Collisionless Boltzmann (Vlasov) Equation and Modeling of Self-Gravitating Systems and Plasmas
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Relaxation and long-time correlation of finite-size fluctuation in thermal equilibrium

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科研費
KAKENHI

Main message

- Vlasov equation has inf. Casimir invariants

$$\Downarrow$$

- Finite-size fluctuation becomes strange even in thermal equilibrium

What is Casimir invariants?

Vlasov equation ($N \to \infty$)

$$\frac{\partial f}{\partial t} + \frac{\partial \mathcal{H}[f]}{\partial q}\frac{\partial f}{\partial q} - \frac{\partial \mathcal{H}[f]}{\partial p}\frac{\partial f}{\partial p} = 0$$

$$\Downarrow$$

Casimir invariant

$$\mathcal{C}[f](t) = \int c(f(q,p,t))dqdp, \qquad \forall c : \mathbb{R} \to \mathbb{R} : C^1$$

Examples

$$\int f dqdp, \quad \int f^2 dqdp, \quad , \cdots , - \int f \log f dqdp$$

What is Casimir invariants?

Vlasov equation ($N \to \infty$)

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$$\Downarrow$$

Casimir invariant

$$\mathcal{C}[f](t) = \int c(f(q,p,t))dqdp, \qquad \forall c : \mathbb{R} \to \mathbb{R} : C^1$$

Note

- Essential block is symplectic structure of Vlasov eq.
- Casimirs are invariants even $\mathcal{H}[f]$ is time-dependent

Finite-size effect

Vlasov equation + Finite-size effect

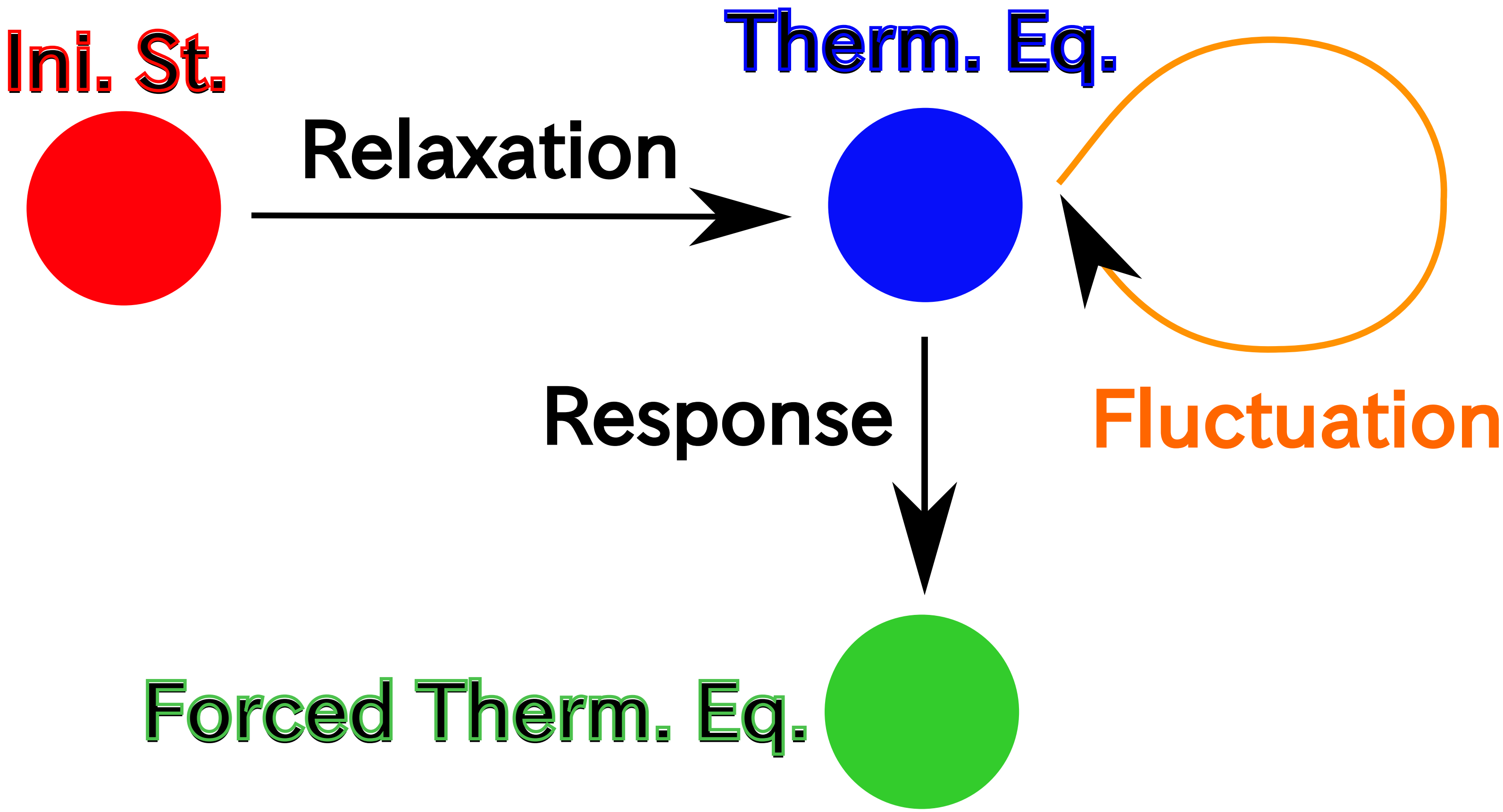
$$\frac{\partial f}{\partial t} + \frac{\partial \mathcal{H}[f]}{\partial q}\frac{\partial f}{\partial q} - \frac{\partial \mathcal{H}[f]}{\partial p}\frac{\partial f}{\partial p} = \frac{1}{N}\mathrm{Col}[f, f]$$

$$\Downarrow$$

Casimirs $\longrightarrow$ Pseudo Casimirs

Effective time scale: $O(N)$

Strange behaviours by pseudo Casimir invariants

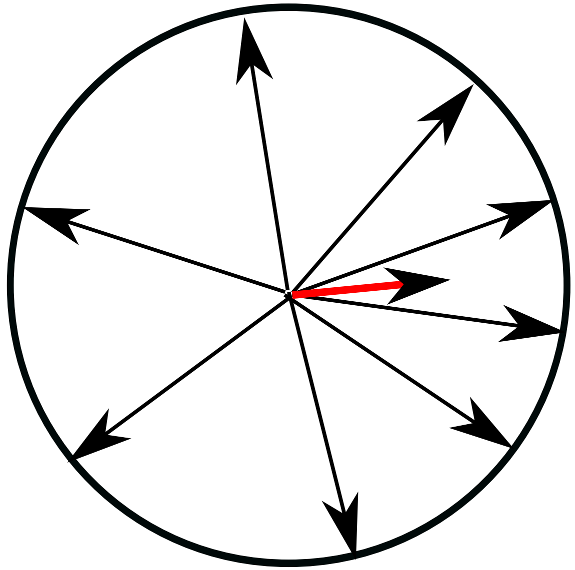


Hamiltonian Mean-Field (HMF) model

$$H = \sum_{j=1}^{N} \frac{p_j^2}{2} + \frac{1}{2N} \sum_{j=1}^{N} \sum_{k=1}^{N} [1 - \cos(q_j - q_k)]$$

Order parameter (mean-field)

$$\boldsymbol{M} = (M_x, M_y) = \frac{1}{N} \sum_{k=1}^{N} (\cos q_k, \sin q_k)$$

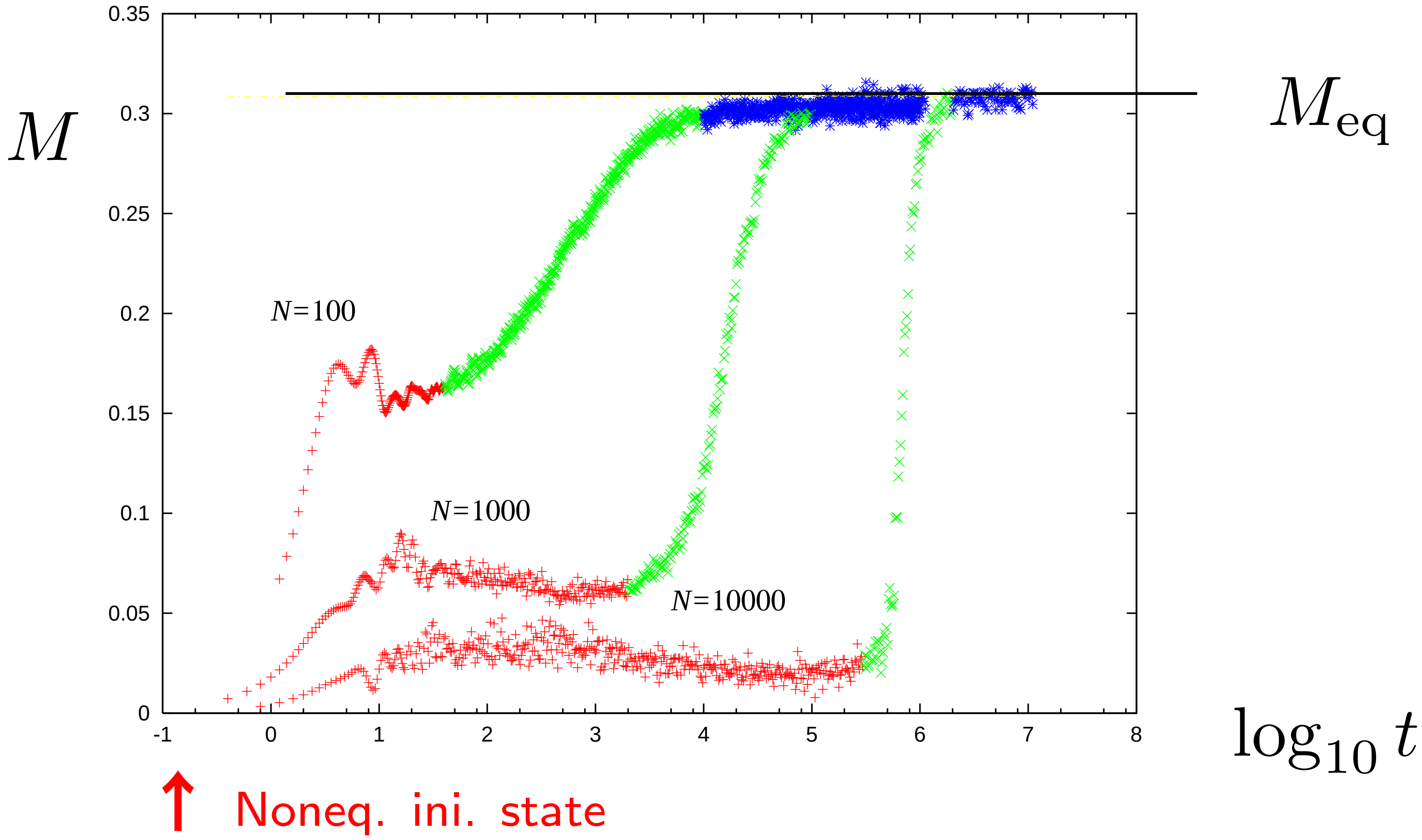


Canonical eq. of motion

$$\frac{\mathrm{d}q_j}{\mathrm{d}t} = \frac{\partial H}{\partial p_j}, \qquad \frac{\mathrm{d}p_j}{\mathrm{d}t} = -\frac{\partial H}{\partial q_j}$$

without thermal noise!

Relaxation TO thermal equilibrium



N : larger $\Longrightarrow$ Relax. Time : larger

cf. 2-body relax. time

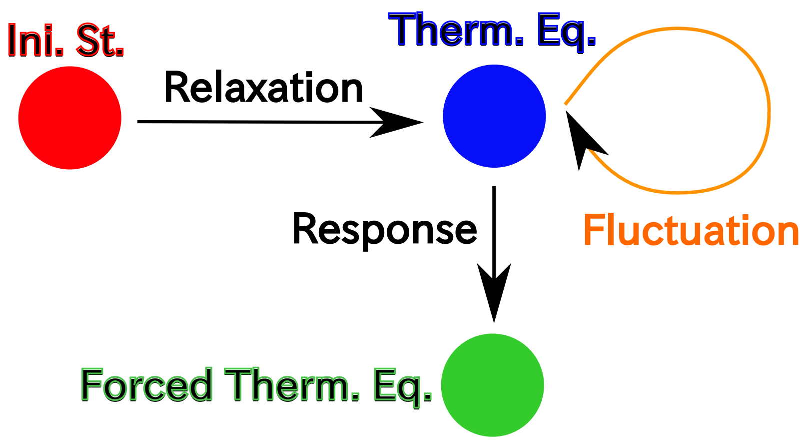
Applying external force in thermal equilibrium

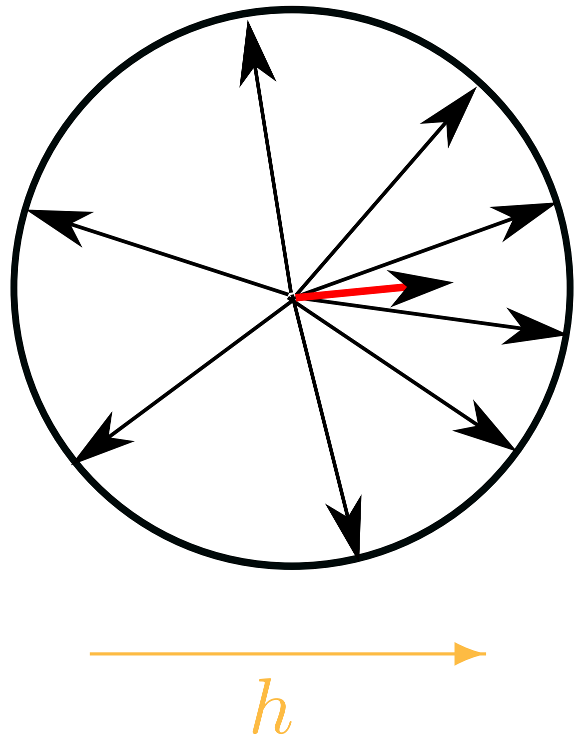
$$H_h = \sum_{j=1}^{N} \frac{p_j^2}{2} + \frac{1}{2N} \sum_{j=1}^{N} \sum_{k=1}^{N} [1 - \cos(q_j - q_k)] - h \sum_{j=1}^{N} \cos q_j$$

Initial conditions:

Randomly draw $\{(q_j, p_j)\}_{j=1}^{N}$ from

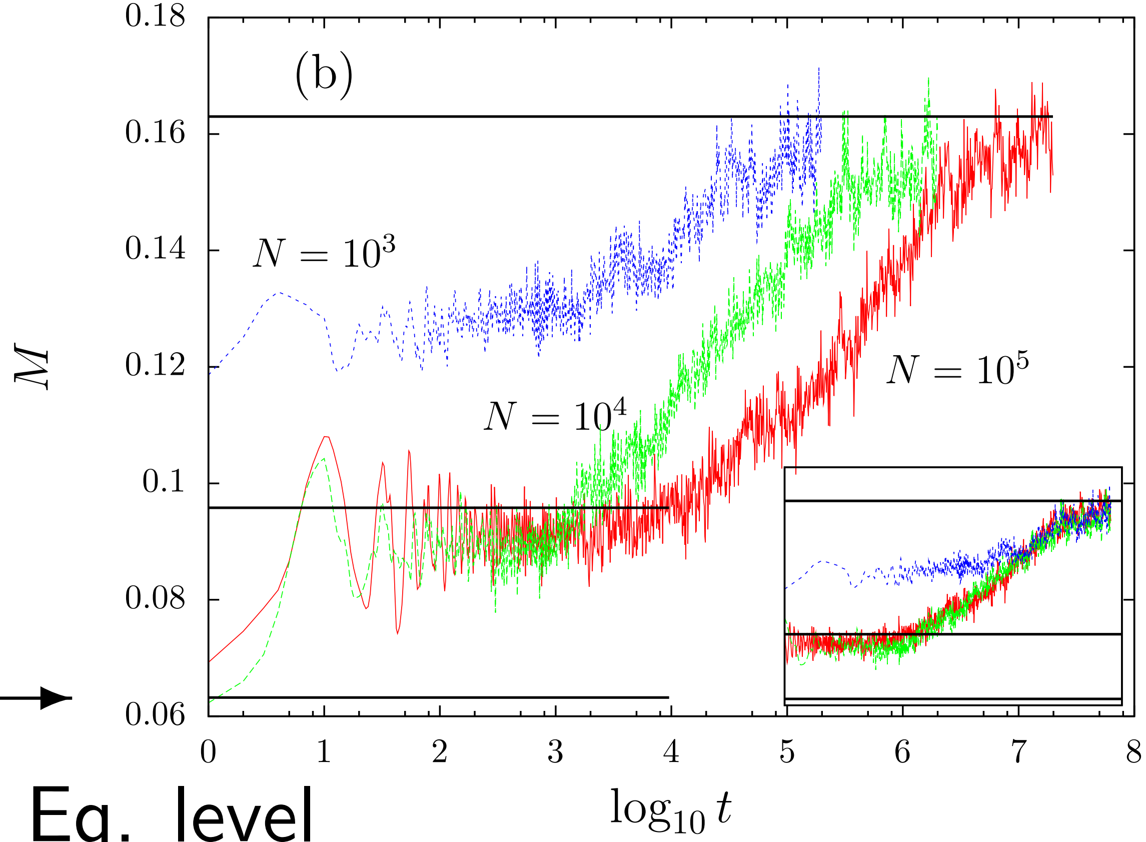
$$f_0(q,p) \propto \exp(-\mathcal{H}_0(q,p)/T)$$





Relaxation TO forced thermal equilibrium

Ogawa-Patelli-YYY, PRE **89**, 032131 (2014)



Forced Th. Eq. level

QSS level : Predicted by Linear Response Th.

Th. Eq. level

Linear response theory in Vlasov eq.

- Linear response theory based on Vlasov eq.

Ogawa-YYY, PRE **85**, 061115 (2012); **91**, 062108 (2015); **92**, 042131 (2015)

$$f_{\mathrm{QSS}} = f_0 + \delta f_{\mathrm{SM}} - \delta f_{\mathrm{Vlasov}}$$

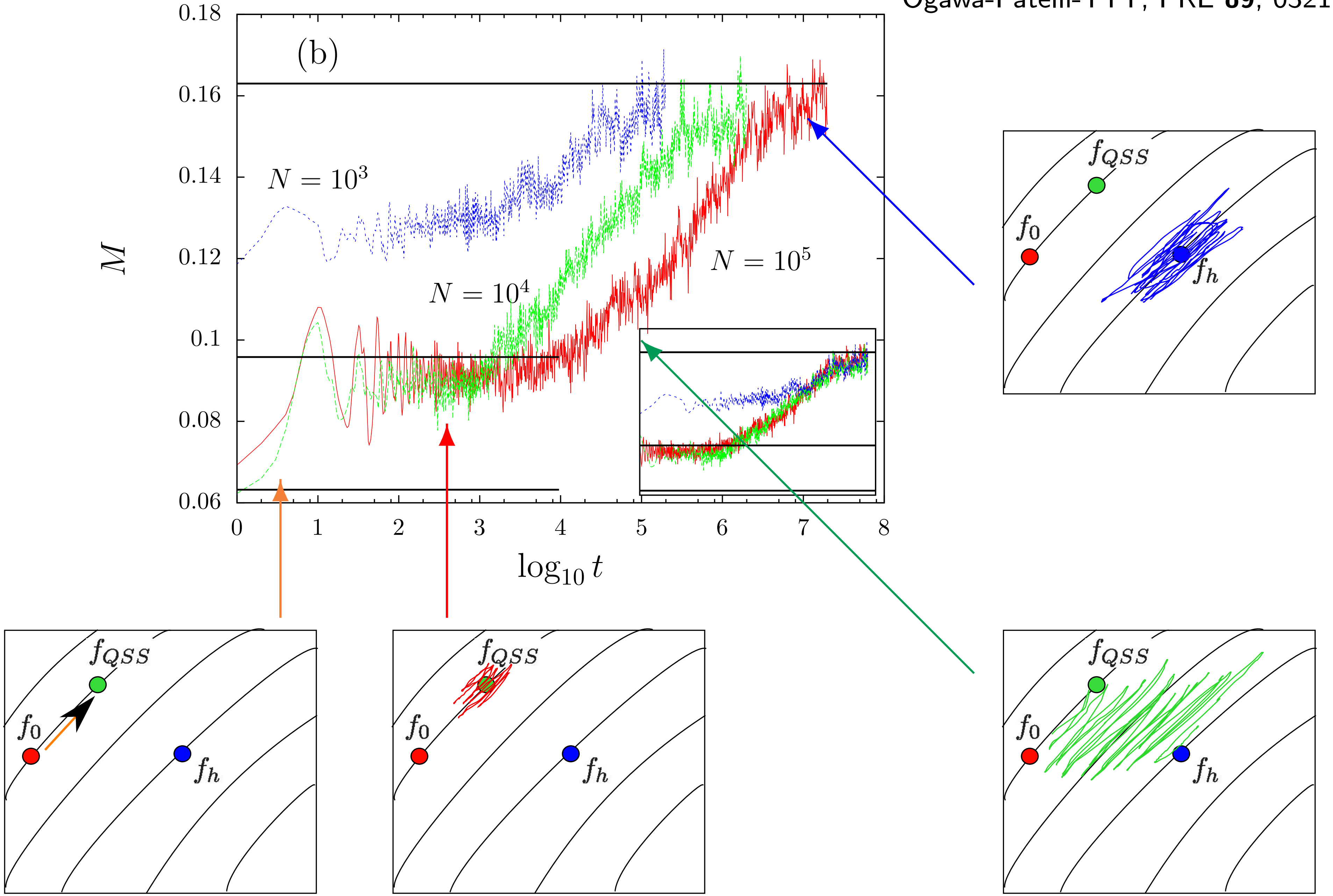
Responsed state (QSS)	Ini. state	Stat. Mech. prediction	Vlasov correction

- Casimirs hold up to the leading order

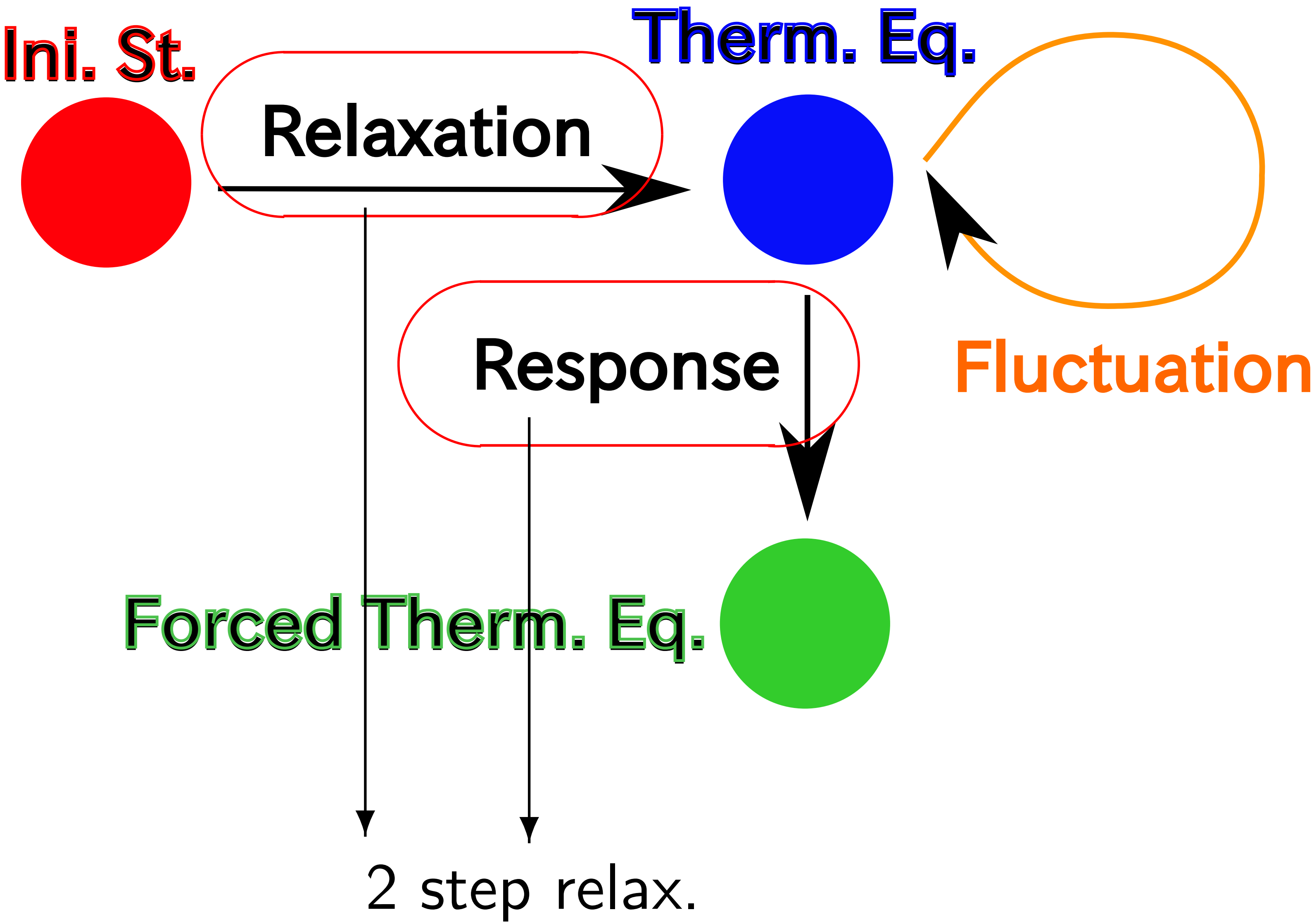
$$\mathcal{C}[f_{\mathrm{QSS}}] = \mathcal{C}[f_0] + (\delta f^2)$$

Relaxation TO forced thermal equilibrium

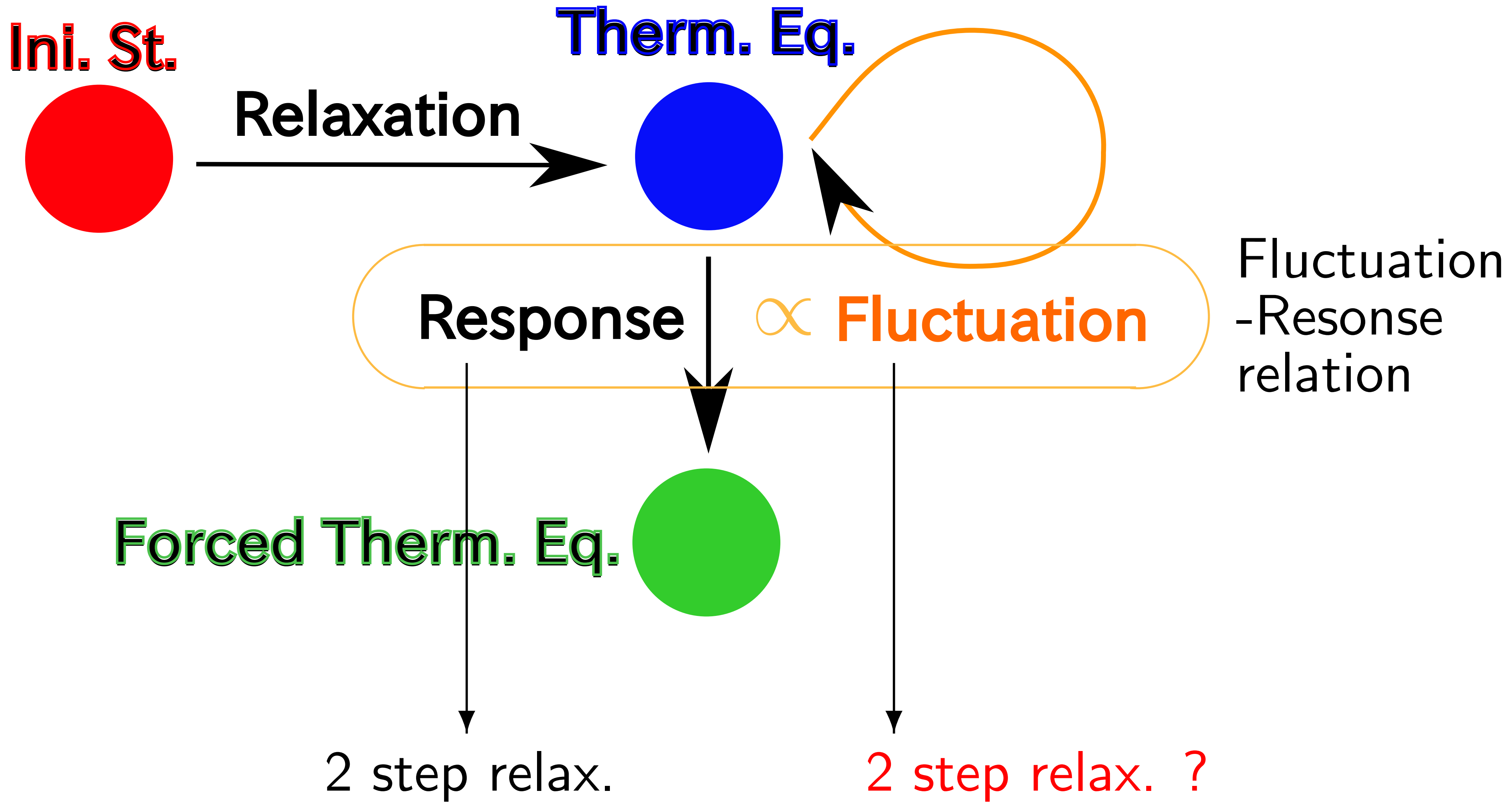
Ogawa-Patelli-YYY, PRE **89**, 032131 (2014)



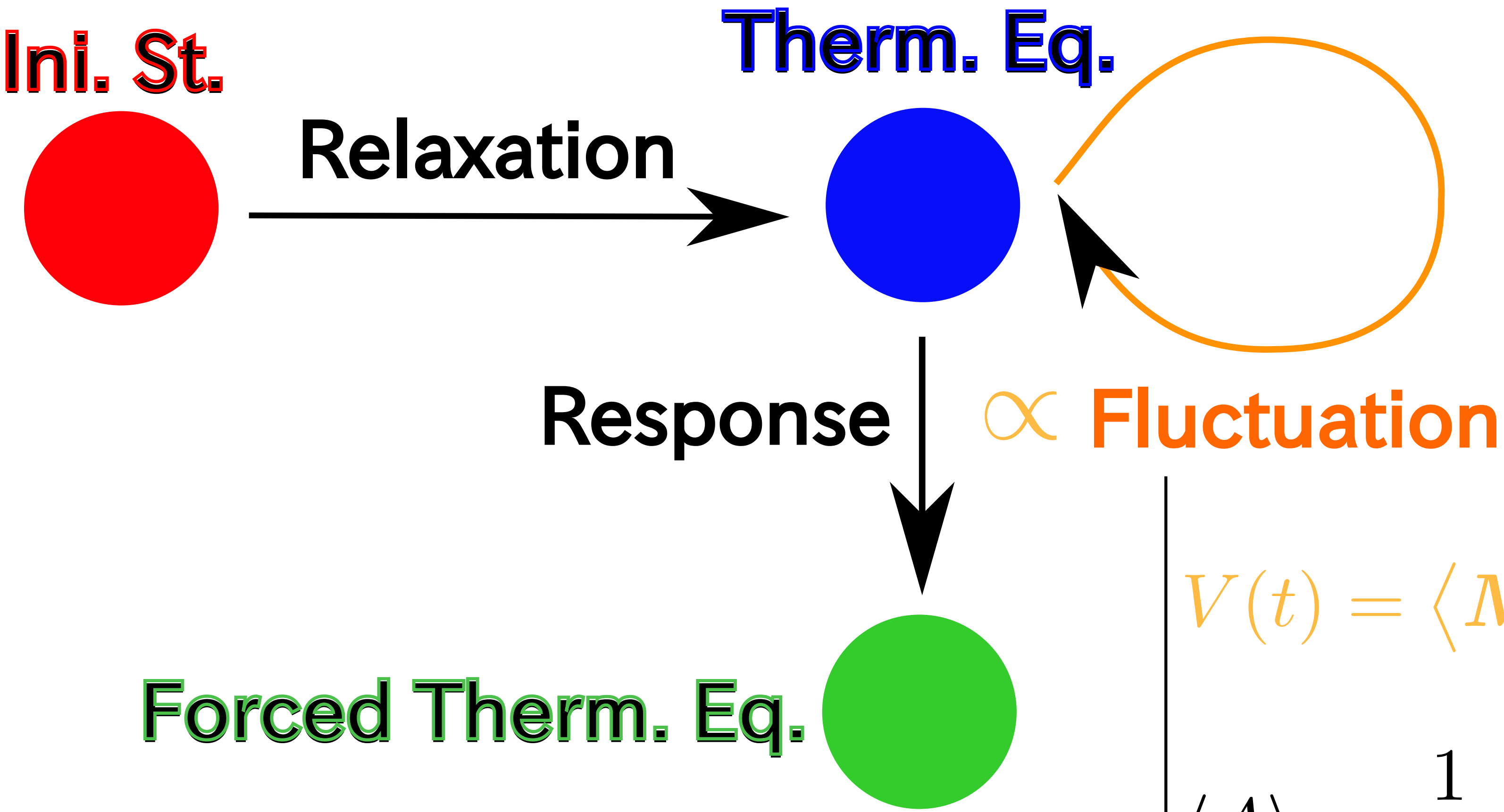
Two step relaxations in Nonequilibrium



Two step relaxations in Equilibrium ?



Time-dependent variance

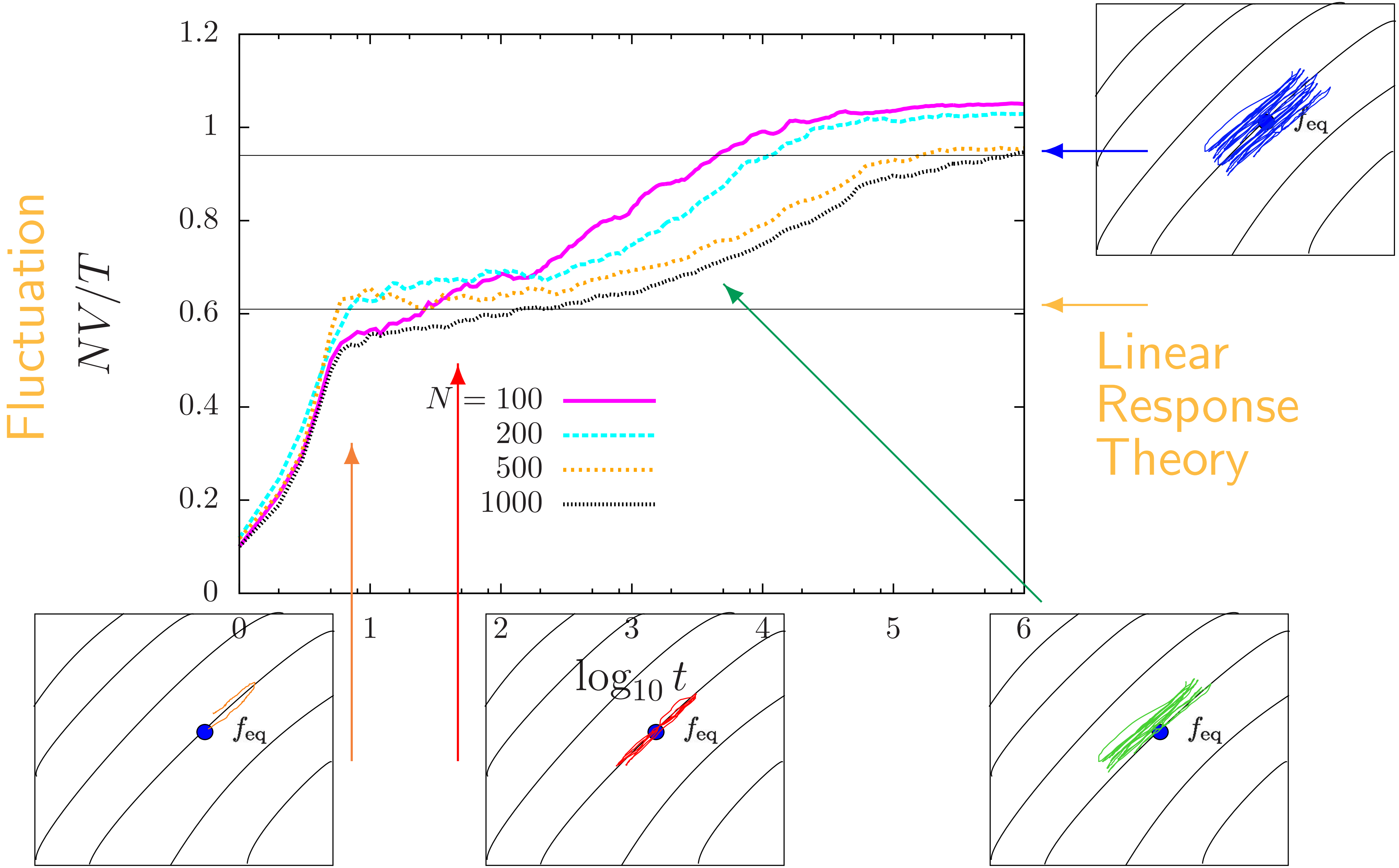


$$V(t) = \langle \boldsymbol{M}^2 \rangle_t - \langle \boldsymbol{M} \rangle_t^2$$

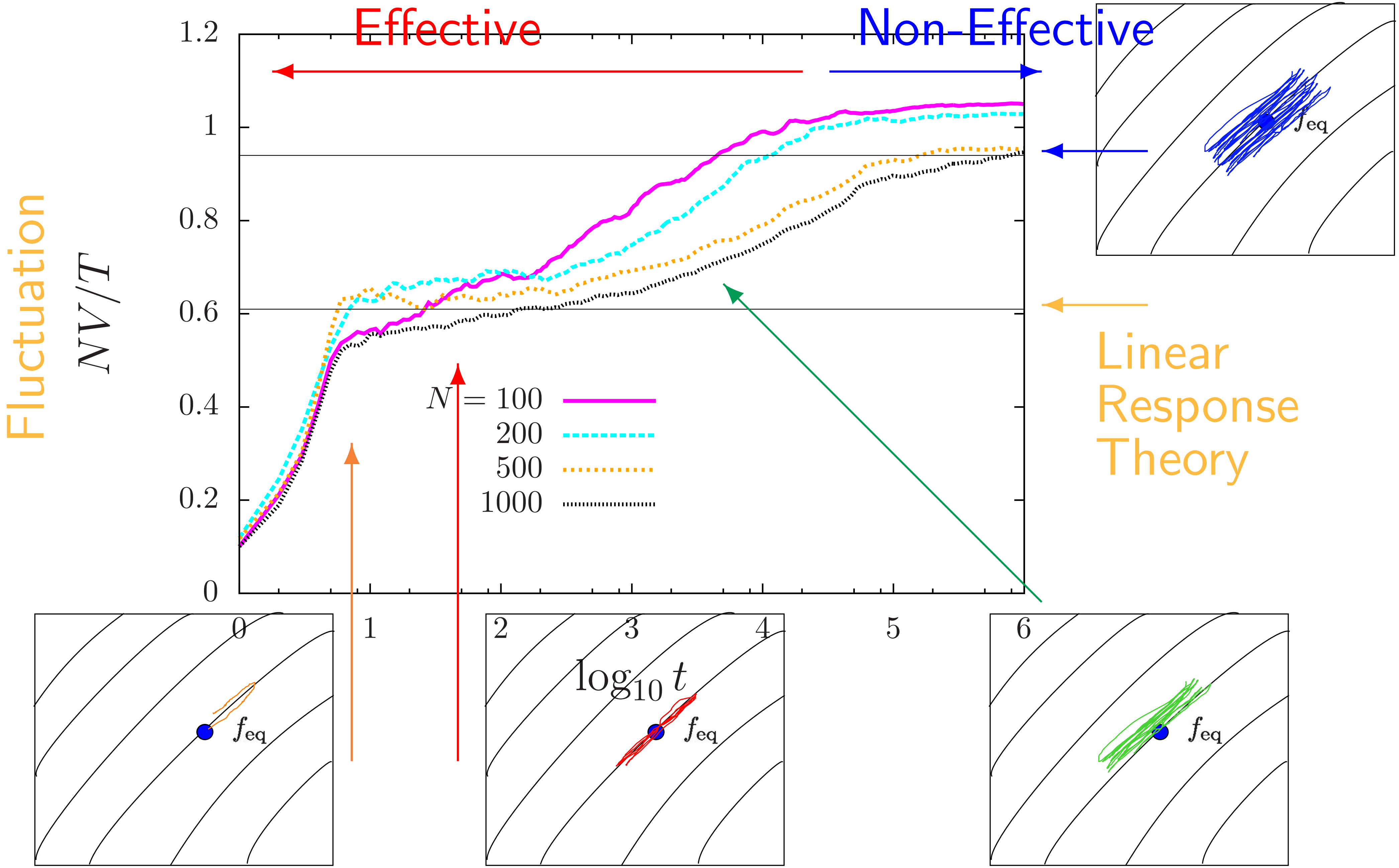
$$\langle A \rangle_t = \frac{1}{t} \int_0^t A(s) ds$$

2 step relax. ?

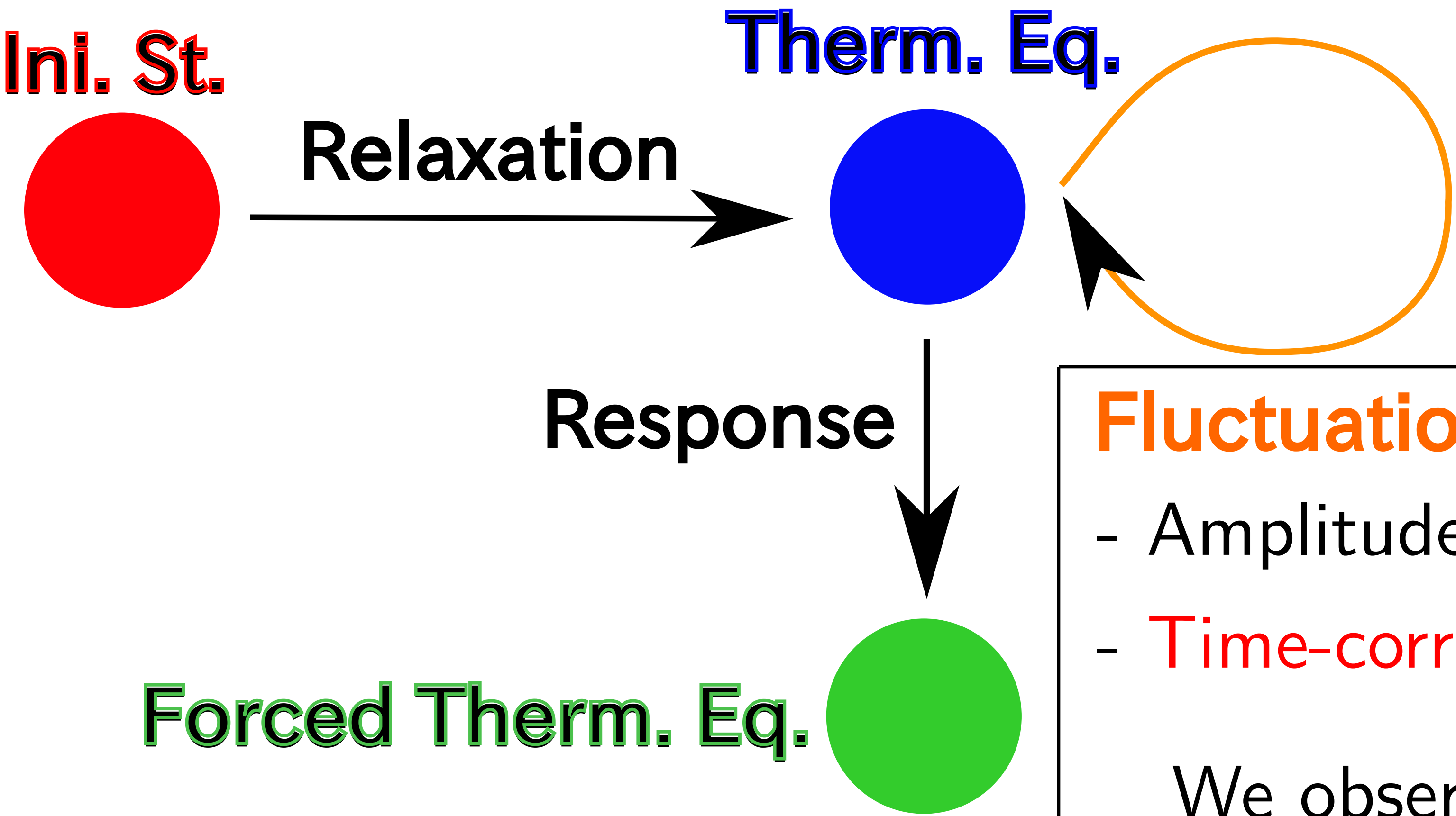
Relaxation of variance IN thermal equilibrium



Relaxation of variance IN thermal equilibrium



Time correlation?

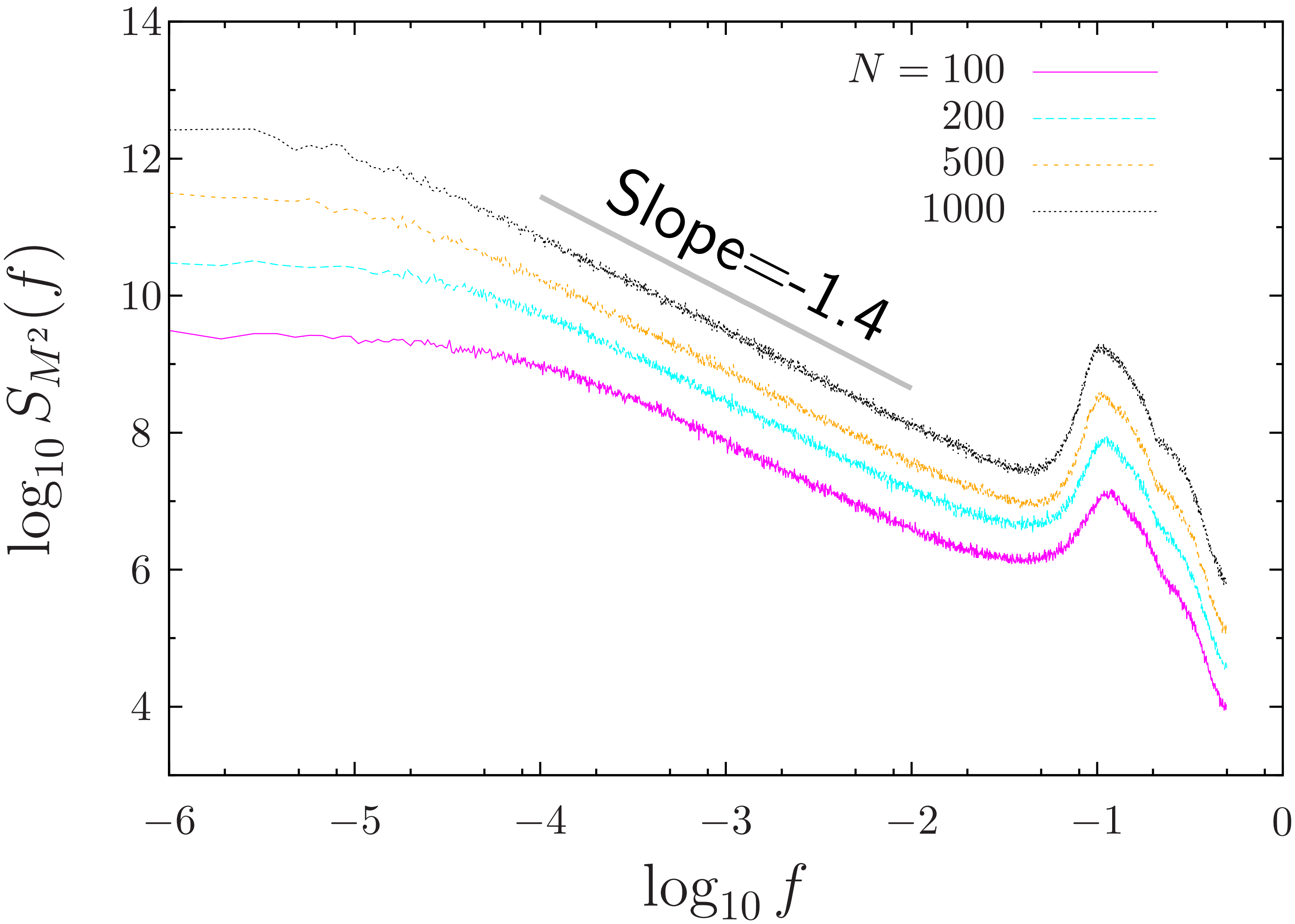


Fluctuation

- Amplitude is strange
- Time-correlation also?

We observe
power spectrum
of $M^2(t)$

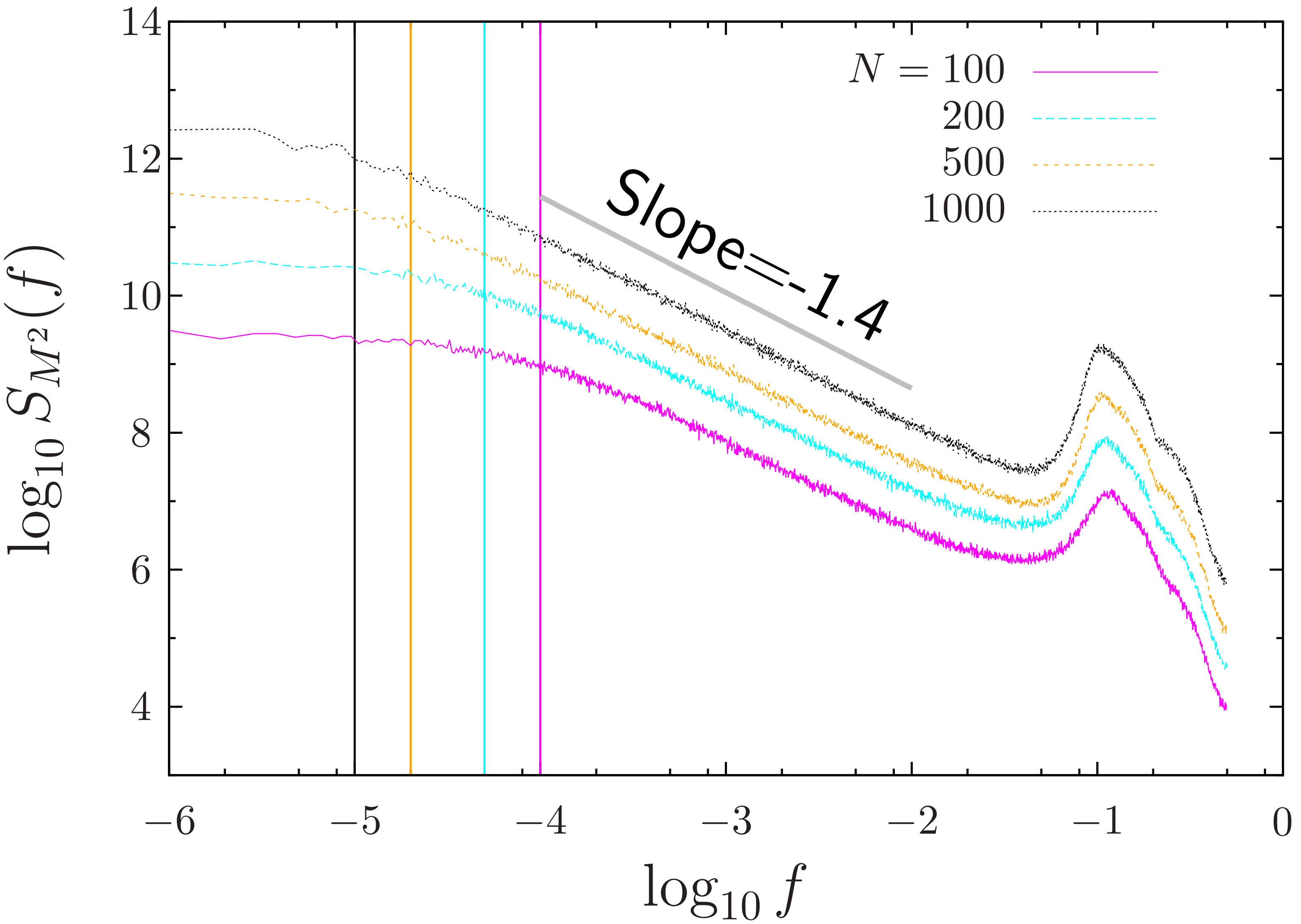
$1/f$ spectrum ($T = 0.45$)



YYY-Kaneko, In preparation

Long-time correlation

$1/f$ spectrum ($T = 0.45$)

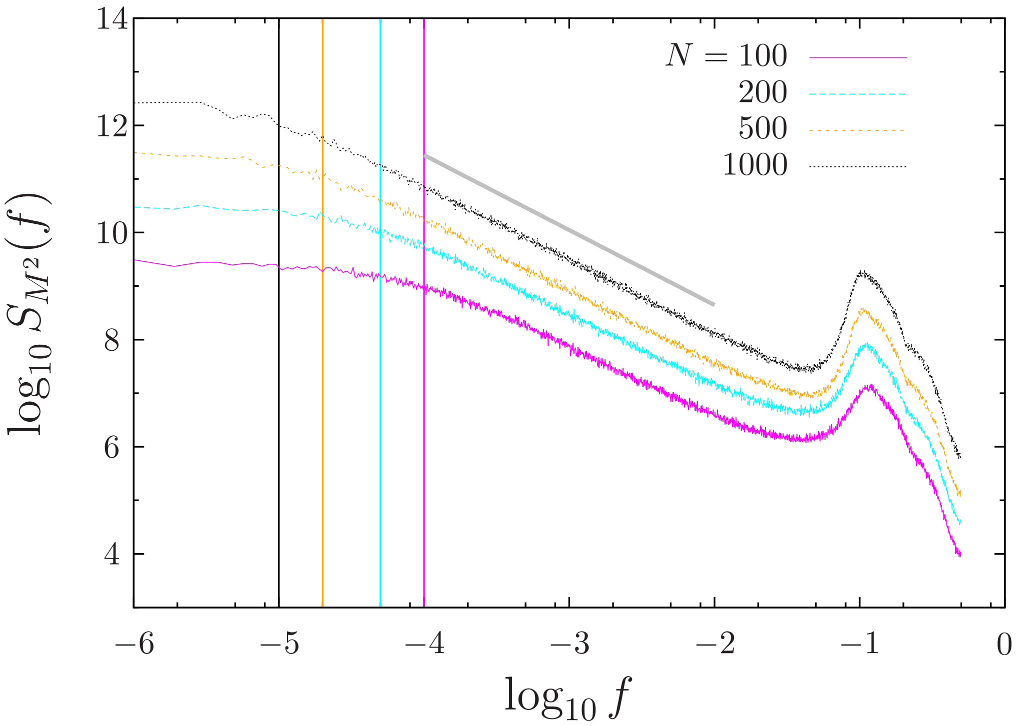


YYY-Kaneko, In preparation

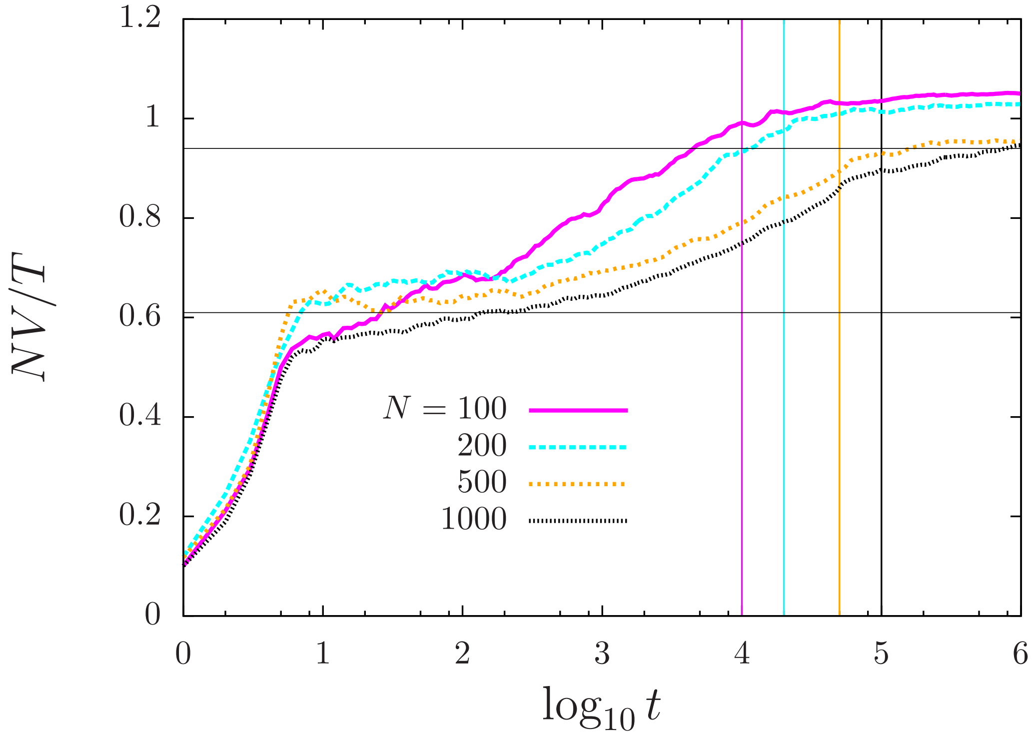
$\tau = 100N$

Link with Casimirs

Power spectrum



Variance



Non-Effective Effective

Effective Non-Effective

Pseudo Casimirs

YYY-Kaneko, In preparation

Pseudo Casimirs $\Longrightarrow$ $1/f$ spectrum

Summary

- Pseudo Casimir induces two time-scales

$\Downarrow$

- 2 step dynamics in relaxation and response
- 2 step relax of fluctuation amplitude (in EQ.)
- $1/f$ fluctuation (in EQ.)

Remarks:

- "Thermal eq." is not essential for $\exists$ pseudo Casimirs
- Several strange things may happen in meta-stable states (Quasi-Stationary States)

Thank you for your attention