Rectifiability of metric spaces via arbitrarily small perturbations

David Bate

University of Helsinki

Rectifiable metric spaces

- (X, d) metric space. $S \subset X$ is n-rectifiable if there exist countably many Lipschitz (equivalently biLipschitz) $f_i \colon A_i \subset \mathbb{R}^n \to X$ such that

$$\mathcal{H}^n(S \setminus \bigcup f_i(A_i)) = 0.$$

Rectifiable metric spaces

- (X, d) metric space. $S \subset X$ is n-rectifiable if there exist countably many Lipschitz (equivalently biLipschitz) $f_i \colon A_i \subset \mathbb{R}^n \to X$ such that

$$\mathcal{H}^n(S \setminus \bigcup f_i(A_i)) = 0.$$

- S is purely n-unrectifiable if every n-rectifiable subset of S has $\mathcal{H}^n$ measure zero. If $\mathcal{H}^n(X) < \infty$ then $X = U \cup R$, U purely n-unrectifiable and R n-rectifiable.

Rectifiable metric spaces

- (X, d) metric space. $S \subset X$ is n-rectifiable if there exist countably many Lipschitz (equivalently biLipschitz) $f_i \colon A_i \subset \mathbb{R}^n \to X$ such that
$$\mathcal{H}^n(S \setminus \bigcup f_i(A_i)) = 0.$$
- S is purely n-unrectifiable if every n-rectifiable subset of S has $\mathcal{H}^n$ measure zero. If $\mathcal{H}^n(X) < \infty$ then $X = U \cup R$, U purely n-unrectifiable and R n-rectifiable.
- Classically (when $X = \mathbb{R}^m$), a fundamental description of rectifiable sets is given by the Besicovitch-Federer projection theorem: $\mathcal{H}^n(S) < \infty$, S purely n-unrectifiable $\Rightarrow$ almost every n-dimensional orthogonal projection of S has Lebesgue measure zero.

BF in non-Euclidean settings I

- Is it possible to obtain a similar characterisation in non-Euclidean settings?

BF in non-Euclidean settings I

- Is it possible to obtain a similar characterisation in non-Euclidean settings?
- Metric spaces have no linear structure $\Rightarrow$ no notion of projection.

BF in non-Euclidean settings I

- Is it possible to obtain a similar characterisation in non-Euclidean settings?
- Metric spaces have no linear structure $\Rightarrow$ no notion of projection.
- In (infinite dimensional) Banach spaces: Projection $=$ continuous linear $T\colon B \to \mathbb{R}^n$ (of full rank).

BF in non-Euclidean settings I

- Is it possible to obtain a similar characterisation in non-Euclidean settings?
- Metric spaces have no linear structure $\Rightarrow$ no notion of projection.
- In (infinite dimensional) Banach spaces: Projection $=$ continuous linear $T\colon B \to \mathbb{R}^n$ (of full rank).
- "Almost every" projection? Prescribe a collection of null sets. Standard examples exist in the theory of GMT in Banach spaces.

BF in non-Euclidean settings II

Theorem (De Pauw)

There exists a purely 1-unrectifiable $S \subset \ell_2$ with $\mathcal{H}^1(S) < \infty$ such that $|T(S)| > 0$ for a non "Aronszajn" null set of projections.

BF in non-Euclidean settings II

Theorem (De Pauw)

There exists a purely 1-unrectifiable $S \subset \ell_2$ with $\mathcal{H}^1(S) < \infty$ such that $|T(S)| > 0$ for a non "Aronszajn" null set of projections.

- BF is false in ℓ_2 for “Aronszajn almost every projection”.

BF in non-Euclidean settings II

Theorem (De Pauw)

There exists a purely 1-unrectifiable $S \subset \ell_2$ *with* $\mathcal{H}^1(S) < \infty$ *such that* $|T(S)| > 0$ *for a non "Aronszajn" null set of projections.*

- BF is false in ℓ_2 for "Aronszajn almost every projection".
- However, this set of projections **is** "Haar" null.

BF in non-Euclidean settings II

Theorem (De Pauw)

There exists a purely 1-unrectifiable $S \subset \ell_2$ *with* $\mathcal{H}^1(S) < \infty$ *such that* $|T(S)| > 0$ *for a non "Aronszajn" null set of projections.*

- BF is false in ℓ_2 for “Aronszajn almost every projection”.
- However, this set of projections **is** “Haar” null.

BF in non-Euclidean settings II

Theorem (De Pauw)

There exists a purely 1-unrectifiable $S \subset \ell_2$ *with* $\mathcal{H}^1(S) < \infty$ *such that* $|T(S)| > 0$ *for a non "Aronszajn" null set of projections.*

- BF is false in ℓ_2 for “Aronszajn almost every projection”.
- However, this set of projections **is** “Haar” null.

Theorem (B, Csörnyei, Wilson)

In any separable infinite dimensional Banach space, there exists a purely 1-unrectifiable S *with* $\mathcal{H}^1(S) < \infty$ *such that* $|T(S)| > 0$ *for* ***every*** *projection.*

BF in non-Euclidean settings II

Theorem (De Pauw)

There exists a purely 1-unrectifiable $S \subset \ell_2$ with $\mathcal{H}^1(S) < \infty$ such that $|T(S)| > 0$ for a non "Aronszajn" null set of projections.

- BF is false in ℓ_2 for "Aronszajn almost every projection".
- However, this set of projections **is** "Haar" null.

Theorem (B, Csörnyei, Wilson)

In any separable infinite dimensional Banach space, there exists a purely 1-unrectifiable S with $\mathcal{H}^1(S) < \infty$ such that $|T(S)| > 0$ for ***every*** *projection.*

- BF is completely false in infinite dimensional spaces.

A new approach

- It is natural to consider Lipschitz mappings of a metric space X into an Euclidean space.

A new approach

- It is natural to consider Lipschitz mappings of a metric space X into an Euclidean space.
- Let $\mathrm{Lip}_1(X, \mathbb{R}^m)$ be the set of all bounded 1-Lipschitz $f \colon X \to \mathbb{R}^m$ equipped with the supremum norm.

A new approach

- It is natural to consider Lipschitz mappings of a metric space X into an Euclidean space.
- Let $\mathrm{Lip}_1(X, \mathbb{R}^m)$ be the set of all bounded 1-Lipschitz $f\colon X \to \mathbb{R}^m$ equipped with the supremum norm.
- This is a complete metric space and so we can consider a typical 1-Lipschitz function (i.e. residual/comeagre in the sense of Baire category: a set that contains a countable intersection of open dense sets).

A new approach

- It is natural to consider Lipschitz mappings of a metric space X into an Euclidean space.
- Let $\mathrm{Lip}_1(X, \mathbb{R}^m)$ be the set of all bounded 1-Lipschitz $f\colon X \to \mathbb{R}^m$ equipped with the supremum norm.
- This is a complete metric space and so we can consider a typical 1-Lipschitz function (i.e. residual/comeagre in the sense of Baire category: a set that contains a countable intersection of open dense sets).
- "A typical 1-Lipschitz function" is a suitable candidate to replace "almost every projection".

A new characterisation

Theorem (B)

Let $S \subset X$ be purely n-unrectifiable with $\mathcal{H}^n(S) < \infty$ and

$$\liminf_{r \to 0} \frac{\mathcal{H}^n(B(x,r))}{r^n} > 0 \qquad (*)$$

for $\mathcal{H}^n$-a.e. $x \in S$.

For any $m \in \mathbb{N}$, a typical $f \in \mathrm{Lip}_1(X, \mathbb{R}^m)$ satisfies

$$\mathcal{H}^n(f(S)) = 0.$$

A new characterisation

Theorem (B)

Let $S \subset X$ be purely n-unrectifiable with $\mathcal{H}^n(S) < \infty$ and

$$\liminf_{r \to 0} \frac{\mathcal{H}^n(B(x,r))}{r^n} > 0 \qquad (*)$$

for $\mathcal{H}^n$-a.e. $x \in S$.

For any $m \in \mathbb{N}$, a typical $f \in \mathrm{Lip}_1(X, \mathbb{R}^m)$ satisfies

$$\mathcal{H}^n(f(S)) = 0.$$

- If $S \subset \mathbb{R}^{m'}$, $(*)$ is not necessary.

A new characterisation

Theorem (B)

Let $S \subset X$ *be purely n-unrectifiable with* $\mathcal{H}^n(S) < \infty$ *and*

$$\liminf_{r\to 0} \frac{\mathcal{H}^n(B(x,r))}{r^n} > 0 \qquad (*)$$

for $\mathcal{H}^n$*-a.e.* $x \in S$.

For any $m \in \mathbb{N}$, *a typical* $f \in \mathrm{Lip}_1(X, \mathbb{R}^m)$ *satisfies*

$$\mathcal{H}^n(f(S)) = 0.$$

- If $S \subset \mathbb{R}^{m'}$, $(*)$ is not necessary.
- Using deep results of the structure of Lebesgue null sets announced by Csörnyei-Jones, $(*)$ is never necessary.

A new characterisation

Theorem (B)

Let $S \subset X$ be purely n-unrectifiable with $\mathcal{H}^n(S) < \infty$ and

$$\liminf_{r \to 0} \frac{\mathcal{H}^n(B(x,r))}{r^n} > 0 \qquad (*)$$

for $\mathcal{H}^n$-a.e. $x \in S$.

For any $m \in \mathbb{N}$, a typical $f \in \mathrm{Lip}_1(X, \mathbb{R}^m)$ satisfies

$$\mathcal{H}^n(f(S)) = 0.$$

- If $S \subset \mathbb{R}^{m'}$, $(*)$ is not necessary.
- Using deep results of the structure of Lebesgue null sets announced by Csörnyei-Jones, $(*)$ is never necessary.
- If $\mathcal{H}^s(S) < \infty$ with $s \notin \mathbb{N}$, then a typical $f \in \mathrm{Lip}_1(X, \mathbb{R}^m)$ satisfies $\mathcal{H}^s(f(S)) = 0$.

The converse statement

As with BF, we get a strong converse.

The converse statement

As with BF, we get a strong converse.

Theorem (B)

Let $S \subset X$ be n-rectifiable. For any $m \geq n$, a typical $f \in \mathrm{Lip}_1(X, \mathbb{R}^m)$ satisfies

$$\mathcal{H}^n(f(S)) > 0.$$

The converse statement

As with BF, we get a strong converse.

Theorem (B)

Let $S \subset X$ be n-rectifiable. For any $m \geq n$, a typical $f \in \mathrm{Lip}_1(X, \mathbb{R}^m)$ satisfies

$$\mathcal{H}^n(f(S)) > 0.$$

- This direction is simpler: uses Kirchheim's description of rectifiable metric spaces.

Idea of the proof of the main direction

Given $f \in \mathrm{Lip}_1(X, \mathbb{R}^m)$, we must make arbitrarily small modifications to obtain a $\tilde{f}$ such that $\mathcal{H}^n(\tilde{f}(S))$ is arbitrarily small. These modifications **cannot** increase the Lipschitz constant.

Idea of the proof of the main direction

Given $f \in \mathrm{Lip}_1(X, \mathbb{R}^m)$, we must make arbitrarily small modifications to obtain a $\tilde{f}$ such that $\mathcal{H}^n(\tilde{f}(S))$ is arbitrarily small. These modifications **cannot** increase the Lipschitz constant.

Theorem (B, Li 2014)

Let $S \subset X$ satisfy $\mathcal{H}^n(S) < \infty + (*)$. If S has n "Alberti representations", then S is n-rectifiable.

Idea of the proof of the main direction

Given $f \in \mathrm{Lip}_1(X, \mathbb{R}^m)$, we must make arbitrarily small modifications to obtain a $\tilde{f}$ such that $\mathcal{H}^n(\tilde{f}(S))$ is arbitrarily small. These modifications **cannot** increase the Lipschitz constant.

Theorem (B, Li 2014)

Let $S \subset X$ satisfy $\mathcal{H}^n(S) < \infty + (*)$. If S has n "Alberti representations", then S is n-rectifiable.

- $\Rightarrow$ for any Lipschitz $f \colon X \to \mathbb{R}^m$, (after removing a set of $\mathcal{H}^n$ measure zero) $\exists$ $n-1$ dimensional "weak tangent field": $V_x \in G(m, n-1)$ s.t. any 1-rectifiable set $\gamma \subset S$ has

$$\mathrm{Tan}_{f(x)} f(\gamma) \in V_x \quad \mathcal{H}^1\text{-a.e. } x \in \gamma.$$

Idea of the proof of the main direction

Given $f \in \mathrm{Lip}_1(X, \mathbb{R}^m)$, we must make arbitrarily small modifications to obtain a $\tilde{f}$ such that $\mathcal{H}^n(\tilde{f}(S))$ is arbitrarily small. These modifications **cannot** increase the Lipschitz constant.

Theorem (B, Li 2014)

Let $S \subset X$ satisfy $\mathcal{H}^n(S) < \infty + (*)$. If S has n "Alberti representations", then S is n-rectifiable.

- $\Rightarrow$ for any Lipschitz $f\colon X \to \mathbb{R}^m$, (after removing a set of $\mathcal{H}^n$ measure zero) $\exists$ $n-1$ dimensional "weak tangent field": $V_x \in G(m, n-1)$ s.t. any 1-rectifiable set $\gamma \subset S$ has

$$\mathrm{Tan}_{f(x)}\, f(\gamma) \in V_x \quad \mathcal{H}^1\text{-a.e. } x \in \gamma.$$

- If $S \subset \mathbb{R}^{m'}$, or using the announcement of Csörnyei-Jones, the theorem can be proved without assuming $(*)$. Similarly, the consequence is true for the case $s \notin \mathbb{N}$.

Idea of the proof of the main direction II

- Have a weak tangent field: $V_x \in G(m, n-1)$ s.t. any 1-rectifiable set $\gamma \subset S$ has

$$\mathrm{Tan}_{f(x)}\, f(\gamma) \in V_x \quad \mathcal{H}^1\text{-a.e. } x \in \gamma.$$

Idea of the proof of the main direction II

- Have a weak tangent field: $V_x \in G(m, n-1)$ s.t. any 1-rectifiable set $\gamma \subset S$ has

$$\mathrm{Tan}_{f(x)} f(\gamma) \in V_x \quad \mathcal{H}^1\text{-a.e. } x \in \gamma.$$

- To construct $\tilde{f}$, we locally squeeze f in all directions orthogonal to V_x.

Idea of the proof of the main direction II

- Have a weak tangent field: $V_x \in G(m, n-1)$ s.t. any 1-rectifiable set $\gamma \subset S$ has

$$\mathrm{Tan}_{f(x)} f(\gamma) \in V_x \quad \mathcal{H}^1\text{-a.e. } x \in \gamma.$$

- To construct $\tilde{f}$, we locally squeeze f in all directions orthogonal to V_x.
  - Since there are no 1-rectifiable sets in these directions, this can be done without perturbing f very much.

Idea of the proof of the main direction II

- Have a weak tangent field: $V_x \in G(m, n-1)$ s.t. any 1-rectifiable set $\gamma \subset S$ has

$$\mathrm{Tan}_{f(x)} f(\gamma) \in V_x \quad \mathcal{H}^1\text{-a.e. } x \in \gamma.$$

- To construct $\tilde{f}$, we locally squeeze f in all directions orthogonal to V_x.
    - Since there are no 1-rectifiable sets in these directions, this can be done without perturbing f very much.
    - $\dim V_x = n-1 \Rightarrow$ can reduce $\mathcal{H}^n(f(S))$ to an arbitrarily small value.

Perturbations of distances I

- The are other targets that are interesting from the point of view of metric spaces.

Perturbations of distances I

- The are other targets that are interesting from the point of view of metric spaces.
- Recall that any separable metric space can be isometrically embedded into ℓ_∞.

Perturbations of distances I

- The are other targets that are interesting from the point of view of metric spaces.
- Recall that any separable metric space can be isometrically embedded into ℓ_∞.
- If S is compact, then for any $\epsilon > 0$ there exists a 1-Lipschitz mapping f into $\ell_\infty^{m(\epsilon)}$ such that $|d(x,y) - \|f(x) - f(y)\|_\infty| < \epsilon$ for each $x, y \in S$.

Perturbations of distances I

- The are other targets that are interesting from the point of view of metric spaces.
- Recall that any separable metric space can be isometrically embedded into ℓ_∞.
- If S is compact, then for any $\epsilon > 0$ there exists a 1-Lipschitz mapping f into $\ell_\infty^{m(\epsilon)}$ such that $|d(x, y) - \|f(x) - f(y)\|_\infty| < \epsilon$ for each $x, y \in S$.
- Applying the Euclidean squeezing argument to f gives a $\tilde{f}$ with a huge Lipschitz constant (because of the ralationship between $\|.\|_2$ and $\|.\|_\infty$ in $\mathbb{R}^m$).

Perturbations of distances I

- The are other targets that are interesting from the point of view of metric spaces.
- Recall that any separable metric space can be isometrically embedded into ℓ_∞.
- If S is compact, then for any $\epsilon > 0$ there exists a 1-Lipschitz mapping f into $\ell_\infty^{m(\epsilon)}$ such that $|d(x, y) - \|f(x) - f(y)\|_\infty| < \epsilon$ for each $x, y \in S$.
- Applying the Euclidean squeezing argument to f gives a $\tilde{f}$ with a huge Lipschitz constant (because of the ralationship between $\|.\|_2$ and $\|.\|_\infty$ in $\mathbb{R}^m$).
- If we are more careful we can obtain something more useful.

Perturbations of distances II

Theorem (B)

Let S be compact purely n-unrectifiable with $\mathcal{H}^n(S) < \infty + (*)$. For any $\epsilon > 0$ $\exists$ $L(n)$-Lipschitz $\sigma \colon S \to \ell_\infty^{m(\epsilon)}$ with

$$|d(x,y) - \|\sigma(x) - \sigma(y)\|| < \epsilon \quad \forall x, y \in S \tag{1}$$

and

$$\mathcal{H}^n(\sigma(S)) < \epsilon.$$

Theorem (B)

Let S be compact purely n-unrectifiable with $\mathcal{H}^n(S) < \infty + (*)$. For any $\epsilon > 0$ $\exists$ $L(n)$-Lipschitz $\sigma\colon S \to \ell_\infty^{m(\epsilon)}$ with

$$|d(x,y) - \|\sigma(x) - \sigma(y)\|| < \epsilon \quad \forall x, y \in S \tag{1}$$

and

$$\mathcal{H}^n(\sigma(S)) < \epsilon.$$

- Since $L(n)$ is independent of ϵ, we have a suitable converse: if S is n-rectifiable, $\inf_{L>0} \liminf_{\epsilon\to 0} \mathcal{H}^n(\sigma(S)) > 0$, $\sigma\colon S \to (Y,\rho)$ L-Lipschitz satisfying (1).

Perturbations of distances II

Theorem (B)

Let S be compact purely n-unrectifiable with $\mathcal{H}^n(S) < \infty + (*)$. For any $\epsilon > 0$ $\exists$ $L(n)$-Lipschitz $\sigma \colon S \to \ell_\infty^{m(\epsilon)}$ with

$$|d(x,y) - \|\sigma(x) - \sigma(y)\|| < \epsilon \quad \forall x, y \in S \tag{1}$$

and

$$\mathcal{H}^n(\sigma(S)) < \epsilon.$$

- Since $L(n)$ is independent of ϵ, we have a suitable converse: if S is n-rectifiable, $\inf_{L>0} \liminf_{\epsilon \to 0} \mathcal{H}^n(\sigma(S)) > 0$, $\sigma \colon S \to (Y, \rho)$ L-Lipschitz satisfying (1).
- $(*)$ is not necessary under the same conditions as before, and have the corresponding statement for $\mathcal{H}^s(S)$, $s \notin \mathbb{N}$.

Perturbations of sets

- If S is a subset of a Banach space B with an *unconditional basis* (ℓ_1, $L^p(\mu)$ $1 < p < \infty$, c_0,...) then σ can be chosen to be a genuine perturbation.

Perturbations of sets

- If S is a subset of a Banach space B with an *unconditional basis* (ℓ_1, $L^p(\mu)$ $1 < p < \infty$, c_0,...) then σ can be chosen to be a genuine perturbation.
- That is, $\exists$ $L(n, B)$-Lipschitz $\sigma \colon B \to B$ with

$$\|x - \sigma(x)\| < \epsilon \quad \forall x \in S$$

and

$$\mathcal{H}^n(\sigma(S)) < \epsilon.$$

Perturbations of sets

- If S is a subset of a Banach space B with an *unconditional basis* (ℓ_1, $L^p(\mu)$ $1 < p < \infty$, $c_0,\ldots$) then σ can be chosen to be a genuine perturbation.
- That is, $\exists$ $L(n,B)$-Lipschitz $\sigma\colon B \to B$ with

$$\|x - \sigma(x)\| < \epsilon \quad \forall x \in S$$

  and

$$\mathcal{H}^n(\sigma(S)) < \epsilon.$$

- Generalises a result of H. Pugh who proved this for Ahlfors regular subsets of Euclidean space. The construction relies on BF.