An application of David-Mattila cubes to non-homogeneous Calderón-Zygmund theory

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Polynomial growth measures

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- A polynomial measure may be doubling. Example: Lebesgue measure on $\mathbb{R}^d$.
- It may also be nondoubling. Example: Hausdorff measure $\mathcal{H}^n|_A$ on a sufficiently bad A. However, there are always many (α, β)-doubling balls, that is, balls B such that

$$\mu(\alpha B) \leq \beta \mu(B)$$

(β needs to be large enough wrt α).

A result by David and Mattila

A compact set $E \subset \mathbb{C}$ is removable for Lipschitz harmonic functions if whenever $U \supset E$ open, $u : U \to \mathbb{R}$ is a Lipschitz function which is harmonic in $U \setminus E$ then u is harmonic in U.

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Let $E \subset \mathbb{C}$ be compact with $\mathcal{H}^1|_E < \infty$. Then E is removable for Lipschitz harmonic functions if and only if E is purely unrectifiable.

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- The proof requires the construction (and handling) of a 1-polynomial measure on a possibly wild set.
- The handling requires the introduction of a filtration that plays the role of the dyadic cubes.

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3. Each $Q \in \mathscr{D}$ is either $(200, \beta)$-doubling or $r(B_Q) = A^{-k}$ and

$$\mu(100B_Q) \leq \beta^{-\ell}\mu(100^\ell B_Q), \ (\ell > 1, \ 100^\ell \leq \beta).$$

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Rmk: (3) means that between consecutive doubling cubes Q and R in $\mathscr{D}$,

$$\int_{100B_R \setminus 100B_Q} \frac{1}{|x - x_{B_Q}|^n} d\mu(x) \lesssim 1.$$

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- Examples:
  - Cauchy transform:

$$\mathcal{C}_\mu f(z) = \text{p.v.} \int_{\mathbb{C}} \frac{f(\zeta)}{z-\zeta} d\mu(\zeta).$$

  - Riesz transform:

$$\mathcal{R}_\mu f(x) = \text{p.v.} \int_{\mathbb{R}^d} \frac{x-y}{|x-y|^{n+1}} f(y) d\mu(y).$$

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$$\|f\|_{\mathrm{RBMO}(\mu)} := \sup_{B \text{ doubling}} \frac{1}{\mu(B)} \int_B |f - \langle f\rangle_B| d\mu + \sup_{B\subset B',\text{ both doubling}} \frac{|\langle f\rangle_B - \langle f\rangle_{B'}|}{K_{B,B'}} < \infty,$$

where

$$K_{B,B'} = 1 + \int_{2B'\setminus 2B} \frac{1}{|x_B - x|^n} d\mu(x).$$

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$$\Longleftrightarrow$$

The filtration $\mathscr{D}$ is **regular**: $\mathsf{E}_k |f| \lesssim \mathsf{E}_{k-1} |f|$.

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Proof

See [David-Mattila] and modify (a little bit). $\square$

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Form the conditional expectations w.r.t. Σ_k:

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Theorem (Tolsa '01)

$$\left(\mathrm{H}^1_{\mathrm{atb}}(\mu)\right)^* = \mathrm{RBMO}(\mu).$$

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$$\|f\|_{\mathrm{H}^1} := \Big\| \Big(\sum_k |\mathsf{E}_k f - \mathsf{E}_{k-1} f|^2 \Big)^{\frac{1}{2}} \Big\|_1 .$$

Theorem (C., Parcet)

$f \in \mathrm{H}^1 \Leftrightarrow f = \sum_j b_j$, where:

- There exists k_j s.t. $\mathsf{E}_{\Sigma_{k_j}}(b_j) = 0$.
- $b_j = \sum_i \lambda_{ij} a_{ij}$. Each a_{ij} is supported in a k_{j+1} measurable set A_{ij}.
- $\|a_{ij}\|_\infty \leq \mu(A_{ij})^{-1}$.

Also,

$$\|f\|_{\mathrm{H}^1} \sim \inf_{f = \sum_j b_j = \sum_{i,j} \lambda_{ij} a_{ij}} \sum_{i,j} |\lambda_{ij}| .$$

A dyadic Calderón-Zygmund decomposition

The filtration Σ allows a version of the Calderón-Zygmund decomposition very similar to the classical one:

Theorem (C., Parcet)

Fix $\lambda > 0$, $f \in L^1(\mu)$, and denote by $\mathcal{Q}$ the family of maximal cubes in Σ w.r.t. $\langle f \rangle_Q > \lambda$. Then

$$f = g + \sum_{Q \in \mathcal{Q}} b_Q = f 1_{(\cup_{Q\in\mathcal{Q}} Q)^c} + \sum_{Q \in \mathcal{Q}} \langle f 1_Q \rangle_{\widehat{Q}} 1_{\widehat{Q}}$$

$$+ \sum_{Q \in \mathcal{Q}} \Big[f 1_Q - \langle f 1_Q \rangle_{\widehat{Q}} 1_{\widehat{Q}} \Big].$$

- $\|g\|_{L^2(\mu)} \lesssim \lambda \|f\|_{L^1(\mu)}$.
- $\int b_Q = 0$, $\sum_{Q\in\mathcal{Q}} \|b_Q\|_{L^1(\mu)} \lesssim \|f\|_{L^1(\mu)}$.

Thank you very much.