Harmonic and elliptic measures, and uniform rectifiability

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Rectifiability

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E is n-AD-regular if

$$\mathcal{H}^n(B(x,r) \cap E) \approx r^n \quad \text{for all } x \in E,\ 0 < r \leq \operatorname{diam}(E).$$

E is uniformly n-rectifiable if it is n-AD-regular and there are $M, \theta > 0$ such that for all $x \in E$, $0 < r \leq \operatorname{diam}(E)$, there exists a Lipschitz map

$$g : \mathbb{R}^n \supset B_n(0,r) \to \mathbb{R}^d, \qquad \|\nabla g\|_\infty \leq M,$$

such that

$$\mathcal{H}^n\big(E \cap B(x,r) \cap g(B_n(0,r))\big) \geq \theta\, r^n.$$

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Uniform n-rectifiability is a quantitative version of n-rectifiability introduced by David and Semmes.

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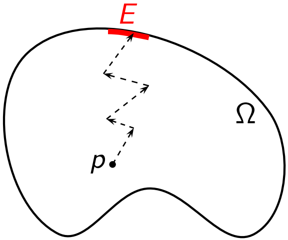
Probabilistic interpretation [Kakutani]:
When Ω is bounded, $\omega^p(E)$ is the probability that a particle with a Brownian movement leaving from $p \in \Omega$ escapes from Ω through E.



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When ω or $\omega_L \in A_\infty(\mu)$, for $\mu = \mathcal{H}^n|_{\partial\Omega}$?
Which is the connection to uniform rectifiability?

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A basic result:
If Ω is a Lipschitz domain, then $\omega \in A_\infty(\mu)$ (Dahlberg).

NTA and uniform domains

NTA domains were introduced by Jerison and Kenig.
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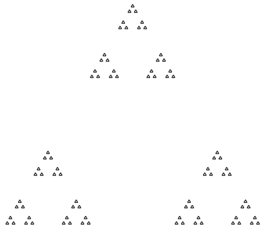
Example: The complement of this Cantor set is uniform but not NTA:



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- Zihui Zhao has shown that $\omega_L \in A_\infty(\mu)$ iff for any u L-harmonic in Ω, continuous in $\overline{\Omega}$, and any ball B centered at $\partial\Omega$,

$$\int_{B\cap\Omega} |\nabla u|^2 \operatorname{dist}(x,\partial\Omega)\, dx \leq C\, \|u\|^2_{BMO(\mu)}\, r(B)^n.$$

  (BMO solvability condition).

- Related result by Hofmann and Le for more general domains.

The Carleson condition on A

For all balls B centered at $\partial\Omega$,

$$\int_{B\cap\Omega}\Big(\sup_{\substack{z\in B(y,4\delta_\Omega(y))\cap\Omega\\ \delta_\Omega(z)\geq\frac14\,\delta_\Omega(y)}}|\nabla A(z)|\Big)\,dy\leq C\,r(B)^n,$$

where $\delta_\Omega(z)=\mathrm{dist}(z,\partial\Omega)$.

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- (b) $\Rightarrow$ (c) $\Rightarrow$ (a) by Garnett, Mourgoglou and T.
- (c) should be understood as a substitute of $\omega \in A_\infty(\mu)$, which fails in general.

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- (d) Hofmann, Le, Martell and Nyström showed $\omega \in A_\infty^{weak}(\mu) \Rightarrow \partial\Omega$ is uniformly n-rectifiable, but (a) $\not\Rightarrow \omega \in A_\infty^{weak}(\mu)$.

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Up to now there was no characterization of uniform rectifiability in terms of harmonic measure.

But there was a characterization in terms of harmonic measure of big pieces of NTA domains by Bortz and Hofmann.

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- Connection with ε-approximability and work of Kenig, Kirchheim, Pipher and Toro.
- Condition (b) is related to the "area integral".
  We cannot replace $\|u\|_{L^\infty(\Omega)}$ by $\|u\|_{BMO(\mu)}$, with $u \in C(\overline{\Omega})$.
  Related work by Hofmann and Le.

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- For each $\mathcal{T} \in I$ with $R = \mathrm{Root}(\mathcal{T})$, there exist points $p^1_{\mathcal{T}}, p^2_{\mathcal{T}} \in \Omega$ with
$$c^{-1}\ell(R) \leq \mathrm{dist}(p^k_{\mathcal{T}}, R) \leq \mathrm{dist}(p^k_{\mathcal{T}}, \partial\Omega) \leq c\,\ell(R)$$
such that, for all $Q \in \mathcal{T}$, $\omega_L^{p^1_{\mathcal{T}}}(5Q) \approx \omega_{L^*}^{p^2_{\mathcal{T}}}(5Q) \approx \dfrac{\mu(Q)}{\mu(R)}$.

Corona decomposition in terms of ω_L and ω_{L^*}

Condition (c) means that there exists a partition of $\mathcal{D}_\mu$ into **trees** $\mathcal{T} \in I$ satisfying:

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The Carleson condition on A:
$$\int_{B\cap\Omega} \Bigg(\sup_{\substack{z_1,z_2\in B(y,M\delta_\Omega(y))\cap\Omega \\ \delta_\Omega(z_k)\geq \frac14\,\delta_\Omega(y)}} \frac{|A(z_1)-A(z_2)|}{|z_1-z_2|} \Bigg)\,dy \leq C\,r(B)^n,$$
for all balls B centered at $\partial\Omega$, where $\delta_\Omega(z) = \mathrm{dist}(z,\partial\Omega)$.

Proof of (c) $\Rightarrow$ (a)

The case $L = \Delta$ can be proved using the connection of the Riesz transform with harmonic measure and uniform rectifiability:

$$K(x-p) - \int K(x-y)\, d\omega^p(y) = c\, \nabla_x G(x,p).$$

for $x \in \Omega$ and $K(z) = \dfrac{z}{|z|^{n+1}}$.

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Use that $\mathcal{R}_\mu$ bounded in $L^2(\mu)$ implies uniform rectifiability (Nazarov-T-Volberg).

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Some ingredients of the proof of the general case:

- Integration by parts, using an idea originated from Lewis and Vogel.
- An appropriate variant of the Alt-Caffarelli-Friedman (ACF) monotonicity formula.
- A topological criterion for uniform rectifiability.

The ACF formula for elliptic operators

Theorem (AGMT)

Let $B(x,R) \subset \mathbb{R}^{n+1}$, and let $u_1, u_2 \in W^{1,2}(B(x,R)) \cap C(B(x,R))$ be nonnegative L-subharmonic functions. Suppose that $A(x) = Id$ and that $u_1(x) = u_2(x) = 0$ and $u_1 \cdot u_2 \equiv 0$, u_i Hölder continuous at x. Set

$$J(x,r) = \left(\frac{1}{r^2}\int_{B(x,r)} \frac{|\nabla u_1(y)|^2}{|y-x|^{n-1}}dy\right) \cdot \left(\frac{1}{r^2}\int_{B(x,r)} \frac{|\nabla u_2(y)|^2}{|y-x|^{n-1}}dy\right).$$

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Then $J(x,\cdot)$ is absolutely continuous and

$$\frac{J'(x,r)}{J(x,r)} \geq -c\, \frac{w(x,r)}{r}, \qquad \text{for a.e. } 0 < r < R,$$

where

$$w(x,r) = \sup_{y \in B(x,r)} |A(y) - A(x)|.$$

Remarks about the ACF formula

- u_i Hölder continuous at x means that there exists $\alpha > 0$ such that

$$u_i(y) \leq C \left(\frac{|y - x|}{r} \right)^{\alpha} \|u\|_{\infty, B(x,r)},$$

for all $0 < r \leq R$ and $y \in B(x, r)$.

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- There are less precise variants for parabolic equations and with weaker assumptions by Caffarelli-Jerison-Kenig, or by Matevosyan-Petrosyan.
- These formulas are a basic tool in free boundary problems.
- The ACF formula is necessary to deal with the case when

$$\{x : G_L(x,p) > \lambda\} \cap \{x : G_{L^*}(x,p) > \lambda\} = \varnothing$$

  (in this case, the integration by parts technique does not work).
  We take $u_1 = G_L(\cdot, p_1)\chi_{U_1}$ and $u_2 = G_L(\cdot, p_1)\chi_{U_2}$.

The topological criterion

Using the ACF formula we prove some quantitative connectivity conditions which allow the application of a suitable criterion for uniform rectifiability.

The topological criterion

A cube $Q \in \mathcal{D}_\mu$ is called weak topologically satisfactory (WTS) if there are $A_0, \alpha, \tau > 0$ and connected open sets $U_1(Q), U_2(Q) \subset A_0 B_Q$ so that

- $\{x \in 10B_Q : \text{dist}(\text{supp}\,\mu) > \tau\ell(Q)\} \subset U_1(Q) \cup U_2(Q)$,
- $U_1(Q) \cap U_2(Q) = \varnothing$.
- For $i = 1, 2$ and all $x \in 10B_Q \cap \text{supp}\,\mu$ and $c_1\ell(Q) < r < 10\ell(Q)$, there is a ball $B(y, c_2\ell(Q)) \subset U_i(Q) \cap B(x, r)$.
- For $x, y \in U_i(Q)$, there is a curve $\Gamma \subset U_i(Q)$ containing x, y so that $\text{dist}(\Gamma, E) \geq \alpha\ell(Q)$.

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The property WTS is a variant of the *weak topologically nice* condition from David and Semmes.

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Theorem (AGMT)

Let μ be n-AD-regular in $\mathbb{R}^{n+1}$. Suppose a suitable compatibility condition holds for the WTS cubes. Suppose that for every $R \in \mathcal{D}_\mu$

$$\sum_{\substack{Q \subset R \\ Q \notin WTS}} \mu(Q) \leq C\,\mu(R).$$

For appropriate choice of constants, μ is uniformly rectifiable.

The precise definitions

A cube $Q \in \mathcal{D}_\mu$ is called weak topologically satisfactory (WTS) if there are $A_0, \alpha, \tau > 0$ and connected open sets connected sets $U_1(Q), U_2(Q), U_1'(Q), U_2'(Q) \subset A_0 B_Q$ so that

- $\{x \in 10B_Q : \mathrm{dist}(\mathrm{supp}\,\mu) > \tau\ell(Q)\} \subset U_1(Q) \cup U_2(Q)$,
- $U_i(Q) \subset U_i'(Q)$ and $U_1'(Q) \cap U_2'(Q) = \varnothing$.
- For $i = 1, 2$ and all $x \in 10B_Q \cap \mathrm{supp}\,\mu$ and $c_1\ell(Q) < r < 10\ell(Q)$, there is a ball $B(y, c_2\ell(Q)) \subset U_i(Q) \cap B(x, r)$.
- For $x, y \in U_i(Q)$, there is a curve $\Gamma \subset U_i(Q)$ containing x, y so that $\mathrm{dist}(\Gamma, E) \geq \alpha\ell(Q)$.

Given $a_0 \geq 1$, we say that the compatibility condition holds for some family $\mathcal{F} \subset WTS$ if for all $P, Q \in \mathcal{F}$ such that $2^{-a_0}\ell(Q) \leq \ell(P) \leq \ell(Q)$, it holds that $U_i(P) \cap 10B_Q \subset U_i'(Q)$.

Happy birthday, Guy.

Thank you!