# An inverse spectral theorem on compact Hankel operators linked to the dynamic of some half wave equation

Sandrine Grellier

Université d'Orléans- Fédération Denis Poisson
CIRM october 2017

*joint works with Patrick Gérard (Université Paris sud-Orsay)*

# Aim and motivation

- **Aim 1**: Establish an inverse spectral result on compact Hankel operator:
  Given a sequence of non negative real numbers, decreasing to 0, describe the set of compact Hankel operators having this sequence as singular values.
- **Motivation:** Allow to construct a **non linear Fourier transform** for some model of a degenerate non-dispersive Schrödinger equation: The cubic Szegő equation.
- **Aim 2**: Understand the Sobolev regularity through this transform.

# Hankel operators

$$\mathcal{H}^2(\mathbb{D}) = \{u \in L^2(\mathbb{S}) : u(z) = \sum_{n=0}^{\infty} c_n z^n, \sum_{n=0}^{\infty} |c_n|^2 < \infty\} \ ,$$

$$\Pi : L^2(\mathbb{S}) \longrightarrow \mathcal{H}^2(\mathbb{D}) \text{ the Szegö projector },$$

Given $u \in \mathcal{H}^2(\mathbb{D})$ "smooth", define $H_u$ on $\mathcal{H}^2(\mathbb{D})$ by

$$H_u(h) = \Pi\left(u\overline{h}\right) \ .$$

$H_u$ is an antilinear operator.

# Hankel operators

$$\mathcal{H}^2(\mathbb{D}) = \{u \in L^2(\mathbb{S}) : u(z) = \sum_{n=0}^{\infty} c_n z^n, \sum_{n=0}^{\infty} |c_n|^2 < \infty\} ,$$

$\Pi : L^2(\mathbb{S}) \longrightarrow \mathcal{H}^2(\mathbb{D})$ the Szegö projector ,

Given $u \in \mathcal{H}^2(\mathbb{D})$ "smooth", define $H_u$ on $\mathcal{H}^2(\mathbb{D})$ by

$$H_u(h) = \Pi\left(u\overline{h}\right) .$$

A Hankel matrix on the Fourier side (non self-adjoint in general):

$$\begin{pmatrix} \hat{u}(0) & \hat{u}(1) & \hat{u}(2) & \dots \\ \hat{u}(1) & \hat{u}(2) & \dots & \dots \\ \hat{u}(2) & \dots & \dots & \dots \\ \vdots & \dots & \dots & \dots \end{pmatrix}$$

# Smoothness properties

Well known:

- $H_u$ is of finite rank iff $u$ is a rational (holomorphic) function in $\mathbb{D}$ (Kronecker 1881).
- $H_u$ is Hilbert-Schmidt iff $u \in W^{1/2,2}(\mathbb{S})$

$$\sum_{j,k\geq 0} |\hat{u}(j+k)|^2 = \sum_{\ell\geq 0}(1+\ell)|\hat{u}(\ell)|^2 \simeq \|u\|^2_{W^{1/2,2}}.$$

- $H_u$ belongs to the Schatten class of order $p > 0$ iff $u$ belongs to the Besov space $B_p$ (Peller/Semmes 1984).
- $H_u$ is compact iff $u = \Pi(f)$, $f$ continuous on $\mathbb{S}$– equivalent to $u \in VMOA(\mathbb{S})$– (Hartman 1958).

# Smoothness properties

In all cases (finite rank, Hilbert-Schmidt, Schatten, compact), $H_u$ has a discrete spectrum which consists of the square-roots of the eigenvalues of $H_u^2$.

Inverse spectral problem:

given a sequence of non-negative real numbers, does there exist a Hankel operator having this sequence as singular values?

# The Megretski–Peller–Treil theorem

**Theorem (Megretski–Peller–Treil, 1995)**

*If $(\lambda_j)_{j\geq 1}$ is the sequence of eigenvalues of some selfadjoint compact Hankel operator, then, for every $\lambda \in \mathbb{R} \setminus \{0\}$,*

$$|\#\{j : \lambda_j = \lambda\} - \#\{j : \lambda_j = -\lambda\}| \leq 1 \ .$$

*Conversely, any sequence $(\lambda_j)_{j\geq 1}$ of real numbers satisfying the above condition and tending to $0$ is the sequence of eigenvalues of some selfadjoint compact Hankel operator.*

# An example

Let $|p| < 1$, $v_p(z) = \frac{1}{1-pz} = \sum_{n\geq 0} p^n z^n$, $\alpha \in \mathbb{C}$ and $u = \alpha v_p$.

$$\text{Mat } H_u \text{ " } = \text{ " } \alpha \begin{pmatrix} 1 & p & p^2 & \dots \\ p & p^2 & \dots & \dots \\ p^2 & \dots & \dots & \dots \\ \vdots & \dots & \dots & \dots \end{pmatrix}$$

and the range of $H_u$ is spanned by $v_p$.

# An example

Let $|p| < 1$, $v_p(z) = \frac{1}{1-pz} = \sum_{n\geq 0} p^n z^n$, $\alpha \in \mathbb{C}$ and $u = \alpha v_p$.

$$H_u(v_p) = \frac{\alpha}{1 - |p|^2} v_p; \quad H_u^2(v_p) = \frac{|\alpha|^2}{(1 - |p|^2)^2} v_p.$$

- If $\alpha, p \in \mathbb{R}$, $H_u$ is self-adjoint and $\frac{\alpha}{1-p^2}$ is its eigenvalue. The knowledge of it does not allow to recover $u$.
- If $\alpha, p \in \mathbb{C}$, $H_u$ has one singular value $\frac{|\alpha|}{(1-|p|^2)}$.

# An example – continued

Add $K_u = H_u S = S^* H_u$ the shifted Hankel operator.

$$K_u(v_p) = S^* H_u(v_p) = \frac{\alpha p}{1-|p|^2} v_p, \quad K_u^2(v_p) = \frac{|\alpha p|^2}{(1-|p|^2)^2} v_p.$$

- if $\alpha, p \in \mathbb{R}$, the knowledge of the eigenvalues of $H_u$ and $K_u$ allows to recover $u$.
- If $\alpha, p \in \mathbb{C}$, the arguments of $\alpha$ and $p$ are lacking.

# The inverse spectral result

- $\Omega_n := \{s_1 > s_2 > \cdots > s_n > 0\} \subset \mathbb{R}^n$ .
- $\Omega_\infty = \{(s_n)_{n\geq 1}\ , s_1 > s_2 > \cdots > s_n \to 0\}$ .
- $\mathcal{B} := \cup_{k=0}^{\infty} \mathcal{B}_k$, $\mathcal{B}_k :=$ Blaschke products of degree $k$.

**Theorem (P. Gérard-S.G., 2013)**

*There exists a map*

$$\begin{aligned} \Phi : VMOA(\mathbb{S}) \setminus \{0\} &\to \cup_{n=1}^{\infty} \Omega_n \times \mathcal{B}^n \cup \Omega_\infty \times \mathcal{B}^\infty, \\ u &\mapsto ((s_j), (\Psi_j)), \end{aligned}$$

*$(s_j)$ being the sequence of singular values of $H_u$ and $K_u$, in decreasing order, which is bijective. Moreover, explicit formula.*

# Generic subset

Symbols $u$ corresponding to simple singular values are generic. In that case, the Blaschke products are just angles $\Psi = \mathrm{e}^{i\psi}$:

- The singular values intertwin: $(s_{2j-1})$ are the simple singular values of $H_u$, $(s_{2j})$ are the ones of $K_u$.
- Let $u_j$ be the orthogonal projection of $u$ on $\ker(H_u^2 - s_{2j-1}^2 I)$. Then $H_u(u_j) = s_{2j-1}\mathrm{e}^{-i\psi_{2j-1}} u_j$.
- Let $\tilde{u}_k$ be the orthogonal projection of $u$ on $\ker(K_u^2 - s_{2k}^2 I)$. Then $K_u(\tilde{u}_k) = s_{2k}\mathrm{e}^{i\psi_{2k}} \tilde{u}_k$.

# Example $u = \alpha v_p$, $v_p(z) = \frac{1}{1-pz}$, $s_1 = \frac{|\alpha|}{1-|p|^2}$, $s_2 = |p| s_1$

- Orthogonal projection of $u$ on $\ker(H_u^2 - s_1^2 I) = \mathrm{span}\{v_p\}$:
  $u_{s_1} = u$.
  $H_u(u_{s_1}) = \frac{|\alpha|}{\alpha} s_1 u_{s_1}$; $\frac{|\alpha|}{\alpha} = \mathrm{e}^{-i\psi_1}$.
- Orthogonal projection of $u$ on $\ker(K_u^2 - s_2^2 I) = \mathrm{span}\{v_p\}$
  $\tilde{u}_{s_2} = u$.
  $K_u(\tilde{u}_{s_2}) = s_2 \frac{\overline{\alpha} p}{|\alpha p|} \tilde{u}_{s_2}$; $\frac{\overline{\alpha} p}{|\alpha p|} = \mathrm{e}^{i\psi_2}$.

# Explicit formula

Let $s_1 > s_2 > \cdots > s_{2q-1} > s_{2q} \geq 0$. Introduce the $q \times q$ matrix

$$\mathcal{C}(z) := \left( \frac{s_{2j-1} \mathrm{e}^{i\psi_{2j-1}} - s_{2k} z \mathrm{e}^{i\psi_{2k}}}{s_{2j-1}^2 - s_{2k}^2} \right)_{1 \leq j,k \leq q} .$$

Then, the matrix $\mathcal{C}(z)$ is invertible for any $z \in \mathbb{D}$ and

$$u(z) = \langle \mathcal{C}(z)^{-1} (\mathbf{1}) , \mathbf{1} \rangle .$$

For infinite sequence, limiting procedure.
In our example, $\mathcal{C}(z) = \frac{1-pz}{\alpha}$.

# Explicit formula

Let $s_1 > s_2 > \cdots > s_{2q-1} > s_{2q} \geq 0$. Introduce the $q \times q$ matrix

$$\mathcal{C}(z) := \left( \frac{s_{2j-1} \mathrm{e}^{i\psi_{2j-1}} - s_{2k} z \mathrm{e}^{i\psi_{2k}}}{s_{2j-1}^2 - s_{2k}^2} \right)_{1 \leq j,k \leq q} .$$

Then, the matrix $\mathcal{C}(z)$ is invertible for any $z \in \mathbb{D}$ and

$$u(z) = \langle \mathcal{C}(z)^{-1} (\mathbf{1}), \mathbf{1} \rangle .$$

For infinite sequence, limiting procedure.
In our example, $\mathcal{C}(z) = \frac{1-pz}{\alpha}$.

# Link with cubic Szegö equation

The simultaneous consideration of operators $H_u$ and $K_u$ was suggested by the study of the equation

$$i\frac{\partial}{\partial t}u = \Pi(|u|^2 u),\;\; u = u(t,z);\;\; t \in \mathbb{R},\;\; z \in \mathbb{S}\,.$$

A Hamiltonian system on $\mathcal{H}^2(\mathbb{D})$ endowed with the symplectic structure $\omega(u,v) := \mathrm{Im}\int_{\mathbb{S}} u\bar{v}\frac{dx}{2\pi}$ for

$$E(u) = \frac{1}{4}\int_{\mathbb{S}} |u|^4\,\frac{dx}{2\pi},$$

wellposed on $W^{s,2}(\mathbb{S})$, $s \geq \frac{1}{2}$.
Conservation law: $\|H_u\|_{HS}^2 = \sum_\ell (1+\ell)|\hat{u}(\ell)|^2 \simeq \|u\|^2_{W^{1/2,2}(\mathbb{S})}$ .

# Link with cubic Szegö equation

The simultaneous consideration of operators $H_u$ and $K_u$ was suggested by the study of the equation

$$i\frac{\partial}{\partial t}u = \Pi(|u|^2 u), \;\; u = u(t,z); \;\; t \in \mathbb{R}, \;\; z \in \mathbb{S} \; .$$

This system enjoys a double Lax pair structure,

$$\frac{dH_u}{dt} = [B_u, H_u] \; , \; \frac{dK_u}{dt} = [C_u, K_u] \; .$$

Consequence :

$$H_{u(t)} = U(t) H_{u(0)} U^*(t), \; \frac{dU}{dt} = B_u U,$$

analogous with $K_{u(t)}$.

# Explicit solution

**Theorem (P. Gérard-S.G., 2013)**

*Let* $u_0 = \Phi((s_j), (\Psi_j))$ *then the solution to the Szegő equation with initial datum* $u_0$ *denoted by* $Z(t)u_0$ *is given by*

$$Z(t)u_0 = \Phi((s_j), (\mathrm{e}^{i(-1)^j s_j^2 t}\Psi_j)).$$

The map $\Phi$ is a "non linear Fourier transform" for the cubic Szegő equation.

# Regularity in the Sobolev scale

Let $\sigma = (s_r)$, $(s_r)$ strictly decreasing to 0, with $\sum_{r=1}^{\infty} s_r^p < \infty$, for any $p > 0$. Consider

$$\mathcal{T}(\sigma) = \{u,\ u = \Phi\left(\sigma, (\mathrm{e}^{i\psi_j})\right),\ (\mathrm{e}^{i\psi_j})_{j\geq 1} \in \mathbb{T}^{\infty}\}$$

From Peller/Semmes

$$\mathcal{T}(\sigma) \subset (\cap_{p>0} B_p).$$

What about Sobolev regularity? $B_2 \cap L^2(\mathbb{S}) = W^{1/2,2}$ but higher smoothness?

# Regularity in the Sobolev scale

Let $\sigma = (s_r)$, $(s_r)$ strictly decreasing to 0, with $\sum_{r=1}^{\infty} s_r^p < \infty$, for any $p > 0$. Consider

$$\mathcal{T}(\sigma) = \{u,\ u = \Phi\left(\sigma, (\mathrm{e}^{i\psi_j})\right),\ (\mathrm{e}^{i\psi_j})_{j\geq 1} \in \mathbb{T}^\infty\}$$

**Theorem (P. Gérard-S.G., 2017)**

- $\exists \sigma;\ \mathcal{T}(\sigma)$ *is unbounded in $W^{s,2}$ for any $s > \frac{1}{2}$ (may be it is neither included in it).*
- *If $s_{r+1} = \varepsilon_r s_r$, $\varepsilon_r \in [0, \delta]$, $\delta < 1$ small enough,*
  *$\mathcal{T}(\sigma)$ is bounded in the set of holomorphic functions on the disc of radius $1 + \rho$ for some $\rho > 0$.*

# Unboundedness in the Sobolev spaces: consequence on the Szegő dynamics

**Theorem (P. Gérard-S.G., 2015)**

*The set of* $u_0 \in C^\infty(\mathbb{S}) \cap \mathcal{H}^2(\mathbb{D})$ *such that* $\forall s > \frac{1}{2}$ ,

$$\limsup_{|t|\to+\infty} \|Z(t)u_0\|_{W^{s,2}} = +\infty$$

*is a dense* $G_\delta$ *subset of* $\mathcal{C}^\infty(\mathbb{S}) \cap \mathcal{H}^2(\mathbb{D})$.

Weak turbulence phenomenon!

# Unboundedness in the Sobolev spaces: consequence on the Szegő dynamics

**Theorem (P. Gérard-S.G., 2015)**

*The set of* $u_0 \in C^\infty(\mathbb{S}) \cap \mathcal{H}^2(\mathbb{D})$ *such that* $\forall s > \frac{1}{2}$ ,

$$\limsup_{|t|\to+\infty} \|Z(t)u_0\|_{W^{s,2}} = +\infty$$

*is a dense* $G_\delta$ *subset of* $\mathcal{C}^\infty(\mathbb{S}) \cap \mathcal{H}^2(\mathbb{D})$.

In particular, there exists $u_0 \in \mathcal{C}^\infty(\mathbb{S}) \cap \mathcal{H}^2(\mathbb{D})$ with,

$$\forall s > \frac{1}{2}, \ \limsup_{|t|\to+\infty} \|Z(t)u_0\|_{W^{s,2}} = +\infty.$$

All these functions are in the same $\mathcal{T}(\sigma)$ for some $\sigma$.

# Proof of the weak turbulence phenomenon

Consider

$$\mathcal{O}_p := \{u_0 \in \mathcal{C}^\infty(\mathbb{S}) \cap \mathcal{H}^2(\mathbb{D});\ \ \exists t_p > p, \qquad \|Z(t_p)u_0\|_{W^{1/2+1/p,2}} > p\}$$

- Wellposedness of Szegő: for any $p \geq 1$, $\mathcal{O}_p$ is open.
- Density argument via explicit formula:

$$\text{for any } p \geq 1,\ \mathcal{O}_p \text{ is dense.}$$

- Baire category argument.

# The density argument

Let $v_0 \in \mathcal{C}^\infty(\mathbb{S}) \cap \mathcal{H}^2(\mathbb{D})$. By genericity, one may assume that

$$\Phi(v_0) = ((s_j)_{1\leq j\leq 2q}, (\mathrm{e}^{i\psi_j})_{1\leq j\leq 2q})$$

for some $q$. One has to approximate $v_0$ by a function in $\mathcal{O}_p$. We construct $v_0^{\varepsilon,\delta} =$
$\Phi\left((s_1, \dots, s_{2q}, \delta(1+\varepsilon), \delta, \delta(1-\varepsilon)), (\mathrm{e}^{i\psi_1}, \dots, \mathrm{e}^{i\psi_{2q}}, 1, 1, -1)\right)$.
CLAIM: $v_0^{\varepsilon,\delta} \to v_0$ and $v_0^{\varepsilon,\delta} \in \mathcal{O}_p$ for good choice of parameters.

# A baby example with large Sobolev norm

$$s_1 = (1+\varepsilon)\ ,\ s_2 = 1\ ,\ s_3 = (1-\varepsilon)$$
$$e^{i\psi_1} = 1\ ,\ e^{i\psi_2} = 1\ ,\ e^{i\psi_3} = -1\ (\text{state } 1) \text{ or } 1\ (\text{state } 2)\ .$$

State 1

$$\left\langle \begin{pmatrix} \frac{1+\varepsilon-z}{(1+\varepsilon)^2-1} & \frac{1}{1+\varepsilon} \\ \frac{-(1-\varepsilon)-z}{(1-\varepsilon)^2-1} & \frac{-1}{1-\varepsilon} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle_{\mathbb{C}^2\times\mathbb{C}^2} = \frac{2z(1-\varepsilon^2)+3\varepsilon}{2-\varepsilon z}$$

State 2

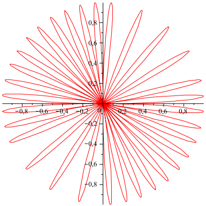
$$\left\langle \begin{pmatrix} \frac{1+\varepsilon-z}{(1+\varepsilon)^2-1} & \frac{1}{1+\varepsilon} \\ \frac{(1-\varepsilon)-z}{(1-\varepsilon)^2-1} & \frac{1}{1-\varepsilon} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle_{\mathbb{C}^2\times\mathbb{C}^2} = \frac{2+\varepsilon^2-2z(1-\varepsilon^2)}{2-(2-\varepsilon^2)z}$$

CLAIM: "State 2" may be reached from "state 1" with Szegő.

# A similar example

Start with the datum $u_0^\varepsilon(z) = z + \varepsilon$ then $Z(t)u_0^\varepsilon(z) = \frac{a^\varepsilon(t)z + b^\varepsilon(t)}{1 - p^\varepsilon(t)z}$

and $1 - |p^\varepsilon(t_n^\varepsilon)| \sim \varepsilon^2$ , $t_n^\varepsilon \sim \frac{(2n+1)\pi}{2\varepsilon}$.




$$\|Z(t_n^\varepsilon)u_0^\varepsilon(z)\|_{W^{s,2}}^2 \sim \frac{1}{\varepsilon^{2(2s-1)}} \sim |t_n^\varepsilon|^{2(2s-1)}, \; s > \frac{1}{2}.$$

# Bounded analytic symbols: ”geometrically spaced singular values”

**Theorem (P. Gérard-S.G., 2017)**

1. *Let $(\varepsilon_r)$ real numbers in $]0,\delta]$, $\delta \in ]0,1[$. Assume $s_{r+1} = \varepsilon_r s_r$. Then, for $\delta$ sufficiently small, $\exists \rho > 0$, $u := \Phi\left((s_r),(\mathrm{e}^{i\psi_r})\right)$ is analytic in the disc of radius $1+\rho$, and is uniformly bounded in this disc for any choice of $(\psi_r)$.*
2. *Let $h > 0$ and $\theta \in \mathbb{R}$. There exists $\rho > 0$, so that the function $u := \Phi\left((\mathrm{e}^{-rh}),(\mathrm{e}^{ir\theta h})\right)$ is analytic in the disc of radius $1+\rho$, and is uniformly bounded in this disc.*

**Note:** The second setting was suggested by J.P. Kahane.

# Idea of proof : $s_{r+1} = \varepsilon_r s_r$, $0 \le \varepsilon_r \le \delta < 1$

Recall

$$\begin{aligned}
\Phi((s_r),(\mathrm{e}^{i\psi_r})) &= \lim_{N\to\infty} u_N \\
u_N(z) &:= \langle \mathcal{C}_N(z)^{-1}(\mathbf{1}), \mathbf{1} \rangle \\
\mathcal{C}_N(z) &= \left( \frac{s_{2j-1}\mathrm{e}^{i\psi_{2j-1}} - z s_{2k}\mathrm{e}^{i\psi_{2k}}}{s_{2j-1}^2 - {s_{2k}}^2} \right)_{1\le j,k\le N} .
\end{aligned}$$

Estimate $\mathcal{C}_N(z)^{-1}$ independently of $N$.

## Idea of proof : $s_{r+1} = \varepsilon_r s_r, \quad 0 \leq \varepsilon_r \leq \delta < 1$

$$\mathcal{C}_N(z) = \left( \frac{s_{2j-1} \mathrm{e}^{i\psi_{2j-1}} - z s_{2k} \mathrm{e}^{i\psi_{2k}}}{s_{2j-1}^2 - {s_{2k}}^2} \right)_{1 \leq j,k \leq N} .$$

For $|z| = 1$, it is a ”Complex” Cauchy matrix

$$\left( \frac{a_j - b_k}{|a_j|^2 - |b_k|^2} \right)_{1 \leq j,k \leq N}$$

which coincides with the Cauchy matrix $\left( \frac{1}{a_j + b_k} \right)_{1 \leq j,k \leq N}$ when $a_j, b_k$ are real. Explicit formula for inverse of Cauchy matrices.

# Idea of proof : $s_{r+1} = \varepsilon_r s_r, \quad 0 \leq \varepsilon_r \leq \delta < 1$

Write $\mathcal{C}_N(z) = \mathcal{C}_N(0) - z\dot{\mathcal{C}}_N$ with $\mathcal{C}_N(0)$ a Cauchy matrix.
Hence,

$$\langle \mathcal{C}_N(z)^{-1}(\mathbf{1}), \mathbf{1}\rangle = \langle (I - z\mathcal{C}_N(0)^{-1}\dot{\mathcal{C}}_N)^{-1}\mathcal{C}_N(0)^{-1}, \mathbf{1}\rangle .$$

Establish $\|(\mathcal{C}_N(0)^{-1})(\mathbf{1})\|_{\ell^1} \leq C$.
and prove $\|\mathcal{C}_N(0)^{-1}\dot{\mathcal{C}}_N\|_{\ell^1 \to \ell^1} < \frac{1}{(1+\rho)}$ for $\delta$ small enough.
$(u_N(z))_N$ defines a uniformly convergent sequence of bounded analytic functions on $|z| < 1 + \rho$.

# Totally geometric case: $(s_r, \Psi_r) = (\mathrm{e}^{-rh}, \mathrm{e}^{ir\theta h})$.

Recall

$$\mathcal{C}_N(z) = \left(\frac{s_{2j-1}\mathrm{e}^{i\psi_{2j-1}} - zs_{2k}\mathrm{e}^{i\psi_{2k}}}{s_{2j-1}^2 - {s_{2k}}^2}\right)_{1\le j,k\le N}.$$

For $\omega = \mathrm{e}^{-h(1-i\theta)}$

$$\begin{aligned}\mathcal{C}_N(z) &= \left(\frac{\omega^{2j-1} - z\omega^{2k}}{|\omega|^{4j-2} - |\omega|^{4k}}\right)_{j,k=1}^N = \left(\frac{1}{\overline{\omega}^{2j-1}}\frac{1 - z\omega^{2(k-j)+1}}{1 - |\omega|^{4(k-j)+2}}\right)_{j,k=1}^N \\ &= \left(\frac{1}{\overline{\omega}^{2j-1}}\varphi_z(k-j)\right)_{j,k=1}^N\end{aligned}$$

a truncated Toeplitz matrix.

# Totally geometric case: $(s_r, \Psi_r) = (\mathrm{e}^{-rh}, \mathrm{e}^{ir\theta h})$.

$$\mathcal{C}_N(z) \quad = \quad \left(\frac{1}{\overline{\omega}^{2j-1}}\varphi_z(k-j)\right)_{j,k=1}^N .$$

Hence

$$u_N(z) = <^t T_N(\varphi_z)^{-1}(\overline{\omega}^{2j-1}), (\mathbf{1}) >$$

where $T_N(\varphi_z)$ is the truncated Toeplitz matrix

$$\left(\frac{1 - z\omega^{2(j-k)+1}}{1 - |\omega|^{4(j-k)+2}}\right)_{j,k=1}^N = (\varphi_z(j-k))_{1\leq j,k\leq N} \cdot$$

# Totally geometric case: $(s_r, \Psi_r) = (\mathrm{e}^{-rh}, \mathrm{e}^{ir\theta h})$.

Use a theorem from Baxter (63):
The sequence ot truncated Toeplitz matrices

$$T_N(\varphi) := (\varphi(j-k))_{1\leq j,k\leq N}$$

is stable: $\sup_{N\geq N_0} \| T_N(\varphi)^{-1} \|_{\ell^2\to\ell^2} < \infty$
iff $T(\varphi) : f \mapsto \Pi(\varphi f)$ is invertible on $\mathcal{H}^2(\mathbb{D})$
or iff

$$\varphi(\zeta) := \sum_{j=0}^{\infty} \varphi(j)\zeta^j$$

has index 0 and does not vanish on the unit sphere.

# Invertibility of $T(\varphi_z)$

Recall $u_N(z) = <^t T_N(\varphi_z)^{-1}(\overline{\omega}^{2j-1}), (\mathbf{1})_{1\leq k\leq N} >$ where

$$\varphi_z(\zeta) = \sum_{\ell\in\mathbb{Z}} \frac{1 - z\omega^{2\ell+1}}{1 - |\omega|^{4\ell+2}} \zeta^\ell .$$

## Lemma

1. *There exists $r < 1$ close to 1, such that, the function $\zeta \mapsto \varphi_0(r\zeta)$ has index zero.*
2. *There exists $\rho > 0$ such that $\varphi_z(\zeta)$ does not vanish in a neighborhood of the circle $|\zeta| = 1$ for any $|z| < 1 + \rho$.*

# $\varphi_0$ has index $I(R) := \frac{1}{2i\pi}\int_{\mathcal{C}_R} \frac{\varphi_0'(\zeta)}{\varphi_0(\zeta)} d\zeta = 0$ for $R \to 1^-$

$$
\begin{aligned}
\varphi_0(\zeta) &= \sum_{j\in\mathbb{Z}} \frac{\zeta^j}{1-\gamma^{2j+1}} = \sum_{\ell\in\mathbb{Z}} \frac{\gamma^\ell}{1-\zeta\gamma^{2\ell}},\ \gamma = |\omega|^2. \\
\varphi_0\left(\frac{1}{\zeta}\right) &= -\zeta\varphi_0(\zeta),\ I(R) + I\left(\frac{1}{R}\right) = -1, \\
\varphi_0\left(\frac{\zeta}{\gamma^2}\right) &= \gamma\varphi_0(\zeta),\ I(R\gamma^2) = I(R). \\
\varphi_0(\gamma^{2\ell+1}) &= 0,\ \ell\in\mathbb{Z}.
\end{aligned}
$$

The poles of $\varphi_0$ are $\gamma^{2\ell},\ \ell\in\mathbb{Z}$.
$I$ is valued in $\mathbb{Z}$, continuous on the intervals corresponding to circles avoiding the zeroes and the poles of $\varphi_0$.

# Technical point

$$\mathcal{C}_N(z) = \left(\frac{r^j}{\overline{\omega}^{2j-1}} \frac{1 - z\omega^{2(k-j)+1}}{1 - |\omega|^{4(k-j)+2}} \frac{r^{k-j}}{r^k}\right)_{j,k=1}^N = \left(\frac{r^j}{\overline{\omega}^{2j-1}} \varphi_z(k-j) \frac{1}{r^k}\right)_{j,k=1}^N .$$

Hence

$$u_N(z) = <^t T_N(\varphi_z^{(r)})^{-1}(r^{-j}\overline{\omega}^{2j-1}), (r^k)_{k=1}^N >$$

where $T_N(\varphi_z^{(r)})$ is the truncated Toeplitz matrix

$$\left(\frac{1 - z\omega^{2(j-k)+1}}{1 - |\omega|^{4(j-k)+2}} r^{j-k}\right)_{j,k=1}^N = \left(\varphi_z^{(r)}(j-k)\right)_{1 \le j,k \le N} .$$

THANKS FOR YOUR ATTENTION!


BONNE FÊTE D'ANNIVERSAIRE GUY!