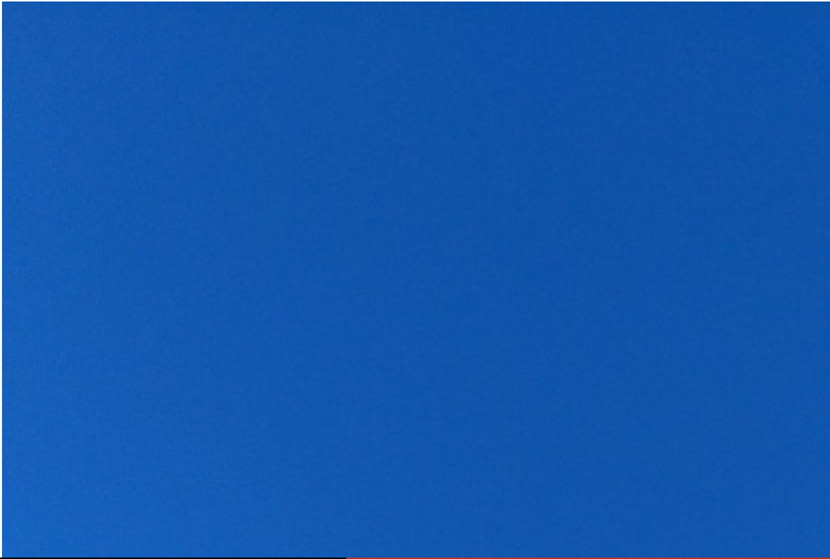

Two theorems on vortex patches

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Guy David Conference, Marseille, October 2017

The Euler equation in the plane

$$(E) \quad \begin{cases} \partial_t v(z,t) + (v \cdot \nabla) v(z,t) = -\nabla p(z,t) \\ \operatorname{div} v = 0 \\ v(z,0) = v_0(z) \end{cases}$$

$$v \cdot \nabla = v_1 \partial_1 + v_2 \partial_2$$

$$\operatorname{div}(v) = \partial_1 v_1(z,t) + \partial_2 v_2(z,t)$$

Well posedness of Euler's equation, Wolibner 1933

Euler's equation in the plane is globally well-posed in

$$C^{1+\gamma}, \quad 0 < \gamma < 1.$$

Bourgain, D.Li; Elguindi, Masmoudi

There exists a function u_0 in $C^1(\mathbb{R}^2)$ such the solution $u(x,t)$ of Euler's equation with initial condition u_0 blows up instantaneously in C^1, that is, for each $t_0 > 0$

$$\sup_{0<t<t_0} \|u(\cdot,t)\|_{C^1} = \infty$$

Vorticity

$$\omega = \mathrm{curl}(v) = \partial_1 v_2 - \partial_2 v_1$$

$$\text{circulation around blob } D = \int_{\partial D} v(z,t) \cdot \vec{\tau}\, ds$$
$$= \int_D \partial_1 v_2(z,t) - \partial_2 v_1(z,t)\, dA(z)$$

$$2\partial = 2\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$$

$$2\,\partial v = 2\,\frac{\partial}{\partial z} v = \mathrm{div}\, v + i\,\mathrm{curl}\, v = i\omega$$

Biot-Savart law : Velocity from Vorticity

$\frac{1}{\pi \bar{z}}$ is the fundamental solution of $\frac{\partial}{\partial z}$

$$v(z,t) = \frac{1}{\pi \bar{z}} * \frac{i}{2}\omega = \frac{i}{2\pi} \int \frac{\omega(\zeta,t)}{\bar{z}-\bar{\zeta}}\, dA(\zeta)$$

$$= \frac{1}{2\pi} \int \frac{(z-\zeta)^{\perp}}{|z-\zeta|^2}\, \omega(\zeta,t) dA(\zeta)$$

How do you compute ∇v ?

$$\partial v = \frac{i}{2}\,\omega \qquad \text{and} \qquad \overline{\partial} v = -\frac{i}{2\pi}\,\mathrm{p.\,v.}\,\frac{1}{\overline{z}^2} * \omega$$

If $\omega = \chi_D$ $\quad D$ a bounded domain with smooth boundary,

then v is a Lipschitz field

The vorticity equation

$$\begin{cases} \partial_t \omega + (v \cdot \nabla)\omega = 0 \\ v = \dfrac{i}{2\pi}\dfrac{1}{\bar{z}} * \omega = \nabla^{\perp} N * \omega, \qquad N = \frac{1}{2\pi}\log|z| \\ \omega(z,0) = \omega_0(z) \end{cases}$$

The Flow : particle trajectories

$$\frac{dX(z,t)}{dt} = v(X(z,t),t), \quad X(z,0) = z$$

$$\frac{d\omega(X(z,t),t)}{dt} = \partial_t \omega(X(z,t),t) + \partial_1 \omega(X(z,t),t) v_1(X(z,t),t) + \dots$$

Yudovich's Theorem

The vorticity equation is well posed in L^∞ : For each $\omega_0 \in L^\infty_c(\mathbb{C})$ there is a unique "weak" solution to the vorticity equation with initial condition ω_0.

"Proof" : Solve

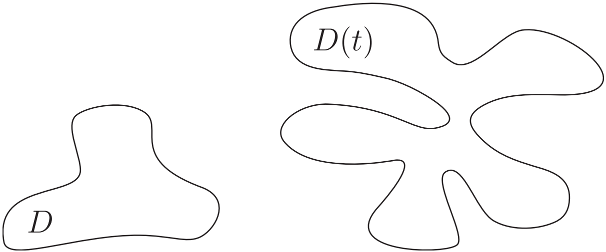
$$\frac{dX(z,t)}{dt} = v(X(z,t),t), \quad X(z,0) = z$$

and set

$$\omega(z,t) = \omega_0(X^{-1}(z,t)) \quad \text{or} \quad \omega(X(z,t),t) = \omega_0(z)$$

Vortex patches

$\omega_0 = \chi_D$, D a domain $\qquad \omega(z,t) = \chi_{D(t)}(z)$



The two known explicit examples

If $D = D(0,1)$ is the unit disc, then

$$D_t = D(0,1), \quad 0 < t,$$

$\chi_{D(0,1)}(z)$ is a steady solution to the vorticity equation

If $D_0 = \{(x,y) : x^2/a^2 + y^2/b^2 = 1\}$ is an ellipse then

$$\textbf{Kirchhoff:} \quad D_t = e^{i\Omega t}\, D_0, \quad 0 < t, \quad \Omega = \frac{ab}{(a+b)^2}$$

(vortex)

Rotating vortex patches or V-states

Definition

A V-state is a vortex patch that rotates with constant angular velocity. If the center of mass of the initial domain D_0 is the origin, then $D_t = e^{it\Omega} D_0$ for a certain angular velocity Ω

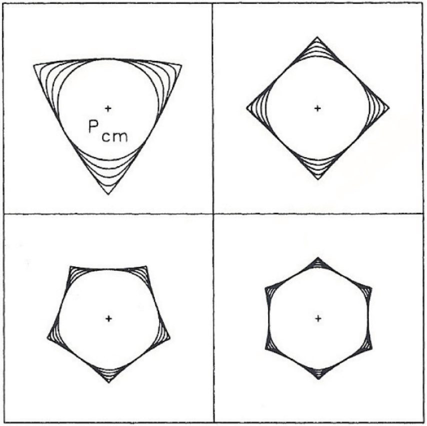
A disc rotates with any angular velocity

Kirchhoff : ellipses are V-states

$$\Omega = \frac{ab}{(a+b)^2}$$

Deem–Zabuski (1978) : numerical discovery of existence of V-states with m-fold symmetry

The pictures in the next frame are from a paper by Wu,Overman and Zabusky (1984)


P cm

Burbea (1982) : analytical proof, by bifurcation

$$\mathrm{Re}\left[\{(2\Omega)\overline{z} + I(z)\}\, z'\right] = 0, \quad z \in \partial D_0$$

$$I(z) = \frac{1}{2\pi i} \int_{\partial D_0} \frac{\overline{\zeta - z}}{\zeta - z}\, d\zeta$$

Conformal mapping

$$\Phi(z) = z + a_o + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots = z + f(z)$$

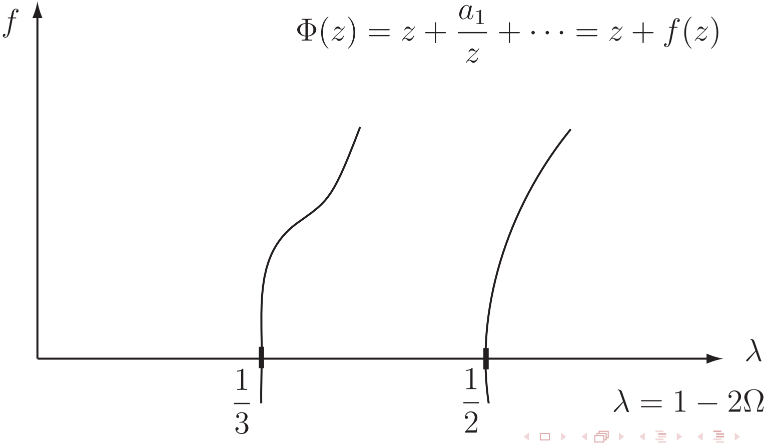
$$\mathrm{Im}\left[\left\{(2\Omega)\overline{\Phi(w)} + I(\Phi(w))\right\}\Phi'(w)w\right] = 0, \quad |w| = 1$$

$$I(\Phi(\omega)) = \frac{1}{2\pi i}\int_{|\tau|=1} \frac{\overline{\Phi(\tau) - \Phi(w)}}{\Phi(\tau) - \Phi(w)}\Phi'(\tau)\,d\tau, \quad |w| = 1$$

Bifurcation a la Crandall-Rabinovitz

$$F(\lambda, f) = 0 \quad f \in C^{1+\alpha}(\mathbb{T})$$

$$\Phi(z) = z + \frac{a_1}{z} + \cdots = z + f(z)$$



$$\lambda = 1 - 2\Omega$$

Hmidi, Mateu, V (2011)

If the V–state is close enough to the circle of bifurcation
then the boundary of the V–state is of class C^∞

Recent improvement by Castro, Córdoba and Gómez (2013)

If the V–state is close enough to the circle of bifurcation
then the boundary of the V–state is real analytic

They work in the context of the surface quasi-geostrophic equation

$$\frac{\overline{\Phi'(\omega)}}{\Phi'(\omega)} = \omega^2 \, \frac{(1-\lambda)\overline{\Phi(\omega)} + I(\Phi(w))}{(1-\lambda)\Phi(\omega) + \overline{I(\Phi(w))}}, \quad \lambda = \frac{1}{m}$$

$$\frac{\overline{\Phi'(\omega)}}{\Phi'(\omega)} = \left(\frac{\overline{\Phi'(\omega)}}{|\Phi'(\omega)|} \right)^2 = \left(\overline{\text{unit tangent vector}}\right)^2$$

Kellog-Warchawsky regularity theory

$$I(\Phi(\omega)) = \frac{1}{2\pi i}\int_{|\tau|=1} \frac{\overline{\Phi(\tau)-\Phi(w)}}{\Phi(\tau)-\Phi(w)}\Phi'(\tau)\,d\tau, \quad |w|=1$$

$$\frac{dI(\Phi(w))}{dw}(w) = \frac{\overline{\Phi'(w)}}{2w^2} + \frac{\Phi'(w)}{2\pi i}\,\text{p. v.}\int_{|\tau|=1}\frac{\overline{\Phi(\tau)-\Phi(w)}}{(\Phi(\tau)-\Phi(w))^2}\Phi'(\tau)d\tau$$

$$I_2(w) := \frac{1}{2\pi i}\int_{|\tau|=1}\frac{\overline{\Phi(\tau)-\Phi(w)-\Phi'(w)(\tau-w)}}{(\Phi(\tau)-\Phi(w))^2}\Phi'(\tau)d\tau, \quad |w|=1$$

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$$\frac{dI(\Phi(w))}{dw}(w) = \frac{\overline{\Phi'(w)}}{w^2} + \Phi'(w)\, I_2(w)$$

$$I_n(w) := \frac{1}{2\pi i}\int_{|\tau|=1} \frac{\overline{\Phi(\tau) - P_{n-1}(\Phi)(\tau, w)}}{\big(\Phi(\tau) - \Phi(w)\big)^n}\Phi'(\tau)d\tau, \quad |w| = 1$$

$$\frac{dI_n(w)}{dw} = n\Phi'(w)I_{n+1}(w), \quad |w| = 1, \quad n \geq 2.$$

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Persistence of boundary smoothness

If ∂D_0 is smooth, is it true that ∂D_t remains smooth for all $t > 0$?

Majda's Conjecture (1986)

There exists an initial "smooth" vortex patch which becomes of infinite length in finite time.

Chemin's Theorem (1993)

If $\partial D_0 \in C^{1+\varepsilon}$ then $\partial D_t \in C^{1+\varepsilon}$ for all $t > 0$.

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Patches for the aggregation equation

$$\partial_t \rho + \operatorname{div}(\rho\, v) = 0, \quad v(x,t) = (-\nabla N * \rho(\cdot,t))(x)$$

$$\partial_t \rho + v \cdot \nabla \rho = 0, \quad v = -\nabla N * \rho, \quad \rho_0 = \chi_{D_0}$$

Bertozzi, Garnett, Laurent, JV

If $\partial D_0 \in C^{1+\varepsilon}$ then $\partial D_t \in C^{1+\varepsilon}$ for all $t > 0$.

Boundary smoothness for short times

View the flow equation as an equation on a Banach space on ∂D_0

$$\frac{dX(z,t)}{dt} = v(X(z,t),t) \qquad v = \nabla^{\perp} N * \chi_{D_t} = N * \vec{\tau}\, d\sigma_{\partial D_t}$$

$$\frac{dX(z,t)}{dt} = \frac{1}{2\pi} \int_{\partial D_t} \log|X(z,t) - \zeta|\, d\zeta \qquad \zeta = X(w,t)$$

$$= \frac{1}{2\pi} \int_{\partial D_0} \log|X(z,t) - X(w,t)|\, \frac{dX(w,t)}{dw}\, dw$$

$$\frac{dX(w,t)}{dw} = DX(w(\theta),t)(w'(\theta))$$

$$B = C^{1+\gamma}(\partial D_0, \mathbb{R}^2)$$

$$U = \{X \in B : |X(z) - X(w)| \geq \frac{1}{M}\,|z - w| \text{ for some } M > 1\}$$

$$\frac{dX}{dt} = F(X), \quad F(0) = I$$

$$F(X)(z) = \frac{1}{2\pi} \int_{\partial D_0} \log|X(z) - X(w)|\, \frac{dX(w)}{dw}\, dw$$

$$\frac{F(X)(z)}{dz} = \frac{1}{2\pi} \int_{\partial D_0} \mathrm{Re}\left(\frac{dX(z)}{dz}\, \frac{1}{X(z) - X(w)}\right) \frac{dX(w)}{dw}\, dw$$

The boundary of ∂D_t is smooth for all times

Assume that ∂D_t is of class $C^{1+\gamma}$ for $0 < t < T$ and T is maximal with this property. If $T < \infty$ then some of the quantities that control the smoothness of ∂D_t become unbounded on $[0, T)$. Hence one has to prove a priori estimates on $[0, T)$ and conclude that these quantities are indeed bounded on $[0, T)$.

A defining function for a domain D is a function $\phi \in C^{1+\gamma}(\mathbb{R}^2)$ such that $D = \{z : \phi(z) < 0\}$, $\partial D = \{z : \phi(z) = 0\}$ and $\nabla\phi(z) \neq 0, \;\; z \in \partial D$.

The relevant quantities are

$$\|\nabla\phi\|_\gamma \qquad \text{and} \qquad \inf_{z\in\partial D} |\nabla\phi(z)|$$

A priori estimates

ϕ_0 defining function for D_0 and $\phi(z,t) = \phi_0(X^{-1}(z,t))$

$$\inf_{z\in\partial D_t} |\nabla\phi(z,t)| \geq \inf_{z\in\partial D_0} |\nabla\phi_0(z)| \, \exp\left(\int_0^T -\|\nabla v(\cdot,s)\|_\infty \, ds\right)$$

$$\|\nabla\phi(\cdot,t)\|_\gamma \leq \|\nabla\phi_0\|_\gamma \exp\left(C \int_0^T \|\nabla v(\cdot,s)\|_\infty \, ds\right)$$

End of proof

$$\|\nabla v(\cdot,t)\|_\infty \le C\left(1+\log^+ \frac{\|\nabla\phi(\cdot,t)\|_\gamma}{\inf_{z\in\partial D_t}|\nabla\phi(z,t)|}\right)$$

$$\le C\left(1+\int_0^t \|\nabla v(\cdot,s)\|_\infty\, ds\right)$$

$$\|\nabla v(\cdot,t)\|_\infty \le C e^{C\,t}, \quad 0<t<T.$$

Proof of the a priori estimates

$$\partial_t \phi(z,t) + v \cdot \nabla \phi(z,t) = 0$$

$$\partial_t \nabla^\perp \phi(z,t) + v \cdot \nabla \nabla^\perp \phi(z,t) - \nabla v(\nabla^\perp \phi(z,t)) = 0$$

$$\frac{D}{Dt}\left(\nabla^\perp \phi(z,t)\right) = \nabla v(\nabla^\perp \phi(z,t))$$

The commutator

$$\nabla v(\nabla^{\perp}\phi(z,t)) = \int_{D_t} \nabla\nabla^{\perp}N(z-w)\,\left(\nabla^{\perp}\phi(z,t) - \nabla^{\perp}\phi(w,t)\right)\,dA(w)$$

THANK YOU VERY MUCH FOR YOUR ATTENTION

AND PATIENCE

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