Rectifiability of measures

Raanan Schul (Stony Brook)

October, 2017

Length

Let $f : [0,1] \to \mathbb{R}$ be Lipschitz function. Let $G \subset \mathbb{R}^2$ be the graph of f. Then the length of G is

$$\mathcal{H}^1(G) = \int_0^1 \sqrt{1 + \left|\frac{df}{dx}\right|^2} dx \approx 1 + c\int_0^1 \left|\frac{df}{dx}\right|^2 dx$$

Key player:

$$\left\|\frac{df}{dx}\right\|_2^2$$

Curvature

- Let J be a dyadic interval, and $J = J_L \cup J_R$ be its decomposition into its left and right parts.
- Let
$$H_J(x) = |J|^{-\frac{1}{2}}\left(\chi_{J_L}(x) - \chi_{J_R}(x)\right)$$
Then $\{H_J\}_{J\in\Delta}$ is an orthonormal basis for $L^2(\mathbb{R})$. ($\Delta =$ all dyadic intervals)
- Extend f as a constant right of 1 and left of 0. Write
$$\frac{df}{dx} = \sum a_J H_J(x).$$
$$\left\|\frac{df}{dx}\right\|_2^2 = \sum_{J\in\Delta} |a_J|^2.$$
- What does $|a_J|$ mean? If J=[0,1]
$$a_{[0,1]} = \langle \frac{df}{dx}, H_{[0,1]}\rangle = \left(f(\frac{1}{2}) - f(0)\right) - \left(f(1) - f(\frac{1}{2})\right)$$
= "**change in slope between the two halves**"

Length and Curvature

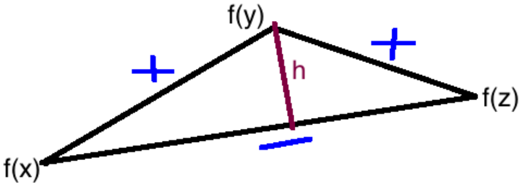
- $\|\frac{df}{dx}\|_2^2 = \sum |a_J|^2 = L^2$ quantity which measures curvature.
- Length '$=$'
  diam + L^2 quantity which measures curvature.
- The above is a *quantitative* connection between length and curvature. It comes into play when working on *qualitative* questions.

(if you fall asleep now, then at least remember that)

Curvature - II

- Let

$$b(x,y,z) := |f(x) - f(y)| + |f(y) - f(z)| - |f(x) - f(z))|$$



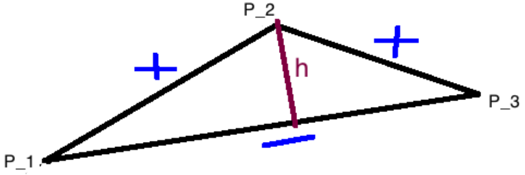
- If no edge is much larger than the other two, then

$$\frac{b(x,y,z)}{\mathrm{diam}^3} \sim \frac{h^2}{\mathrm{diam}^4} \sim \frac{1}{R^2}$$

  where $R = R(x,y,z)$ is radius of circle through $f(x), f(y), f(z)$ (Menger curvature:$= \frac{1}{R}$).
- Note: we don't need f anymore to make these definitions.

Curvature - III

- Non-paramtric version of b:
  $b_{\min}(P_1, P_2, P_3) :=$
  $$\min_{\sigma \in S_3} \left(|P_{\sigma(1)} - P_{\sigma(2)}| + |P_{\sigma(2)} - P_{\sigma(3)}| - |P_{\sigma(1)} - P_{\sigma(3)}| \right)$$



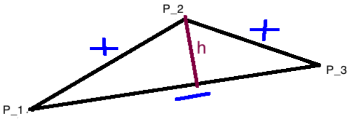
- If no edge is much larger than the other two, then

$$\frac{b_{\min}(A, B, C)}{\mathrm{diam}(A, B, C)^3} \sim \frac{h_{\min}^2}{\mathrm{diam}^4} \sim \frac{1}{R^2}$$

Length and Curvature - II

Suppose G is a graph of an L-Lipschitz function f. Then

$$\mathcal{H}^1(G) \sim \mathrm{diam}(G) + c_L \sum_J h^2(J)/|J|$$
$$\sim \mathrm{diam}(G) + c \iiint \frac{b_{min}}{\mathrm{diam}^3}$$



- $\sum$: over dyadic intervals J. For $J = [a, b]$,

$$h(J) := \sup_{z \in [a,b]} \mathrm{dist}\big(f(z), P\big)$$

  where for each J we choose P as the line minimizing $h(J)$.
- $\iiint$: over all triples in G, $(d\text{length})^3$.

True in much more generality... (many contributors)

1-Rectfiability

Slightly non-standard way of saying it

- Let μ be a measure on $\mathbb{R}^n$. We say that μ is 1-rectifiable if there is a countable collection of Lipschitz curves

$$f_i : [0,1] \to \mathbb{R}^n$$

such that

$$\mu\Big(\mathbb{R}^n \setminus \bigcup_{i=1}^{\infty} f_i[0,1]\Big) = 0.$$

- If $E \subset \mathbb{R}^n$ and $\mu = \mathcal{H}^1|_E$ then E is called a "1-rectifiable set".
- m-rectifiability uses $[0,1]^m$ as domain...

Questions

- Let μ be a measure on $\mathbb{R}^n$. We say that μ is 1-rectifiable if there is a countable collection of Lipschitz curves
$$f_i : [0,1] \to \mathbb{R}^n$$
such that
$$\mu\Big(\mathbb{R}^n \setminus \bigcup_{i=1}^{\infty} f_i[0,1]\Big) = 0.$$

- When is μ 1-rectifiable?
- When is one curve enough to capture all of μ?
- When does one curve capture a significant part of μ?

The case $\mu = \mathcal{H}^1|_E$ (or $\mu \ll \mathcal{H}^1|_E$) is very well studied, and the case and $\mu \perp \mathcal{H}^1$ is not.

Example 1

- Let $\mu =$ Lebesgue measure on $[0,1]^2 \subset \mathbb{R}^2$.
- For any $f : [0,1] \to \mathbb{R}^2$ Lipschitz,

$$\mu\Big(f[0,1]\Big) = 0.$$

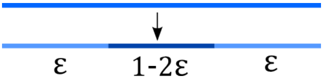
- μ is NOT 1-rectifiable.

Example 2

- Let μ be a more eccentric version of Example 1:

Example 2

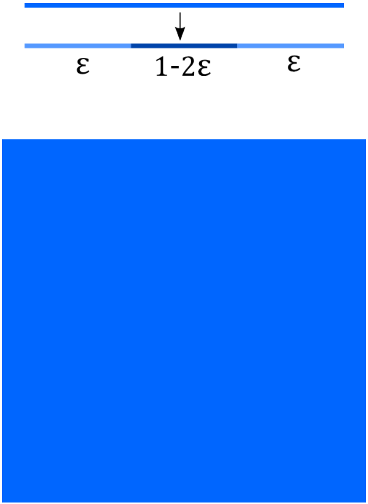
- Let μ be a more eccentric version of Example 1:



pic by M. Badger

Example 2

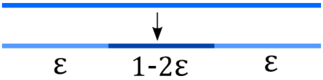
- Let μ be a more eccentric version of Example 1:

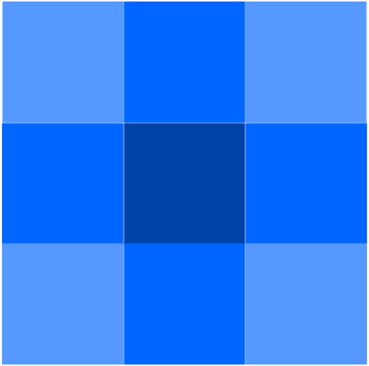


pic by M. Badger

Example 2

- Let μ be a more eccentric version of Example 1:

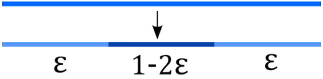


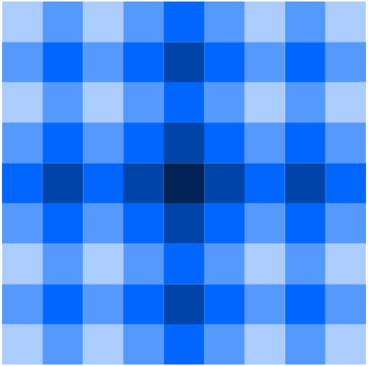


pic by M. Badger

Example 2

- Let μ be a more eccentric version of Example 1:

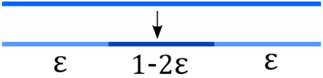


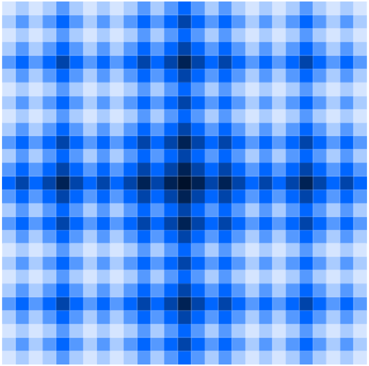


pic by M. Badger

Example 2

- Let μ be a more eccentric version of Example 1:

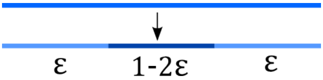


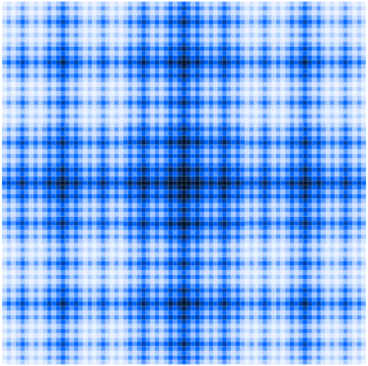


pic by M. Badger

Example 2

- Let μ be a more eccentric version of Example 1:

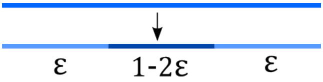


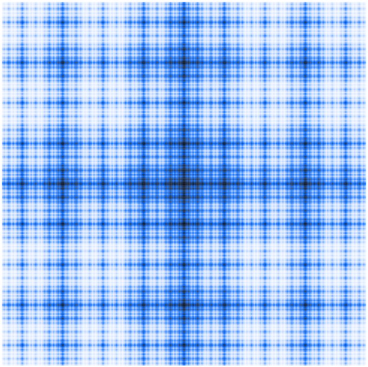


pic by M. Badger

Example 2

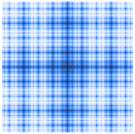
- Let μ be a more eccentric version of Example 1:





pic by M. Badger

Example 2 - continued



(If $\epsilon = \frac{1}{3}$ we recover 2-dim. Leb. meas.)

If $\epsilon > 0$ is small enough, then

- $\mu \perp \mathcal{H}^1|_E$ for any $E \subset \mathbb{R}^2$ with $H^1(E) < \infty$.
- μ is doubling on $\mathbb{R}^2$. $\mu(L) = 0$ for any line L.
- $\mu(G) = 0$ for any G, an isometric copy of a Lipschitz graph .
- μ is 1-rectifiable (Theorem [Garnett-Killip-S. 2009])

A measure μ is *"doubling on $\mathbb{R}^n$"* if there is a $C > 0$ such that for any $x \in \mathbb{R}^n$ and $r > 0$ we have

$$\mu(B(x, 2r)) < C\mu(B(x, r)).$$

Rectifiable Measures

$$\{\ m\text{-rectifiable measures } \mu \text{ on } \mathbb{R}^n\ \}$$

$$\supsetneq$$

$$\{\ m\text{-rectifiable measures } \mu \text{ on } \mathbb{R}^n \text{ such that } \mu \ll \mathcal{H}^m\ \}$$

$$\supsetneq$$

$$\{\ m\text{-rectifiable measures } \mu \text{ on } \mathbb{R}^n \text{ of the form } \mu = \mathcal{H}^m|_E\ \}$$

- How do you tell if a ‘generic’ measure is 1-rectifiable?
- What about 2-rectifiable? m-rectifiable?

$\mu \ll \mathcal{H}^1|_E$

Lower and upper (Hausdorff) *m*-density:

$$\underline{D}^m(\mu, x) = \liminf_{r \downarrow 0} \frac{\mu(B(x,r))}{c_m r^m} \quad \overline{D}^m(\mu, x) = \limsup_{r \downarrow 0} \frac{\mu(B(x,r))}{c_m r^m}$$

Write $D^m(\mu, x)$, the *m*-density of μ at x, if $\underline{D}^m(\mu, x) = \overline{D}^m(\mu, x)$.

Theorem 1 (Mattila 1975)

Suppose that $E \subset \mathbb{R}^n$ is Borel and $\mu = \mathcal{H}^m|_E$ is locally finite. Then μ is m-rectifiable if and only if $D^m(\mu, x) = 1$ μ-a.e.

Theorem 2 (Preiss 1987)

Suppose that μ is a locally finite Borel measure on $\mathbb{R}^n$. Then μ is m-rectifiable and $\mu \ll \mathcal{H}^m$ if and only if $0 < D^m(\mu, x) < \infty$ μ-a.e.

$\mu \ll \mathcal{H}^1|_E$

- For $s > 0$, $a \in \mathbb{R}^n$ and P an m-plane in $\mathbb{R}^n$ (through 0) define the two sided cone

$$X(a, P, s) = \{x \in \mathbb{R}^n : d(x - a, P) < s|x - a|\}.$$

- We say that P above is an *approximate tangent of E at a* if for $\mu = \mathcal{H}^m|_E$, $\overline{D}^m(\mu, a) > 0$, and for all $s \geq 0$

$$\lim_{r \downarrow 0} \frac{\mu(B(a, r) \setminus X(a, P, s))}{r^m} = 0$$

Theorem 3 (Marstrand-Mattila)

Suppose that $E \subset \mathbb{R}^n$ is Borel and $\mu = \mathcal{H}^m|_E$ is locally finite. TFAE

- *μ is m-rectifiable*
- *For μ a.e. $a \in E$ there is a unique approx. tangent plane for E at a.*
- *For μ a.e. $a \in E$ there is a some approx. tangent plane for E at a.*

$\mu \ll \mathcal{H}^1|_E$

General fact:

If $\mu \ll \mathcal{H}^m$, then you can change the def. we gave:

- *We say that μ is rectifiable if there is a countable collection of Lipschitz maps*

$$f_i : [0,1]^m \to \mathbb{R}^n$$

  such that

$$\mu\Big(\mathbb{R}^n \setminus \bigcup_{i=1}^{\infty} f_i[0,1]^m\Big) = 0.$$

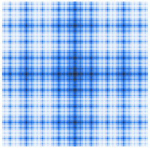
- TO
- *We say that μ is rectifiable if there is a countable collection $\{G_i\}$ of isometric copies of graphs of Lipschitz functions*

$$g_i : [0,1]^m \to \mathbb{R}^{n-m}$$

  such that

$$\mu\Big(\mathbb{R}^n \setminus \bigcup_{i=1}^{\infty} G_i\Big) = 0.$$

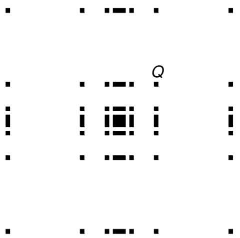
Properties of μ from Example 4

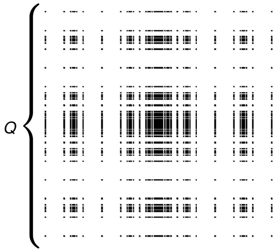


μ is 1-rectifiable, however:

- For μ almost every x,
  - $D^1(\mu, x) = \infty$
  - NO 1-dimensional TANGENT (in any sense)
- For any graph G, $\mu(G) = 0$.
- How does one tell if a measure μ on $\mathbb{R}^2$ is 1-rectifiable?
- We will give some answers in the following slides...
- $$\frac{\text{“deviation from tangent”}}{\text{“density”}}$$

1-rectifiable





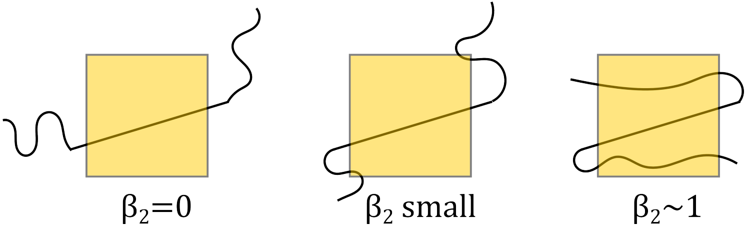
Some Results

Preliminaries - L^2 Beta Numbers

Let μ be a locally finite Borel measure on $\mathbb{R}^n$ and $Q \subset \mathbb{R}^n$ a cube. Define the L^2 beta number $\beta_2(\mu, Q) \in [0,1]$ by

$$\beta_2(\mu, Q)^2 = \inf_{\ell} \int_Q \left(\frac{\mathrm{dist}(x, \ell)}{\mathrm{diam} Q} \right)^2 \frac{d\mu(x)}{\mu(Q)}$$

where the infimum runs over all lines ℓ in $\mathbb{R}^n$.



pic by M. Badger

Preliminaries - L^2 Jones Functions

Ordinary L^2 Jones function

$$J_2(\mu, x) = \sum_{\substack{\mathrm{diam} Q \leq 1 \\ Q \text{ dyadic}}} \beta_2(\mu, 3Q)^2 \chi_Q(x).$$

Density-normalized L^2 Jones function

$$\widetilde{J}_2(\mu, x) = \sum_{\substack{\mathrm{diam} Q \leq 1 \\ Q \text{ dyadic}}} \beta_2(\mu, 3Q)^2 \frac{\mathrm{diam} Q}{\mu(Q)} \chi_Q(x).$$

Note:

- If $\overline{D}^1(\mu, a) < \infty$, then
$$\widetilde{J}(\mu, a)) < \infty \implies J(\mu, a) < \infty.$$
- If $\underline{D}^1(\mu, a) > 0$, then
$$J(\mu, a) < \infty \implies \widetilde{J}(\mu, a) < \infty.$$

Some results

Can have m-dimensional version of β-numbers, J_2, etc.

Theorem 4 (Azzam-Tolsa (IF) + Tolsa (ONLY IF))

Suppose μ is locally finite Borel, and
$0 < \overline{D}^m(\mu, x) < \infty$ *μ-a.e..*
Then μ is m-rectifiable **if and only if** $J_2(\mu, x) < \infty$ *μ-a.e.*

(other "if" work by Pajot and Badger-S)

Theorem 5 (Edelen-Naber-Valtorta)

Suppose μ is locally finite Borel, and
$0 < \overline{D}^m(\mu, x)$, $\underline{D}^m(\mu, x) < \infty$ *μ-a.e..*
Then μ is m-rectifiable **if** $J_2(\mu, x) < \infty$ *μ-a.e.*

$m = 1$

Theorem 6 (Badger-S)

Suppose μ be a locally finite doubling Borel measure on $\mathbb{R}^n$. Then μ is 1-rectifiable if and only if

$$\widetilde{J}_2(\mu, x) = \sum_{\substack{\operatorname{diam} Q \leq 1 \\ Q \text{ dyadic}}} \beta_2(\mu, 3Q)^2 \frac{\operatorname{diam} Q}{\mu(Q)} \chi_Q(x) < \infty \quad \mu\text{-a.e.}$$

(discuss example 4!!!)
Note: more work by Azzam-Mourgoglou, Martikainen-Orponen and others.

Theorem 7 (Badger-S)

Can remove doubling assumption with more technical definition of β

(details next slide)

$$\beta_2^*(\mu, Q)^2 := \inf_{\ell} \max_{R \in \Delta^*(Q)} \beta_2(\mu, 3R, \ell)^2 \min\left(\frac{\mu(3R)}{\mathrm{diam} 3R}, 1\right),$$

$\Delta^*(Q)$ are cubes of similar size and location, and ℓ is a line.

$$J_2^*(\mu, x) := \sum_{\substack{\mathrm{diam} Q \leq 1 \\ Q \text{ dyadic}}} \beta_2^*(\mu, Q)^2 \frac{\mathrm{diam} Q}{\mu(Q)} \chi_Q(x)$$

Theorem 8 (Badger-S.)

Let $n \geq 2$, and μ a Radon measure on $\mathbb{R}^n$. Then

$$1 - \mathrm{rect} = \left\{x \in \mathbb{R}^n : \underline{D}^1(\mu, x) > 0 \text{ and } J_2^*(\mu, x) < \infty\right\},$$

$$1 - \mathrm{pur.unrect.} = \left\{x \in \mathbb{R}^n : \underline{D}^1(\mu, x) = 0 \text{ or } J_2^*(\mu, x) = \infty\right\}.$$

$m \geq 1$

- Basic tool in Theorem 8 is a variant of Jones' TST.
- But what if... $m > 1$?!

$m \geq 1$

- Basic tool in Theorem 8 is a variant of Jones' TST.
- But what if... $m > 1$?!
- For an m-plane ℓ and Ball B or radius r_B:

$$\beta_E^m(B,\ell) = \frac{1}{r_B}\int_0^1 \mathcal{H}_\infty^m\{x \in B \cap E : \mathrm{dist}(x,\ell) > tr_B\}dt$$

and $\beta_E^m(B) = \inf_\ell \beta_E^m(B,\ell)$.
Note: uses Hausdorff content.
IF assume Ahlfors-regularity get David-Semmes β_1

$m \geq 1$

- Basic tool in Theorem 8 is a variant of Jones' TST.
- But what if... $m > 1$?!
- For an m-plane ℓ and Ball B or radius r_B:

$$\beta_E^m(B,\ell) = \frac{1}{r_B}\int_0^1 \mathcal{H}_\infty^m\{x \in B \cap E : \mathrm{dist}(x,\ell) > tr_B\}dt$$

  and $\beta_E^m(B) = \inf_\ell \beta_E^m(B,\ell)$.
  Note: uses Hausdorff content.
  IF assume Ahlfors-regularity get David-Semmes β_1
- *Lower content regular*: for all $x \in E \cap B(0,1)$ and $r < 1$

$$\mathcal{H}_\infty^m(E \cap B(x,r)) \geq cr^m$$

$m \geq 1$

- $$d_B(E,\ell) = \frac{1}{r_B} \max\left\{ \sup_{y \in E \cap B} \mathrm{dist}(y,\ell), \sup_{y \in \ell \cap B} \mathrm{dist}(y,E) \right\}.$$
- $$\vartheta_E^m(B) = \inf_{\ell \text{ an } m\text{-plane}} d_B(E,\ell)$$

$m \geq 1$

- $$d_B(E,\ell) = \frac{1}{r_B} \max\left\{ \sup_{y\in E\cap B} \mathrm{dist}(y,\ell), \sup_{y\in \ell\cap B} \mathrm{dist}(y,E) \right\}.$$
- $$\vartheta_E^m(B) = \inf_{\ell \text{ an } m\text{-plane}} d_B(E,\ell)$$
- $$\Theta_E(B(0,1)) := \sum \{\mathrm{diam}(Q)^m : Q \in \Delta, \; Q \cap E \cap B(0,1) \neq \emptyset \text{ and } \vartheta_E(3Q) \geq \epsilon\}$$

Theorem 9 (Azzam-S.)

Let $1 \leq m < n$, $C_0 > 1$. *Let* $\emptyset \neq E \subseteq B(0,1)$ *is Lower-content-regular. There is* $\epsilon_0 = \epsilon_0(n, c) > 0$ *such that for* $0 < \epsilon < \epsilon_0$ *we have:*

$$1 + \sum_{\substack{Q \in \Delta \\ Q \cap E \cap B(0,1) \neq \emptyset}} \beta_E^m(C_0 Q)^2 \mathrm{diam}(Q)^m \lesssim_{C_0, n, \epsilon, c} \mathcal{H}^m(E \cap B(0,1)) + \Theta_E(B(0,1))$$

Theorem 10 (Azzam-S.)

Same assumptions.

$$\mathcal{H}^m(E \cap B(0,1)) + \Theta_E(B(0,1)) \lesssim_{C_0,n,\epsilon,c} 1 + \sum_{\substack{Q \in \Delta \\ Q \cap E \cap B(0,1) \neq \emptyset}} \beta_E^m(C_0 Q)^2 \mathrm{diam}(Q)^m.$$

Furthermore, if the right hand side is finite, then E is m-rectifiable

Examples

Fang-Jones (90's?): β_∞ not good enough.
Graph of Lipschitz function

$$f : [0,1]^3 :\to \mathbb{R}$$

Draw picture.
If

$$\sum \epsilon_i^3 < \frac{1}{2}$$

then

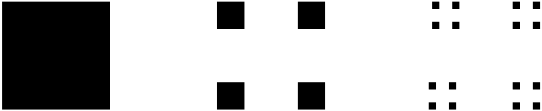
$$\sum \beta_\infty^2(Q)\mathrm{diam}(Q)^3 \sim \sum \epsilon_i^2 = \infty$$

Wanted:

$$\mathcal{H}^m(E \cap B(0,1)) + \Theta_E(B(0,1)) \qquad \sim \qquad 1 + \sum \beta(C_0 Q)^2 \mathrm{diam}(Q)^m.$$

Examples

Need Θ: Take the *Boundary* of the N^{th} stage in the 4 corner cantor set.



- $\sum \beta(C_0 Q)^2 \mathrm{diam}(Q) \sim N$
- On the other side $\mathcal{H}^1 \sim 1$ and $\Theta \sim N$
- Note: if we had Condition B, it would guarantee

$$\Theta \text{ controlled by } \mathcal{H}^m$$

Wanted:

$$\mathcal{H}^m(E \cap B(0,1)) + \Theta_E(B(0,1)) \qquad \sim \qquad 1 + \sum \beta(C_0 Q)^2 \mathrm{diam}(Q)^m .$$

Examples

Accumulating lines: Average betas don't work
Draw picture.
Two options:

$$\beta_{\mu,2}(Q,L)=\left(\frac{1}{\ell(Q)^m}\int_Q\left(\frac{\mathrm{dist}(y,L)}{\ell(Q)}\right)^2 d\mu(y)\right)^{1/2}$$

(yields $\beta(3Q)\lesssim\frac{1}{N-j(Q)}$ and $\sum\beta(3Q)^2\mathrm{diam}(Q)\lesssim\log(N)$) OR

$$\beta_{\mu,2}(Q,L)=\left(\frac{1}{\mu(Q)}\int_Q\left(\frac{\mathrm{dist}(y,L)}{\ell(Q)}\right)^2 d\mu(y)\right)^{1/2}$$

(yields $\beta(3Q)\gtrsim 2^N\ell(Q)$ and $\sum\beta(3Q)^2\mathrm{diam}(Q)\gtrsim N2^N$)
Either way, $\mathcal{H}^1\sim 2^N\sim\Theta$, so no chance!

Wanted:

$$\mathcal{H}^m(E\cap B(0,1))+\Theta_E(B(0,1))\quad\sim\quad 1+\sum\beta(C_0Q)^2\mathrm{diam}(Q)^m.$$

A sketch of a proof

Simple direction:

$$\mathcal{H}^m(E \cap B(0,1)) + \Theta_E(B(0,1)) \quad \lesssim \quad 1 + \sum \beta(C_0 Q)^2 \mathrm{diam}(Q)^m.$$

- A stopping time which reduces to using David-Toro (RF with holes) used to build biLipschitz surfaces whose union $\supset E$

Complicated direction:

$$\mathcal{H}^m(E \cap B(0,1)) + \Theta_E(B(0,1)) \quad \gtrsim \quad 1 + \sum \beta(C_0 Q)^2 \mathrm{diam}(Q)^m.$$

- A stopping time which produces graphs getting closer to E (coronization). (use DT here too!)
- Use Dorronsoro for graphs.
- Show that the upper bound did not grow too much...

QUESTION: what about analogue of Theorems 7 or 8??
(with no abs. cont. assumption!)

$C^{k,\alpha}$-rectifiability

Theorem 11 (Silvia Ghinassi, on arxiv. See poster)

Let $E \subset B(0,1) \subset \mathbb{R}^n$ be a d-dimensional Reifenberg flat set With Holes. Let $\alpha \in (0,1]$. Assume: there is $M < \infty$ such that for all $x \in E$

$$\sum_{k \geq 0} \beta_{E,1}(x, 2^{-k})^2 / 2^{-k\alpha} < M.$$

Then there is a map $f : \mathbb{R}^m \to \mathbb{R}^n$ such that $E \subset f(\mathbb{R}^d)$, f is invertible, and both f and its inverse have directional derivatives which are α-Hölder.

- $\alpha = 0$: David-Toro (get f is bi-Lipschitz).
- $\alpha > 0$: David-Kenig-Toro (no holes), Blatt-Kolasiński (small holes),
- Characterization of $C^{k,\alpha}$ rectifiable measures?? (some work by Kolasiński and collaborators on $k = 1$)

Other notions of rectifiability

For a measure μ:

- Is it Lipschitz-Graph-rectifiable?
- BiLipschitz-rectifiable?
- [your-favorite-class-of-functions]-rectifiable?...
- How do these relate to each other?

Happy Birthday, Guy.

Thank you, Organizers.