Localization of Eigenfunctions via an Effective Potential

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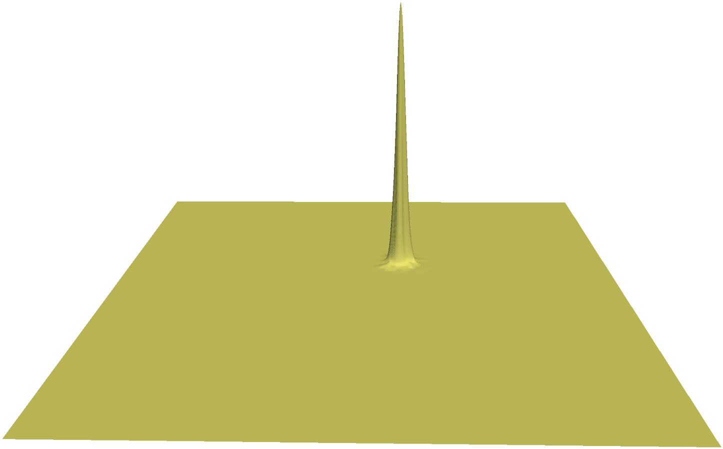
Joint work with Douglas Arnold, Guy David, Marcel Filoche, and Svitlana Mayboroda — **en l'honneur de Guy!**

Anderson Localization

Example:

$$L = -\Delta + V \quad \text{on } 80 \times 80 \text{ square}$$

Choose V constant on unit squares i. i. d. uniformly in $0 \le V \le 4$.



First periodic eigenfunction on 80×80 square

Where is the eigenfunction?

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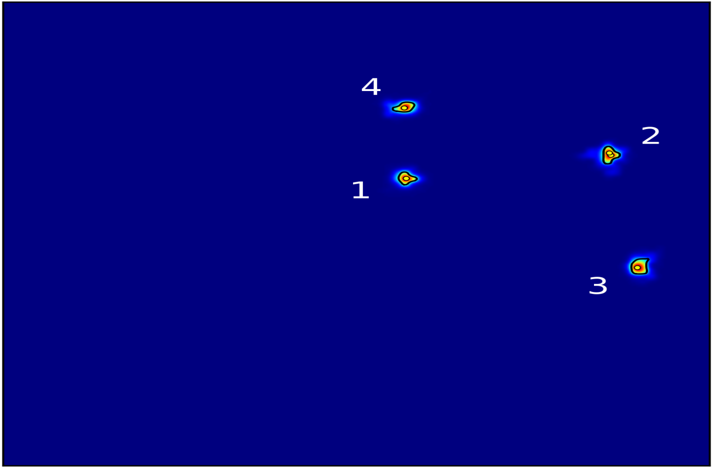
$$u = \sum_j \frac{\langle 1, \psi_j \rangle}{\lambda_j} \psi_j$$



Top view of first four eigenfunctions versus prediction using u

Divide by u: $\tilde{\psi} = \psi/u$.

$$L\psi = \lambda\psi \iff \tilde{L}\tilde{\psi} = \lambda\tilde{\psi}$$

$Lf = -\text{div}\,(\nabla f) + Vf$ implies

$$\tilde{L}g := \frac{1}{u}L(ug) = -\frac{1}{u^2}\text{div}\,(u^2\nabla g) + \frac{1}{u}g$$

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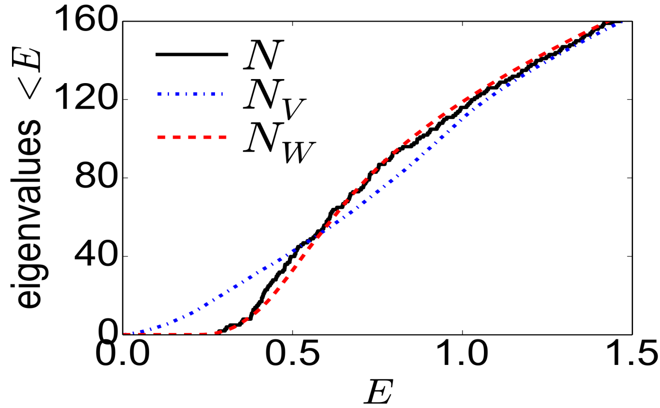
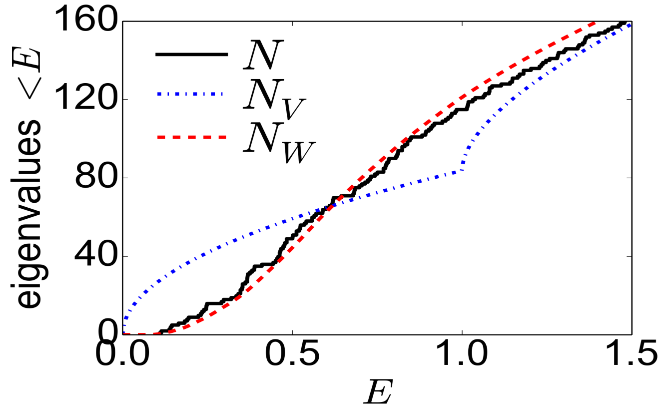
Effective potential $1/u(x)$
Prinicipal symbol of $\tilde{L}$:

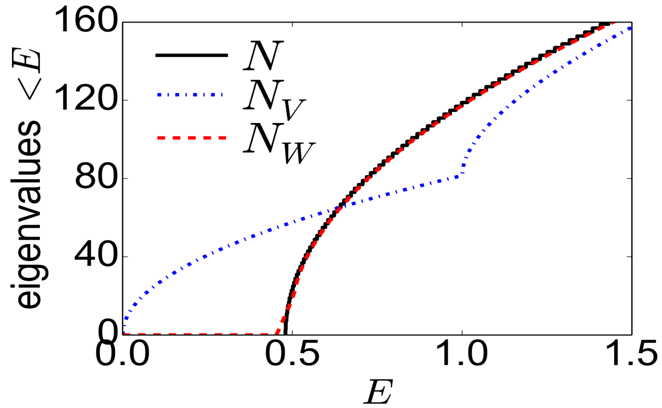
$$\xi^2 + \frac{1}{u(x)}$$



"Weyl law" with $0 \leq V \leq 1$ uniform iid on 512 unit intervals.



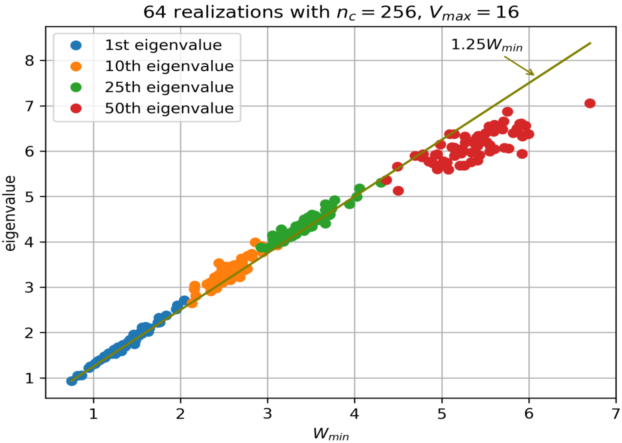
“Weyl law” with $V = 0$ or 1 Bernoulli iid 512 unit intervals.



"Weyl law" with $V = 0$ and $V = 1$ alternating, on 512 unit intervals.

Filoche, Mayboroda and other collaborators used this eigenvalue counting to speed up algorithms to simulate performance of LEDs by a factor of 100 to 1000:

$$\text{2 days} \;\rightarrow\; \text{2 minutes}$$



Prediction of eigenvalues by counting minima of $1/u$

Exponential decay in $\mathbb{R}$

If $\quad -\psi'' + (V(x) - \lambda)\psi \geq \alpha^2\psi,$

and $\psi(x) \to 0$ as $x \to \infty$, then

$$|\psi(x)| \lesssim e^{-\alpha x}, \quad x \to \infty$$

Classical Confinement in $\mathbb{R}^n$

Claim: Eigenfunctions with eigenvalue λ decay exponentially in $\{V(x) - \lambda > 0\}$.

$$\int [|\nabla f|^2 + (V - \lambda) f^2]\, dx \geq \int (V - \lambda) f^2\, dx.$$

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$$\int [|\nabla f|^2 + (V - \lambda) f^2]\, dx \geq \int (V - \lambda) f^2\, dx.$$

$$w(x) = (V(x) - \lambda)_+,$$

$$\langle (L - \lambda) f, f \rangle \geq \langle wf, f \rangle, \quad \text{all } f \in C_0^\infty(\{w > 0\}).$$

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Agmon distance to $\{w = 0\}$ **:**

$$h(x) = \min_\gamma \int_0^1 \sqrt{w(\gamma(t))}|\dot\gamma(t)|\, dt$$

$$\gamma(0) \in \{w=0\},\ \gamma(1) = x.$$

Thm (Agmon) If $L\psi = \lambda\psi$, $\psi \in L^2(\mathbb{R}^n)$, then

$$|\psi| \lesssim e^{-(1-\varepsilon)h(x)}$$

Divide by u: $\tilde{\psi} = \psi/u$.

$$L\psi = \lambda\psi \iff \tilde{L}\tilde{\psi} = \lambda\tilde{\psi}$$

$Lf = -\mathrm{div}\,(\nabla f) + Vf$ implies

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$$\langle Lf, f\rangle = \langle u\nabla(f/u), u\nabla(f/u)\rangle + \langle \frac{1}{u}f, f\rangle \geq \langle \frac{1}{u}f, f\rangle$$

$$\langle (L-\lambda)f, f\rangle \geq \langle \left(\frac{1}{u} - \lambda\right) f, f\rangle$$

The Effective Potential $1/u$

$$w_\lambda(x) := \left(\frac{1}{u} - \lambda\right)_+$$

Then

$$\langle (L-\lambda)f, f\rangle \geq \langle w_\lambda f, f\rangle \text{ for all } f \in C_0^\infty(w_\lambda > 0).$$

Thus $1/u$ replaces V and acts as an effective potential. We will prove exponential decay of eigenfunctions outside the potential well $\{w_\lambda = 0\}$.

Lemma. If $0 \leq V(x) \leq \overline{V}$, $M = (\mathbb{R}/T\mathbb{Z})^n$, $L = -\Delta + V$, $Lu = 1$, then

$$\int_M (|\nabla f|^2 + Vf^2)\, dx = \int_M (u^2 |\nabla (f/u)|^2 + \frac{1}{u} f^2)\, dx$$

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Proof: Use $Lu = 1$ in weak form with test function f^2/u, to obtain

$$\int_M (\nabla u \cdot \nabla (f^2/u) + Vu(f^2/u))\, dx = \int_M 1(f^2/u)\, dx\,.$$

Applying the product rule yields the result. (No integration by parts!)

$$w_\lambda(x) = \left(\frac{1}{u(x)} - \lambda\right)_+, \quad E(\lambda+\delta) = \{1/u(x) \le \lambda+\delta\}$$

$$\rho_\lambda(x,y) = \inf_\gamma \int_0^1 \sqrt{w_\lambda(\gamma(t))}|\dot{\gamma}(t)|\, dt$$

$\gamma(0) = x,\ \gamma(1) = y$

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Theorem 1. If $L\psi = \lambda\psi$, then

$$\int_{h\ge 1} e^h(|\nabla\psi|^2 + \overline{V}\psi^2)\,dx \le 100\frac{\overline{V}}{\delta}\int_M \overline{V}\psi^2\,dx$$

Proof

Substitute $f = \chi e^{h/2}\psi$, $\chi = \min(h, 1)$ in the Lemma.

$$|\nabla h(x)|^2 \le w_\lambda(x) \quad (\le 1/u(x) \le \overline{V})$$

Proof

Substitute $f = \chi e^{h/2}\psi$, $\chi = \min(h, 1)$ in the Lemma.

$$|\nabla h(x)|^2 \leq w_\lambda(x) \quad (\leq 1/u(x) \leq \overline{V})$$

Generalizes to closed C^1 manifolds with C^0 metrics and L^∞ densities and also to the Neumann problem in biLipschitz subdomains. Same proof, same constants.

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$$\mathbb{R}/T\mathbb{Z}, \quad T = 2^{19}, \quad 0 \le V(x) \le 4 \text{ (uniform iid)}$$

has 17 ± 2 intervals in $(1/u - \lambda_0) \le 0$.

$$\text{Agmon distance between wells:} \quad S \sim T^{1/5}.$$

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has 17 ± 2 intervals in $(1/u - \lambda_0) \le 0$.

$$\text{Agmon distance between wells:} \quad S \sim T^{1/5}.$$

There is content because $e^{T^{1/5}} >> T$. But to prove that the deepest of the 17 wins, we will need absence of resonance between wells.

Approximate Diagonalization

Choose a threshold $\bar{\mu}$ and divide

$$E(\bar{\mu}+\delta) = \{1/u \le \bar{\mu}+\delta\} = \bigsqcup_{\ell=1}^{R} E_\ell\,.$$

$$S := \min_{\ell \neq \ell'} \bar{\rho}(E_\ell, E_{\ell'}), \quad (\bar{\rho} = \rho_{\bar{\mu}})$$

Choose Ω_ℓ disjoint such that

$$\{x \in M : \bar{\rho}(x, E_\ell) < (S-\varepsilon)/2\} \subset \Omega_\ell\,.$$

Let $\Psi_{(a,b)}$ be the orthogonal projection onto the span of eigenfunctions of L with e-vals in (a, b).

Let $\varphi_{\ell,j}$, $j = 1, 2, \ldots$ be the Dirchlet eigenfunctions of L on Ω_ℓ. Let $\Phi_{(a,b)}$ be the orthogonal projection onto these eigenfunctions with e-vals in (a, b).

Theorem 2. Set $\varphi = \varphi_{\ell,j}$, $\mu = \mu_{\ell,j}$. If $\mu \le \bar{\mu}$, then

$$\|\varphi - \Psi_{(\mu-\delta,\mu+\delta)}\varphi\|^2 \le 300 \left(\frac{\overline{V}}{\delta}\right)^3 e^{-S/2}$$

Similarly, if $\psi = \psi_j$, $\lambda = \lambda_j$ and $\lambda \le \bar{\mu}$, then

$$\|\psi - \Phi_{(\mu-\delta,\mu+\delta)}\psi\|^2 \le 300 \left(\frac{\overline{V}}{\delta}\right)^3 e^{-S/2}$$

Proof: Choose a cutoff $\eta \in C_0^1(\Omega_\ell)$ so that

$$L(\eta\psi) = \lambda\eta\psi + r$$

satisfies

$$\|r\|_{H^{-1}}^2 \leq 18e^2 \frac{\overline{V}}{\delta} e^{-S/2} \|\psi\|^2$$

And similarly for the Dirichlet eigenfunctions φ.

Corollary. If

$$300N\left(\frac{\overline{V}}{\delta}\right)^3 < e^{S/2},$$

then $\lambda_1, \ldots, \lambda_N$ are within $\pm\delta$ of the first N eigenvalues among $\mu_{\ell,j}$ on Ω_ℓ, $\ell = 1, \ldots, R$.

CONCLUSION

The effective potential

$$\frac{1}{u} \qquad (Lu = 1)$$

yields

- Eigenvalue distribution (bottom half)
- Location and exponential decay of eigenfunctions
- Approximate diagonalization of L

Merci à Guy!