2$ this is $\int |f|^2 d\gamma - \left(\int |f| d\gamma\right)^2 \leq \int_{\mathbb{R}^n} \|\nabla f\|^2 d\gamma$. This shows that the spectral gap i.e. the first nontrivial eigenvalue of the self-adjoint positive operator $L = -\Delta + x \cdot \nabla$ in $L^2(\mathbb{R}^n, d\gamma)$ is bounded from below by 1.
$M(x,y) = x^p - \frac{p(p-1)}{2} x^{p-2} y^2$ where $x, y \geq 0$ $1 \leq p \leq 2$. If $q = 2/p$

$$\begin{bmatrix} M_{xx} + \frac{M_y}{y} & M_{xy} \\ M_{xy} & M_{yy} \end{bmatrix} = \begin{bmatrix} -\frac{2(2-q)(1-q)(2-3q)x^{\frac{2}{q}-4}y^2}{q^4} & -\frac{4(2-q)(1-q)x^{\frac{2}{q}-3}y}{q^3} \\ -\frac{4(2-q)(1-q)x^{\frac{2}{q}-3}y}{q^3} & -\frac{4(2-q)x^{\frac{2}{q}-2}}{q^2} \end{bmatrix} \leq 0 \qquad (7)$$

4. Improving Beckner's inequality for $p = 3/2$.

Consider

$$M(x,y) = \frac{1}{\sqrt{2}}\left(2x - \sqrt{x^2+y^2}\right)\sqrt{x+\sqrt{x^2+y^2}} \quad \text{where} \quad x, y \geq 0.$$

We have

$$\begin{pmatrix} M_{xx} + \frac{M_y}{y} & M_{xy} \\ M_{xy} & M_{yy} \end{pmatrix} = \frac{3\sqrt{2}}{8\sqrt{x^2+y^2}} \begin{pmatrix} -\frac{y^2}{(x+\sqrt{x^2+y^2})^{3/2}} & \frac{y}{\sqrt{x+\sqrt{x^2+y^2}}} \\ \frac{y}{\sqrt{x+\sqrt{x^2+y^2}}} & -\sqrt{x+\sqrt{x^2+y^2}} \end{pmatrix} \tag{8}$$

5. Sharper than Beckner–Sobolev inequality.

$$\int_{\mathbb{R}^n} \frac{1}{\sqrt{2}} \left(2f - \sqrt{f^2 + \|\nabla f\|^2}\right) \sqrt{f + \sqrt{f^2 + \|\nabla f\|^2}} d\gamma \leq$$

$$\leq \left(\int_{\mathbb{R}^n} f d\gamma\right)^{3/2}.$$

Notice that

$$x^{3/2} - \frac{3}{8} x^{-1/2} y^2 \leq M(x, y) = \frac{1}{\sqrt{2}} \left(2x - \sqrt{x^2 + y^2}\right) \sqrt{x + \sqrt{x^2 + y^2}}$$

So this inequality is better than the Beckner's one:

$$\int_{\mathbb{R}^n} f^{3/2} d\mu - \frac{3}{8} \int_{\mathbb{R}^n} f^{-1/2} |\nabla f|^2 d\mu \leq \left(\int_{\mathbb{R}^n} f d\gamma\right)^{3/2}.$$

Bobkov:
For a Lipschitz function $f : \mathbb{R}^n \to [0,1]$, we have

$$I\left(\int_{\mathbb{R}^n} f d\gamma\right) \le \int_{\mathbb{R}^n} \sqrt{I^2(f) + \|\nabla f\|^2} d\gamma\,, \tag{9}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2} dx$, and $I(x) := \Phi'(\Phi^{-1}(x))$.
Testing (9) for $f(x) = 1_A$ where A is a Borel subset of $\mathbb{R}^n$ one obtains Gaussian isoperimetry: for any Borel measurable set $A \subset \mathbb{R}^n$

$$\gamma^+(A) \ge \Phi'(\Phi^{-1}(\gamma(A)))\,, \tag{10}$$

where $\gamma^+(A) := \liminf_{\varepsilon \to 0} \frac{\gamma(A_\varepsilon) - \gamma(A)}{\varepsilon}$ denotes Gaussian perimeter of A, here $A_\varepsilon = \{x \in \mathbb{R}^n : \mathrm{dist}_{\mathbb{R}^n}(A, x) < \varepsilon\}$.

$$M(x,y) = -\sqrt{I^2(x)+y^2} \quad \text{where} \quad x \in [0,1], \quad y \geq 0. \tag{11}$$

Then $M_y = \frac{-y}{\sqrt{I^2(x)+y^2}} \leq 0$ and

$$\begin{bmatrix} M_{xx} + \frac{M_y}{y} & M_{xy} \\ M_{xy} & M_{yy} \end{bmatrix} = \begin{bmatrix} -\frac{(I'(x))^2y^2}{(I^2(x)+y^2)^{3/2}} + \frac{I(x)I''(x)+1}{\sqrt{I^2(x)+y^2}} & y\frac{I(x)I'(x)}{(I^2(x)+y^2)^{3/2}} \\ y\frac{I(x)I'(x)}{(I^2(x)+y^2)^{3/2}} & -\frac{I^2(x)}{(I^2(x)+y^2)^{3/2}}. \end{bmatrix} \tag{12}$$

Notice that $I''(x)I(x) = -1$, therefore (12) is negative semidefinite.

In general finding $M(x,y)$ will be based purely on solving PDEs. First notice that in log-Sobolev (5) and in Bobkov's inequality (9) determinant of the matrices (4) and (12) are zero. In Beckner–Sobolev inequality (6) determinant of (7) is zero if and only if $p=1,2$. We will seek $M(x,y)$ among those functions which in addition with (1) also satisfy *Monge–Ampére equation with a drift*:

$$\det \begin{bmatrix} M_{xx}+\frac{M_y}{y} & M_{xy} \\ M_{xy} & M_{yy} \end{bmatrix} = M_{xx}M_{yy} - M_{xy}^2 + \frac{M_y M_{yy}}{y} = 0 \qquad (13)$$

for $(x,y)\in \Omega \times \mathbb{R}_+$.

9. Reduction to the exterior differential systems and backwards heat equation

Let us make the following observation: consider

$$(x, y, p, q) = (x, y, M_x(x, y), M_y(x, y))$$

in *xypq*-space. This is a surface Σ in 4-space on which $\Upsilon = \mathrm{d}x \wedge \mathrm{d}y$ is nonvanishing but to which the two 2-forms

$$\Upsilon_1 = \mathrm{d}p \wedge \mathrm{d}x + \mathrm{d}q \wedge \mathrm{d}y \quad \text{and} \quad \Upsilon_2 = (y\mathrm{d}p + q\mathrm{d}x) \wedge \mathrm{d}q$$

pull back to be zero. Consider a simply connected surface Σ in *xypq*-space (with $y > 0$) on which Υ is nonvanishing but to which Υ_1 and Υ_2 pullback to be zero. The 1-form $p\mathrm{d}x + q\mathrm{d}y$ pull back to Σ to be closed (since Υ_1 vanishes on Σ) and hence exact, and so there exists a function $m : \Sigma \to \mathbb{R}$ such that $\mathrm{d}m = p\mathrm{d}x + q\mathrm{d}y$ on Σ. We then have, $m = M(x, y)$ on Σ and, by its definition, we have $p = M_x(x, y)$ and $q = M_y(x, y)$ on the surface. Υ_2 vanishes when pulled back to Σ implies that $M(x, y)$ satisfies the desired equation (13) of slide 41.

Thus, we have encoded the given PDE as an exterior differential system on $\mathbb{R}^4$. Note, that we can make a change of variables on the open set where $q < 0$: Set $y = qr$ and let $t = \frac{1}{2}q^2$. then, using these new coordinates on this domain, we have

$$\Upsilon_1 = \mathrm{d}p \wedge \mathrm{d}x + \mathrm{d}t \wedge \mathrm{d}r \quad \text{and} \quad \Upsilon_2 = (r\mathrm{d}p + \mathrm{d}x) \wedge \mathrm{d}t.$$

Now, when we take an integral surface Σ on these 2-forms on which $\mathrm{d}p \wedge \mathrm{d}t$ is not vanishing, it can be written locally as a graph of the form

$$(p, t, x, r) = (p, t, u_p(p, t), u_t(p, t))$$

(since Σ is an integral of Υ_1), where $u(p, t)$ satisfies $u_t + u_{pp} = 0$ (since on Σ $0 = \Upsilon_2 = u_t\mathrm{d}p \wedge \mathrm{d}t + \mathrm{d}u_p \wedge \mathrm{d}t = (u_t + u_{pp})\mathrm{d}p \wedge \mathrm{d}t$). Thus, "generically" our PDE is equivalent to the backwards heat equation, up to a change of variables.

11. Parametrization of Bellman function M

Thus the function $M(x,y)$ can be parametrized as follows:

$$x = u_p\left(p, \frac{1}{2}q^2\right); \quad y = q u_t\left(p, \frac{1}{2}q^2\right); \tag{14}$$

$$M(x,y) = p u_p\left(p, \frac{1}{2}q^2\right) + q^2 u_t\left(p, \frac{1}{2}q^2\right) - u\left(p, \frac{1}{2}q^2\right), \tag{15}$$

where

$$u_t + u_{pp} = 0\,.$$

$M(x, \sqrt{y}) \in C^2(\Omega \times \mathbb{R}_+)$ therefore $M_y(x,0) = 0$. By choosing $q = 0$ in (14), we have $y = 0$, and we obtain the boundary condition:

$$x = u_p(p,0) = u_p(M_x(x,0),0)$$

Or, if to denote boundary function $M(x,0)$ by $f(x)$, then u has initial conditions

$$u'(f'(x),0) = x\,.$$

11a. Definiteness of matrix

Non-negativity of matrix also implies one more condition

$$u_t^2 - 2t(Hess\, u) \geq 0\,. \tag{16}$$

Inequality (5) shows us sharp lower bounds of the expression $\left(\int g d\gamma\right) \ln \left(\int g d\gamma\right)$. Therefore, we should take $M(x, 0) = x \ln x$. Boundary condition then can be rewritten as $u'(\ln x + 1, 0) = x$ or $u(p, 0) = e^{p-1}$ for all $p \in \mathbb{R}$. If we set $D = \frac{\partial^2}{\partial p^2}$ then

$$u(p,t) = e^{-tD} e^{p-1} = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} e^{p-1} = e^{p-t-1} \quad \text{for all} \quad t \geq 0.$$

Clearly $u(p, t)$ satisfies (16) because $\det(\mathrm{Hess}\, u) = 0$. Notice that we have $u_t < 0$,

$$\begin{cases} x = e^{p - \frac{q^2}{2} - 1}; \\ y = -q e^{p - \frac{q^2}{2} - 1}; \end{cases} \quad \text{then} \quad \begin{cases} q = -\frac{y}{x}; \\ p = \ln x + \frac{y^2}{2x^2} + 1. \end{cases}$$

Therefore we obtain
$M(x, y) = xp + qy - u\left(p, \frac{1}{2} q^2\right) = x \ln x + \frac{y^2}{2x} + x - \frac{y^2}{x} - x = x \ln x - \frac{y^2}{2x}$.

In this case we are interested for the sharp lower bounds of the expression $-I(\int f d\gamma)$ in terms of $\int M(f, \|\nabla f\|) d\gamma$. We have $M(x, 0) = -I(x)$. Boundary condition takes the form

$$u(p, 0) = p\Phi(p) + \Phi'(p) \quad \text{for all} \quad p \in \mathbb{R}. \tag{17}$$

In fact, $M_x(x, 0) = -I'(x)$ and $-I'(x) = \Phi^{-1}(x)$: $I'(x) = \left[e^{-\frac{[\Phi^{-1}]^2}{2}}\right]'$ and $(\Phi^{-1})' = e^{\frac{[\Phi^{-1}]^2}{2}}$. First: usual heat extension of $u(p, 0)$, $\tilde{u}_{pp} = \tilde{u}_t$, and then we try to consider the formal candidate $u(p, t) := \tilde{u}(p, -t)$. The heat extension of $\Phi'(p) = \frac{1}{\sqrt{2\pi}} e^{-p^2/2}$ is $\frac{1}{\sqrt{2\pi}\sqrt{1+2t}} e^{-\frac{p^2}{2(1+2t)}}$. Heat extension of $\Phi(p)$ is $\Phi\left(\frac{p}{\sqrt{1+2t}}\right)$. Indeed, the heat extension of the function $1_{(-\infty,0]}(p)$ at time $t = 1/2$ is $\Phi(p)$. By the semigroup property the heat extension of $\Phi(p)$ at time t will be the heat extension of $1_{(-\infty,0]}(p)$ at time $1/2 + t$ which equals to $\Phi\left(\frac{p}{\sqrt{1+2t}}\right)$.

Therefore, the heat extension of $p\Phi(p)$ can be found as follows:

$$\frac{2t}{\sqrt{2\pi}\sqrt{1+2t}} e^{-\frac{p^2}{2(1+2t)}} + p\Phi\left(\frac{p}{\sqrt{1+2t}}\right).$$

Thus we obtain that

$$\tilde{u}(p,t) = \sqrt{1+2t}\,\Phi'\left(\frac{p}{\sqrt{1+2t}}\right) + p\Phi\left(\frac{p}{\sqrt{1+2t}}\right).$$

This expression is well defined even for $t \in (0, -1/2)$. Therefore if we set

$$u(p,t) = \tilde{u}(p,-t) = \sqrt{1-2t}\,\Phi'\left(\frac{p}{\sqrt{1-2t}}\right) + p\Phi\left(\frac{p}{\sqrt{1-2t}}\right)$$

for $\quad p \in \mathbb{R}, \quad t \in \left[0, \frac{1}{2}\right)$,

15. Applications: how to find Bobkov's Bellman function

Direct computations show that $u(p,t)$ satisfies $u_t + u_{pp} = 0$, the boundary condition (17) and (16) because

$\det(\mathrm{Hess}\, u) = -\left(\frac{\Phi'(\frac{p}{\sqrt{1-2t}})}{1-2t}\right)^2 < 0$. We have $u_t = -\frac{\Phi'(\frac{p}{\sqrt{1-2t}})}{\sqrt{1-2t}} < 0$ and $u_p = \Phi\left(\frac{p}{\sqrt{1-2t}}\right)$. Therefore,

$$\begin{cases} x = \Phi\left(\frac{p}{\sqrt{1-q^2}}\right); \\ y = qr = qu_t = \frac{-q}{\sqrt{1-q^2}}\Phi'(\frac{p}{\sqrt{1-q^2}}); \end{cases} \quad \text{then} \quad \begin{cases} \Phi^{-1}(x) = \frac{p}{\sqrt{1-q^2}}; \\ y = \frac{-q}{\sqrt{1-q^2}}\Phi'(\Phi^{-1}(x \end{cases}$$

From the last equalities we obtain $M_y = q = -\frac{y}{\sqrt{I^2(x)+y^2}}$ and $M_x = p = \frac{I(x)\Phi^{-1}(x)}{\sqrt{I^2(x)+y^2}}$ where we remind that $I(x) = \Phi'(\Phi^{-1}(x))$. Then it is clear that

$$M(x,y) = -\sqrt{I^2(x)+y^2}.$$

Let $u(p,0) = g(p)$ then condition $u(f'(x),0) = xf'(x) - f(x)$ where $f(x) = M(x,0)$ implies that $g(f'(x)) = xf'(x) - f(x)$. By taking derivative we obtain

$$g'(f'(x)) = x$$

Thus $u_p(p,0)$ is the *inverse* of $M_x(x,0)$ i.e.,

$$M(x,0) = \int (u_p(p,0))^{-1} dp$$

Example of $u(p,0) = -\sin p$:
Then $u(p,t) = -e^t \sin(p)$. Notice that $u_t \le 0$ for $p \in [0,\pi]$, and

$$u_t^2 - 2t \det(\mathrm{Hess}\, u) = e^{2t}(2t + \sin^2(x)) \ge 0.$$

We also notice that

$$M(x,0) = x \arccos(-x) + \sqrt{1-x^2} \quad \text{for} \quad x \in [-1,1]$$

17. Isoperimetric inequalities for all!

The following conditions

$$x = u_p(p, q^2/2); \;\; y = qu_t(p, q^2/2);$$
$$M(x, y) = px + qy - u(p, q^2/2).$$

can be rewritten as follows

$$x = -e^{q^2/2}\cos(p, \;\; y = -qe^{q^2/2}\sin(p)$$
$$M(x, y) = px + qy + e^{q^2/2}\sin(p) = px + qy - \frac{y}{q}, \quad x \in [-1, 1], \; y \geq 0.$$

It follows that the negative number q satisfies the equation

$$-q\sqrt{e^{q^2} - x^2} = y \tag{18}$$

And then $p = \arccos(-xe^{-q^2/2})$. Thus we obtain

$$M(x, y) = x\arccos(-xe^{-q^2/2}) + (1 - q^2)\sqrt{e^{q^2} - x^2}$$

where a negative number q is the unique solution of (18).

18. Isoperimetric inequalities for all!

Thus we obtain that

$$\int_{\mathbb{R}^n} f \arccos(-f\, e^{-F(f,|\nabla f|)/2}) + (1 - F(f,|\nabla f|))\sqrt{e^{F(f,|\nabla f|)} - f^2} d\gamma_n \leq$$

$$\left(\int f\right) \arccos\left(-\int f\right) + \sqrt{1 - \left(\int f\right)^2}$$

for any $f : \mathbb{R}^n \to (-1,1)$ where $F(u,v) > 0$ solves the equation

$$|\nabla f|^2 = F(e^F - f^2)$$

$$\int_{\mathbb{R}^n} f \arccos(-f\, e^{-F(f,|\nabla f|)/2}) + (1 - F(f,|\nabla f|))\sqrt{e^{F(f,|\nabla f|)} - f^2} d\gamma_n \le$$

$$\left(\int f\right) \arccos\left(-\int f\right) + \sqrt{1 - \left(\int f\right)^2}$$

for any $f : \mathbb{R}^n \to (-1, 1)$ where $F(u, v) > 0$ solves the equation

$$|\nabla f|^2 = F(e^F - f^2)$$

This can be rewritten (since $\arccos(-x) = \pi - \arccos(x)$) as follows:where r solves the equation $|\nabla f|^2 = r(e^r - f^2)$

$$\int [(1-r)\sqrt{1 - (fe^{-r/2})^2} - fe^{-r/2} \arccos(f\, e^{-r/2})] e^{r/2} d\gamma \le$$

$$\sqrt{1 - \left(\int f\right)^2} - \left(\int f\right) \arccos\left(\int f\right)$$

20. Jensen's correction. Poincaré inequality follows.

It is very interesting because $\Psi(x) = \sqrt{1-x^2} - x\arccos(x)$ is decreasing convex function on $[-1,1]$ therefore when $r \to 0$ one should expect opposite integral inequality (By Jensen's inequality) however the condition $r \to 0$ enforces $f \approx const$. For example, the inequality can be rewritten as follows

$$\int \Psi(fe^{-r/2})e^{r/2}d\gamma \leq \Psi\left(\int fd\gamma\right) + \int |\nabla f|\sqrt{r}d\gamma.$$

For example if f is positive then $\Psi(fe^{-r/2})e^{r/2} \geq \Psi(f)e^{r/2} \geq \Psi(f)$ so one obtains the reverse to Jensen's inequality $\int \Psi(f)d\gamma \leq \Psi\left(\int fd\gamma\right) + \int |\nabla f|\sqrt{r}d\gamma$. Since $\sqrt{r} = \frac{|\nabla f|^2}{e^r - f^2} \leq \frac{|\nabla f|^2}{1-f^2}$ one can go further and write

$$\Psi\left(\int fd\gamma\right) \leq \int \Psi(f)d\gamma \leq \Psi\left(\int fd\gamma\right) + \int \frac{|\nabla f|^2}{1-f^2}d\gamma.$$

One can obtain Poincare inequality, indeed notice that $\Psi(x) = 1 - \frac{1}{2}\pi x + \frac{1}{2}x^2 + O(x^3)$ for $|x| < 1$. Take $f_\varepsilon = \varepsilon f$ and send $\varepsilon \to 0$.

21. A shortcut to become an applied mathematician: Two-point inequality for M.

Our primary goal will be to understand for which $M(x, y)$, for any $n \geq 1$ and any $f : \{-1, 1\}^n \to \Omega \subset \mathbb{R}$ the following function

$$B(t) := \mathbb{E}\, M(P_t^{di} f, |\nabla P_t f|), \quad t \in [0, \infty) \tag{19}$$

is monotonically increasing where

$$P_t^{di} f = \sum_{S \subset 2^n} e^{-|S|t} \hat{f}(S) W_S(x)$$

is a semigroup, $W_S(x)$ is the standard Walsh system on $(\{-1, 1\}^n, d\mu)$, and $d\mu$ is the uniform counting measure on the cube $\{-1, 1\}^n$.

Let P_t be Ornestein–Uhlenbeck semigroup: $p_t f = e^{-tL} f$, $L = -\Delta + x \cdot \nabla$. Function

$$t \to \int_{\mathbb{R}^n} M(P_t f, |\nabla P_t f|) d\gamma_n \tag{20}$$

is increasing provided that M is such that $M(x, \sqrt{y}) \in C^2(\Omega \times \mathbb{R}_+)$ and it satisfies PDI

$$\begin{pmatrix} M_{xx} + \frac{M_y}{y} & M_{xy} \\ M_{xy} & M_{yy} \end{pmatrix} \le 0 \tag{21}$$

In fact, we will prove that PDI is equivalent to a stronger statement:

$$P_t M(f, |\nabla f|) \le M(P_t f, |\nabla P_t f|) \tag{22}$$

23. Prove of monotonicity

$$P_t M(f, |\nabla f|) \leq M(P_t f, |\nabla P_t f|) \tag{23}$$

In fact, "concavity" (23) is stronger than monotonicity of (24):

$$t \to \int_{\mathbb{R}^n} M(P_t f, |\nabla P_t f|) d\gamma_n \tag{24}$$

Indeed, Integrating $P_h M(P_t f, |\nabla P_t f|) \leq M(P_{t+h} f, |\nabla P_{t+h} f|)$ we get $\int M(P_t f, |\nabla P_t f|) \leq \int M(P_{t+h} f, |\nabla P_{t+h} f|)$ and (24) follows. To prove that negativity of the matrix implies (23) we put $V(x,t) := P_t M(f, |\nabla f|) - M(P_t f, |\nabla P_t f|)$. Then $V(x,0) = 0$. If we prove that $\left(\frac{\partial}{\partial t} - L\right) V(x,t) \leq 0$ then by maximum principle $V(x,t) \leq 0$

$$\left(\frac{\partial}{\partial t} - L\right) V(x,t) = \left(L - \frac{\partial}{\partial t}\right) M(P_t f, |\nabla P_t f|) = Tr(W\Gamma(DP_t f)) \leq 0\,?$$

where $Dg := (g, \partial_1 g, \ldots, \partial_n g)$, $\Gamma(X) = \langle \nabla X_i, \nabla X_j \rangle$, $g = P_t f$, and

$$W = S\left(W_1 + \frac{M_y}{\|\nabla f\|} W_2\right) S$$

where S is a diagonal matrix with diagonal $(1, \frac{\nabla P_t f}{\|\nabla P_t f\|})$, and W_1, W_2 are corresponding matrices

$$\begin{bmatrix} M_{xx} + \frac{M_y}{\|\nabla f\|} & M_{xy} & \dots & M_{xy} \\ M_{xy} & M_{yy} & \dots & M_{yy} \\ \dots & \dots & \dots & \dots \\ M_{xy} & M_{yy} & \dots & M_{yy} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \dots 0 \\ 0 & \frac{\|\nabla P_t f\|^2}{(P_t f_{x_1})^2} - 1 & -1 & \dots - 1 \\ 0 & -1 & \frac{\|\nabla P_t f\|^2}{(P_t f_{x_2})^2} - 1 & \dots \\ \dots & \dots & \dots & \dots \\ 0 & -1 & \dots - 1 & \frac{\|\nabla P_t f\|^2}{(P_t f_{x_n})^2} - 1 \end{bmatrix}$$

It is clear that $W_1 \leq 0$ because M satisfies (21) of slide 22. $W_2 \geq 0$ by Hölder inequality. And $M_y \leq 0$.

25. Discrete PDE are tough

We saw that

$$\begin{pmatrix} M_{xx} + \frac{M_y}{y} & M_{xy} \\ M_{xy} & M_{yy} \end{pmatrix} \le 0 \qquad (25)$$

ensures that

$$\mathbb{E} M(f, |\nabla f|) \le M(\mathbb{E} f, 0), \text{ where } \mathbb{E} = \int \dots d\gamma_n .$$

If $\mathbb{E}_n = \frac{1}{2^n} \sum \dots$ on discrete cube $\{-1, 1\}^n$, we need the discrete inequality, which become (25) in its infinitesimal version. Then we hope to get

$$\mathbb{E}_n M(f, |\nabla f|) \le M(\mathbb{E}_n f, 0), \text{ where } \frac{1}{2^n} \sum \dots .$$

But there are many ways to discretize (25). We need a correct one.

26. Bobkov's inequality on Hamming cube.

We will see now discrete version of monotonicity on $1D$ discrete cube: $\mathbb{E}_1 M(P_t^{di} f, |\nabla P_t^{di} f|$ increases when $t \to +\infty$. Here $M(x, y) = -\sqrt{I^2(x) + y^2}$. Let us consider the equation

$$I'' I = -1, \quad I(0) = I(1) = 0\,. \tag{26}$$

and its solution $I_0(x) = \phi \circ \Phi^{-1}(x)$, where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2} dx, \quad \phi(x) = \Phi'(x)\,.$$

Bobkov proved (by direct tedious calculations) that function I_0 satisfies not only (26), but also a more general discrete inequality

$$I_0(x) \le \frac{1}{2}\sqrt{I_0^2(x+\varepsilon) + \varepsilon^2} + \frac{1}{2}\sqrt{I_0^2(x-\varepsilon) + \varepsilon^2},\ I_0(0) = I_0(1) = 0\,. \tag{27}$$

Moreover, the RHS decreases in ε. When $\varepsilon \to 0$ one restores (26)–it is an infinitesimal version of more general (27).

27. Full Bobkov's inequality on Hamming cube.

Consider $M(x,y) := -\sqrt{I^2(x)+y^2}$. Then $1D$ Bobkov's inequality (27) from slide 27 is precisely

$$\mathbb{E}_1 M(f, |\nabla f|) \le M(\mathbb{E}_1 f, 0)\,, \tag{28}$$

where $\mathbb{E}_1$ is the expectation with $(1/2, 1/2)$ measure on function on one dimensional Hamming cube (= two points).
Bobkov managed to prove by induction that then

$$\mathbb{E}_n M(f, |\nabla f|) \le M(\mathbb{E}_n f, 0) \tag{29}$$

independently of dimension n. **The precise form of M played very important part in his proof.**
How to have a general class of $M(x,y)$ for which induction works?
Full Bobkov's inequality on discrete cube leads to Gaussian (slide 6)

$$I\left(\int_{\mathbb{R}^n} f d\gamma\right) \le \int_{\mathbb{R}^n} \sqrt{I^2(f) + \|\nabla f\|^2} d\gamma_n \tag{30}$$

by the CLT. But Gaussian version can be proved independently by change of variable in PDE and by monotonicity of flow.

28. Function 3/2.

Consider the function

$$M(x,y)=\frac{1}{\sqrt{2}}\left(2x-\sqrt{x^2+y^2}\right)\sqrt{x+\sqrt{x^2+y^2}} \quad \text{where} \quad x,y\geq 0.$$

We know that in Gaussian world it gives an estimate better than Beckner–Sobolev one.

Question. Does it have a discrete analog on Hamming cube?

Do we have

$$\mathbb{E}M(f,|\nabla f|)\leq M(\mathbb{E}f,0) \tag{31}$$

for function our M above? If $\mathbb{E}$ is Gaussian then YES.

What if $\mathbb{E}=\mathbb{E}_1$?

What if $\mathbb{E}=\mathbb{E}_n$?

(Induction works?)

29. Sly induction works.

We need to invent an inductable claim. It turns out that

$$\mathbb{E}_n M(f, \sqrt{|\nabla f|^2 + |v|^2}) \leq M(\mathbb{E}_n f, |\mathbb{E}_n v|) \tag{32}$$

if true for n can be easily inducted to $n+1$.

The next question would be
What about the base of induction, $n = 1$?
First, let us prove that this induction will finish the proof of our inequality on Hamming cube:

$$M(\mathbb{E}_n f, 0) \geq \mathbb{E}_n M(f, |\nabla f|).$$

30. f as martingale

Define the martingale $\{f_k\}_{k=0}^n$ as follows: let $f_k = \mathbb{E}(f|\mathcal{F}_k)$ to be the average of the function f with respect to the variables $(x_{k+1}, \dots, x_n)$. For example

$$f_n = f;$$

$$f_{n-1} = \frac{1}{2}\left(f(x_1, \dots, x_{n-1}, 1) + f(x_1, \dots, x_{n-1}, -1)\right);$$

$$\dots$$

$$f_0 = \frac{1}{2^n} \sum_{x \in \{-1,1\}^n} f(x) = \mathbb{E}f.$$

Thus f_k lives on $\{-1, 1\}^k$ for $1 \le k \le n$.

30a. Supermartingale appears

Next we would like to know how the next generation $k+1$ is related to the previous generation k. For $x \in \{-1,1\}^{k+1}$ let $x = (x', x_{k+1})$ where $x' \in \{-1,1\}^k$. Notice that

$$f_{k+1}(x', x_{k+1}) = f_k(x') + x_{k+1} \cdot g(x');$$
$$|\nabla f_{k+1}(x', x_{k+1})|^2 = |\nabla_{x'}(f_k(x') + x_{k+1} \cdot g(x'))|^2 + |g(x')|^2.$$

where $g = g^k$ is a function on $\{-1,1\}^k$, and $\nabla_{x'}$ denotes gradient taken in x'.
We claim that the following process

$$z_k = M(f_k, |\nabla f_k|), \quad 0 \le k \le n$$

is a supermartingale.

After which our inequality follows immediately:

$$M(\mathbb{E}f,0)=z_0\geq \mathbb{E}z_n=\mathbb{E}M(f,|\nabla f|). \qquad (33)$$

To verify the claim we notice that

$$\mathbb{E}(z_{k+1}|\mathcal{F}_k)(x')=\frac{1}{2}\left(z_{k+1}(x',1)+z_{k+1}(x',-1)\right)=$$
$$\frac{1}{2}\Bigg(M(f_k(x')+g(x'),\sqrt{|\nabla_{x'}(f_k(x')+g(x'))|^2+|g(x')|^2})+$$
$$M(f_k(x')-g(x'),\sqrt{|\nabla_{x'}(f_k(x')-g(x'))|^2+|g(x')|^2})\Bigg)\leq$$
$$M(f_k(x'),|\nabla f_k(x')|)=z_k.$$

The last inequality follows from (34) (next slide) where we set $x=f_k(x'), a=g(x'), y=\nabla_{x'}f_k(x')$ and $b=\nabla_{x'}g(x')$.

31. The base of induction. Elementary?

Whenever $x+a, x-a, y+b, y-b \geq 0$ we have

$$M(x,y) \geq \frac{1}{2}\left(M(x+a, \sqrt{a^2+(y+b)^2}) + M(x-a, \sqrt{a^2+(y-b)^2})\right) \tag{34}$$

where

$$M(x,y) = \left(2x - \sqrt{x^2+y^2}\right)\sqrt{x+\sqrt{x^2+y^2}} \quad \text{where} \quad x, y \geq 0.$$

Looks like too many square roots.... .Can it be made a rational expression?

Consider the function

$$f(t) := M(x+at, \sqrt{(at)^2+(y+bt)^2}) + M(x-at, \sqrt{(at)^2+(y-bt)^2}$$

It is enough to show that $f(t)$ is decreasing for $t \in [0,1]$. Change variable $at \to t$ and consider $f(t)$ on the interval $[0,a]$ (but now $b \to b/a$) Notice that

$$f'(t) = M_x^+ + M_y^+ \frac{t+b(y+bt)}{\sqrt{t^2+(y+bt)^2}} - M_x^- + M_y^- \frac{t-b(y-bt)}{\sqrt{t^2+(y-bt)^2}} =$$

$$\frac{9}{4M_x^+}\left[(x+t)+\sqrt{(x+t)^2+t^2+(y+bt)^2} - (t+b(y+bt))\right] -$$

$$-\frac{9}{4M_x^-}\left[(x-t)+\sqrt{(x-t)^2+t^2+(y-bt)^2} + (t-b(y-bt))\right]$$

Where M^+ and M^- are computed at the points $(x+t, \sqrt{t^2+(y+bt)^2})$ and $(x-t, \sqrt{t^2+(y-bt)^2})$ correspondingly.

32a. Explanation.

This is why the last equality of slide 32 holds:

$$M_x^+ = \frac{3}{2}\sqrt{\sqrt{(x+t)^2 + t^2 + (y+bt)^2} + (x+t)},$$

$$M_y^+ = -\frac{3}{2}\sqrt{\sqrt{(x+t)^2 + t^2 + (y+bt)^2} - (x+t)},$$

$$M_x^+ M_y^+ = -\frac{9}{4}\sqrt{t^2 + (y+bt)^2}.$$

33. Main.

Next we can always assume (by homogeneity of M and considering new variables $\tilde{x} = xt$, $\tilde{y} = yt$) that we need to show that

$$\frac{x - by - b^2 + \sqrt{(x+1)^2 + 1 + (y+b)^2}}{\sqrt{x + 1 + \sqrt{(x+1)^2 + 1 + (y+b)^2}}} \leq \qquad (35)$$

$$\frac{x - by + b^2 + \sqrt{(x-1)^2 + 1 + (y-b)^2}}{\sqrt{x - 1 + \sqrt{(x-1)^2 + 1 + (y-b)^2}}} \qquad (36)$$

and $|b| \leq y$.
If $b = 0$ then inequality (35) is true.

Let $F(x) := LHS - RHS$.

34. At both infinities

Lemma

We have

$$F(x) = -b^2\sqrt{2}\cdot x^{-1/2} + O(x^{-3/2}) \quad as \quad x\to\infty;$$

$$F(x) =$$

$$-\frac{\sqrt{-2x}\left((1+b^2+by)\sqrt{1+(y-b)^2}+(1+b^2-by)\sqrt{1+(y+b)^2}\right)}{\sqrt{(1+(y+b)^2)(1+(y-b)^2)}}$$

$$+O((-x)^{-1/2})$$

$$as \quad x\to-\infty;$$

And the signs of $f(x)$ are negative at $\pm\infty$.

35. Squaring and squaring.

After squaring (35) of slide 33 and simplifying the expressions we end up with the following inequality

$$C_A \cdot A + C_B \cdot B + C_{AB} \cdot A \cdot B + L = 0 \tag{37}$$

where

$$C_A := 4by - 4b^2x + b^2 - b^2y^2 + 2b^3y - b^4 - 2 - y^2$$
$$C_B := -4b^2x + b^2y^2 + 2b^3y + b^4 + 2 + y^2 + 4by - b^2$$
$$L := -4 - 4b^2x^2 + 4b^3yx - 2b^4 + 8byx - 2b^2 - 2b^2y^2 - 2y^2$$
$$A := \sqrt{(x+1)^2 + 1 + (y+b)^2}$$
$$B := \sqrt{(x-1)^2 + 1 + (y-b)^2}.$$

36. Squaring again.

After moving terms $L, C_{AB} \cdot A \cdot B$ to the RHS and squaring and moving some terms again we finally obtain that

$$(C_A^2 \cdot A^2 + C_B^2 \cdot B^2 - L^2 - C_{AB}^2 \cdot A^2 \cdot B^2)^2 -$$
$$4 \cdot A^2 \cdot B^2 \cdot (C_{AB} \cdot L - C_A \cdot C_B)^2 = 0$$

Lets denote the LHS of the equation by $P(x)$. This is a 3rd degree polynomial in x

37. Here is $P(x; b, y)$.

$$P(x) =$$
$$-128b^3y^3(b^2y^2 + y^2 + 2 + 4by + 3b^2 + 2b^3y+$$
$$b^4)(b^2y^2 + y^2 + 2 - 4by + 3b^2 - 2b^3y + b^4)x^3+$$
$$(-64y^8b^8 + 1088b^6y^6 - 3392b^8y^4 + 8128b^{10}y^2+$$
$$384b^{10}y^6 - 704b^{12}y^4 + 960b^8y^6 - 3136b^{10}y^4$$
$$+3392b^{12}y^2 + 512b^{14}y^2 - 64y^8b^6 + 64y^8b^4+$$
$$64y^8b^2 - 960b^4y^6 + 960b^6y^4 + 64b^2y^6$$
$$-2816b^4y^2 + 1280b^4y^4 + 1088b^6y^2-$$
$$640b^2y^4 + 7872b^8y^2 - 1280b^2y^2 - 10880b^8$$
$$-8960b^{10} - 3072b^4 - 128b^{16} - 7808b^6 - 512b^2-$$
$$4352b^{12} - 1152b^{14})x^2$$
$$(-1792b^5y^3 + 256b^7y^7 - 5504b^7y^3 - 1408b^5y^7+$$

$$3456b^7y^5 - 384y^7b^3 + 640b^9y^5 +$$
$$2752b^5y^5 + 1536b^3y^3 - 5760b^9y^3 - 3840b^{11}y^3 -$$
$$768b^3y^5 + 512by + 3072b^3y +$$
$$1024by^3 + 1984b^{13}y + 384b^{15}y + 32b^{17}y +$$
$$32by^9 + 10272b^9y + 768by^5 +$$
$$5760b^{11}y + 256by^7 + 32b^9y^9 - 128b^{11}y^7 -$$
$$1408b^{13}y^3 - 64b^5y^9$$
$$-640b^9y^7 + 1664b^{11}y^5 + 192b^{13}y^5 - 128b^{15}y^3 +$$
$$7936b^5y + 11520b^7y)x +$$
$$-256 - 144b^{18} - 16y^{10} + 688y^8b^8 + 1504b^6y^6 -$$
$$1920b^8y^4 - 3440b^{10}y^2$$
$$-2304b^{10}y^6 + 2592b^{12}y^4 - 192b^8y^6 + 3264b^{10}y^4 -$$

$$y^2 - 288y^8b^6 - 224y^8b^4 + 48y^8b^2 - 736b^4y^6 -$$
$$1376b^6y^4 - 320b^2y^6 - 2816b^4y^2$$
$$- 480b^4y^4 + 2496b^6y^2 - 1792b^2y^4 + 3056b^8y^2 -$$
$$3072b^2y^2 - 768y^2 - 512y^6 - 896y^4$$
$$- 144y^8 - 3344b^8 + 1584b^{10} - 4992b^4 - 336b^{16} -$$
$$6656b^6 - 1792b^2 + 2528b^{12} +$$
$$608b^{14} - 64b^{16}y^4 + 96b^14y^6 + 16y^{10}b^2 + 32y^{10}b^4 +$$
$$624b^{16}y^2 - 864b^{14}y^4$$
$$+ 416b^{12}y^6 - 64b^{12}y^8 - 16b^{10}y^8 - 16b^8y^{10} +$$
$$16b^{10}y^{10} - 32y^{10}b^6 + 16b^{18}y^2$$

40. $b = 0$.

If $b = 0$ then

$$P(x) = -16(y^2 + 1)(y^2 + 2)^4 < 0.$$

This means that $F(x)$ does not have roots when $b = 0$. Therefore further we assume that $b \neq 0$.

Next if $y = 0$ then

$$P(x) = -16(b^2+1)^5(8b^2(b^2+2)^2x^2 + (3b^2+2)^2(b^2-2)^2) < 0,$$

Which again means that $F(x)$ does not have roots and hence $F(x) < 0$ in this case as well. Next we assume that $b, y \neq 0$.

The discriminant of this polynomial is

$$\Delta = 16777216 \cdot (1+b^2)^2 \cdot (-8 - 16b^2 - 8b^4 - 8y^2 + 20b^2y^2 + b^4y^2 - 2$$
$$(-b^4y^2 + 2b^2y^2 - y^2 - 2 - 3b^2 + b^6)^2(b^2y^2 + y^2 + 2 + 4by + 3b^2 + 2b$$
$$(b^2y^2 + y^2 + 2 - 4by + 3b^2 - 2b^3y + b^4)^2 \cdot$$
$$(4 + 24b^2 + 3b^{12} + 76b^6 + 54b^8 + 20b^{10} + 4y^8 + 14y^6 + 17y^4 + 12y^2 +$$
$$+ 19b^8y^4 - 12b^{10}y^2 + 4y^8b^4 + 8y^8b^2 - 22b^4y^6 + 46b^6y^4 + 6b^2y^6 + 4b$$
$$- 52b^6y^2 + 26b^2y^4 - 48b^8y^2 + 32b^2y^2)^2 \cdot b^6 =$$
$$16777216 \cdot (1+b^2)^2 \cdot T_1 \cdot T_2^2 \cdot T_3^2 \cdot T_4^2 \cdot T_5^2 \cdot b^6.$$

Discriminant does not vanish except when

$$y = \frac{(b^2+1)\sqrt{b^2-2}}{b^2-1};$$

$F(x)$ is the LHS-RHS of the slide 33.
In this case $P(x)$ has a root of multiplicity 2 which is $x = b\sqrt{b^2-2}$. We just need to make sure that at this root $F(x)$ is not zero. Then $F(x)$ may have at most 1 root but since it has negative signs at $\pm\infty$ we are done. So assuming $y = \frac{(b^2+1)\sqrt{b^2-2}}{b^2-1}$ and $x = b\sqrt{b^2-2}$ we obtain that in the LHS of $F(x)$ we have

$$x - by - b^2 + \sqrt{(x+1)^2 + 1 + (y+b)^2} =$$

$$-\frac{b(2\sqrt{b^2-2} + b^3 - b)}{b^2-1} + \sqrt{\frac{b^2(2\sqrt{b^2-2} + b^3 - b)^2}{(b^2-1)^2}} = 0.$$

On the other hand lets see what is the RHS of $F(x)$:

$$x - by + b^2 + \sqrt{(x-1)^2 + 1 + (y-b)^2} =$$

$$-\frac{b(2\sqrt{b^2-2} - b^3 + b)}{b^2-1} + \sqrt{\frac{b^2(2\sqrt{b^2-2} - b^3 + b)^2}{(b^2-1)^2}} =$$

$$-2 \cdot \frac{b(2\sqrt{b^2-2} - b^3 + b)}{b^2-1} > 0 \quad \text{for} \quad |b| \geq \sqrt{2}$$

A new edge-isoperimetric inequality on Hamming cube

Again:

$$\mathbb{E}f^{3/2} - (\mathbb{E}f)^{3/2} \le \frac{1}{\sqrt{2}}\mathbb{E}|\nabla f|^{3/2}, \quad f : \{-1,1\}^N \to \mathbb{R}_+. \qquad (38)$$

Next, let $A \subset \{-1,1\}^n$, and let $w_A(x)$ denotes the number of neighbor vertices from the complement of the set where x belongs, i.e., it counts opposite neighbors. Clearly $w_A(x)$ lives on the *boundary* of the set A: $w_A(x) = 4|\nabla \mathbf{1}_A|^2$. If A has cardinality 2^{n-1} then the classical edge isoperimetric inequality of Harper (J. Combin. Theory, 1996) states that $\sum_{x\in\{-1,1\}^n} w_A(x) \ge 2^n$. On the other hand, taking $f = \mathbf{1}_A$ in (38) gives

$$\sum_{x\in\{-1,1\}^n} w_A(x)^{3/4} \ge (2-\sqrt{2})2^n$$

which is a new edge-isoperimetric inequality and does not follow from the classical one.