A Geometrical Version of the Maxwell-Vlasov Hamiltonian Structure

Michel VITTOT
Centre de Physique Théorique
Aix-Marseille Université, CNRS, UMR 7332
Luminy, Marseille, France
vittot@cpt.univ-mrs.fr
http://www.cpt.univ-mrs.fr/~vittot/Home.htm

Philip J. MORRISON
Department of Physics and Institute for Fusion Studies
University of Texas at Austin, Austin, USA
morrison@physics.utexas.edu
http://www.ph.utexas.edu/~morrison/

Collisionless Boltzmann (Vlasov) Equation
Modeling of Self-Gravitating Systems & Plasmas

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We look for a geometrical version of the Maxwell-Vlasov Hamiltonian structure of Morrison (1980) and Marsden-Weinstein (1982). This Poisson structure was indeed written with a "nabla" operator ∇ and a cross product $(\times)$, which are complicated to use in an arbitrary coordinate system or in a curved space.

Here we start with a 3-d manifold $\mathcal{Q}$ (with a Riemanian structure g) as our configuration space: for instance, the interior of a Tokamak, or some stellar medium

Its cotangent bundle ("phase space") $\mathcal{T} := T^*\mathcal{Q}$, has a natural symplectic structure σ_0 viewed as a 2-form, or a linear map from 1-vector fields to 1-forms.
σ_0 is invertible, and $d\sigma_0 = 0$. The exterior derivative d here acts on the 6-d manifold $\mathcal{T}$. We use the same symbol for the exterior derivative acting on $T^*\mathcal{Q}$ and on $\mathcal{Q}$: no ambiguities.

Hence $q \in \mathcal{Q}, \quad p \in T_q^*\mathcal{Q}$.
These variables will now be viewed as "fixed labels" in a "Lagrangian-Eulerian" duality. And the actual dynamics takes place on the ("Vlasov") distribution function f defined on $\mathcal{T}$, which describes the matter, as well as on the electromagnetic fields E, B defined on $\mathcal{Q}$.
More precisely $f(q, p)$ is the (positive) density of matter at the point $q \in \mathcal{Q}$ with momentum $p \in T_q^*\mathcal{Q}$.

So our actual phase space will be the set Φ of the electromagnetic fields E and B (defined on the whole configuration space $\mathcal{Q}$) along with the ("Vlasov") distribution functions f, defined on:

$$\mathcal{T} := T^*\mathcal{Q}$$

But instead of considering E and B as vectorial functions on $\mathcal{Q}$, we will consider them as differential forms of order 2:

$$E\,,\, B \in \Omega^2\mathcal{Q}$$

We have the same information in $\vec{E}$ and in $(\vec{E} \times dq).dq$ or in $\vec{E}.dq$ but the differential form is coordinate free.

We will restrict B as being a closed 2-form:

$$dB = 0 \qquad \text{i.e.} \qquad B \in \Omega_0^2 \mathcal{Q}$$

since this is the usual "divergence free" constraint (no magnetic monopoles), and it will be essential.

Likewise instead of considering f as a (positive) scalar function (a 0-form), we will choose it as a positive "volume form" (here a 6-form):

$$f \in \Omega_+^6 \mathcal{T}$$

Hence our (infinite dimensional) phase space is:

$$\Phi \ := \ \{(f, E, B) \in \Omega^6_+\mathcal{T} \times \Omega^2\mathfrak{Q} \times \Omega^2_0\mathfrak{Q}\}$$

The set of "observables" is the vector space:

$$\Omega^0\Phi := \mathcal{C}^\infty(\Phi \to \mathbb{R})$$

of "smooth" functions ("functionals", or 0-forms) from this phase space Φ to $\mathbb{R}$.

We will endow it with a Poisson structure.

Some important examples of observables are the "elementary" ones. For any $\alpha \in \Omega^1 \mathcal{Q}$ independent of E, B, f (a fixed parameter):

$$\mathbb{E}_\alpha = \mathbb{E}_\alpha(f, E, B) := \alpha \wedge E = \int_{\mathcal{Q}} \alpha \wedge E$$

The exterior product $\alpha \wedge E \in \Omega^3 \mathcal{Q}$ is a volume form (dependent on q) and so may be integrated on the whole configuration manifold $\mathcal{Q}$. This integration is understood above: we omit the summation index q.

Likewise:

$$\mathbb{B}_\alpha = \mathbb{B}_\alpha(f, E, B) := \alpha \wedge B = \int_{\mathcal{Q}} \alpha \wedge B$$

and for any fixed parameter $\beta \in \Omega^0\mathcal{T}$:

$$\mathbf{f}_\beta = \mathbf{f}_\beta(f, E, B) := \beta \wedge f = \int_{\mathcal{T}} \beta \ f$$

Remark : $$\frac{\partial \mathbb{B}_\alpha}{\partial B} b \ := \lim_{\varepsilon \to 0} \frac{\mathbb{B}_\alpha(B + \varepsilon b) - \mathbb{B}_\alpha(B)}{\varepsilon} = \alpha \wedge b$$

$$\Longrightarrow \quad \partial_B \, \mathbb{B}_\alpha \ := \frac{\partial \mathbb{B}_\alpha}{\partial B} = \alpha \in \Omega^1\mathcal{Q} \qquad \qquad \partial_f \, \mathbf{f}_\beta \ := \frac{\partial \mathbf{f}_\beta}{\partial f} = \beta \in \Omega^0\mathcal{T}$$

On any observable: $\partial_B \, , \, \partial_E \ \in \ \Omega^1\mathcal{Q}$ and $\partial_f \ \in \ \Omega^0\mathcal{T}$

Another important example of observable will be our Hamiltonian:

$$H = H(f, E, B) := \frac{1}{2}(E^* \wedge E + B^* \wedge B) \; + \; f\, m\gamma$$

This Hamiltonian uses the Hodge star duality $*$, build from the Riemanian metric g on $\mathcal{Q}$, and a chosen volume form on $\mathcal{Q}$. For instance $E^*, B^* \in \Omega^1\mathcal{Q}$. So that the exterior product $E^* \wedge E + B^* \wedge B \in \Omega^3\mathcal{Q}$ may again be integrated on the whole configuration manifold $\mathcal{Q}$. This integration is again understood above.
The metric only occurs in the Hamiltonian, not in the Poisson structure.

Likewise the Vlasov 6-form f is integrated over $\mathcal{T}$ after multiplication by the 0-form $m\gamma$ which is the usual (relativistic) kinetic energy:

$$m\gamma = \sqrt{m^2 + p\,\bar{p}}$$

The metric g is used via:

$$\bar{p} := p\,g = p\,g(q) \in T\mathcal{Q}$$

where we use the action of $g : T^*\mathcal{Q} \to T\mathcal{Q}$ on its left. More precisely $g(q) : T_q^*\mathcal{Q} \to T_q\mathcal{Q}$
And we use the contraction $p\,\bar{p}$ between $T^*\mathcal{Q}$ and $T\mathcal{Q}$.

Here we take as units: $c = \varepsilon_0 = \mu_0 = 1$ (permitivity, susceptibility) and the mass m of the particules is a given parameter.

Let us recall that $f(q, p)$ is the density of particules of mass m and charge e, at the point $q \in \mathcal{Q}$ with momentum $p \in T_q^*\mathcal{Q}$.

To describe another type of particule (another “specie”, with mass m' and charge e') we just have to introduce another Vlasov function $f_{e',m'}$ and the above Hamiltonian will be modified by summing over the different species.

The coupling (via the electric charge e) between the matter and the fields, is not introduced in the Hamiltonian, but in the Poisson bracket.

First the Poisson bracket on the "label space" $\mathcal{T} = T^*\mathcal{Q}$:

$$\pi_0 := \sigma_0^{-1}$$

$$\forall \alpha \in \Omega^1\mathcal{T} \qquad \sigma_0(\alpha\pi_0) = \alpha \qquad\qquad \forall X \in \chi^1\mathcal{T} \qquad (\sigma_0 X)\,\pi_0 = X$$

Then the coupling (via e) between the matter and the fields, is introduced in the symplectic form, so in the Poisson bracket:

$$\sigma_B := \sigma_0 - eB_0 \in \Omega^2_0\mathcal{T} \qquad\qquad \pi_B := \sigma_B^{-1} = (1 + e\hat{B})\,\pi_0$$

with $B_0 := \hat{q}^* B$ the pullback of B by the projection $\hat{q} : \mathcal{T} \to \mathcal{Q}$. This is a 2-form on $\mathcal{T}$, which may be added to σ_0.

$$\hat{B} := B_0\,(...\pi_0) \qquad \text{i.e.} \qquad \forall\alpha \in \Omega^1\mathcal{T} \qquad \alpha\;\hat{B} := B_0(\alpha\;\pi_0)$$

i.e. $\hat{B} : \Omega^1\mathcal{T} \to \Omega^1\mathcal{T}$ (acts on its left). So $\hat{B}\;\pi_0 : \Omega^1\mathcal{T} \to \chi^1\mathcal{T}$

Indeed: $\sigma_B := \sigma_0\,(1 - e\hat{B}^*)$ with $\hat{B}^* := (B_0...)\,\pi_0 \;: \chi^1\mathcal{T} \to \chi^1\mathcal{T}$ (acts on its right) and $\hat{B}$ is nilpotent: $\hat{B}^2 = 0$ so that $(1 - e\hat{B}^*)^{-1} = (1 + e\hat{B})$.

Remind: $\sigma_B = \sigma_0 - e(\vec{B} \times dq).dq$ and $\pi_B = \pi_0 + e(\vec{B} \times \partial_p).\partial_p$

Since $d\,B = 0$ we get $d\,B_0 = 0$ and so $d\,\sigma_B = 0$.
Hence its inverse $\pi_B = (1 + e\hat{B})\,\pi_0$ satisfies the Jacobi identity.

Theorem:
The Maxwell-Vlasov Poisson 2-vector $\pi \in \chi^2\Phi$ is:

$$\pi := d\,\partial_B \wedge \partial_E + f\,.[(d\,\partial_f \wedge d\,\partial_f).(1 + e\hat{B}) + e\,\partial_E \wedge d\,\partial_f].\,\pi_0$$

The first term is a 3-form on $\mathcal{Q}$ and the bracketed term is a 2-form on $\mathcal{T}$, which is converted into a 0-form after contraction with π_0 .

The second part of π satisfies the Jacobi identity, by the Lie-Poisson theorem.

The first term ("Born-Infeld") also satisfies the Jacobi identity, as easily seen.
The third term is the coupling term.

π satisfies the Jacobi identity since it is a translation of the similar 2-vector (of Morrison, Marsden-Weinstein) by the diffeomorphism $\vec{E} \to E$, $\vec{B} \to B$ and similarly for f. The direct proof (done by Morrison) is long...

The Poisson structure is independent of the metric g but the Hamiltonian depends on it!

More extensively we have $\forall\; H, G \;\in\; \Omega^0\Phi$:

$$\{H, G\} \;:=\; \mathbf{d}H \wedge \mathbf{d}G\;\pi \;=\; d\,\partial_E\,H \wedge \partial_B\,G \;-\partial_B\,H \wedge d\,\partial_E\,G$$
$$+\;\partial_f\,H\;.\,L_{\Delta\,G\,\pi_0}.f \;-\; L_{\Delta\,H\,\pi_0}.f\;.\,\partial_f\,G$$

where:

$$\Delta\,G \;:=\; \frac{1}{2}(d\,\partial_f\,G)(1\,+\,e\,\hat{B})\;+\;e.\partial_E\,G \;\in\; \Omega^1\mathcal{T}\otimes\Omega^0\Phi$$

The tensor product is over $\Omega^0\mathcal{T}$

The flow generated by the Hamiltonian H is the exponential of the vector field $\mathbf{d}H\,\pi$ where:

$$\mathbf{d}H = E^* \wedge \mathbf{d}E \;+\; B^* \wedge \mathbf{d}B \;+\; m\gamma\,\mathbf{d}f \quad \in\; \Omega^1\Phi$$

The evolution of any observable $V \in \Omega^0\Phi$ by this flow is given by:

$$\dot{V} \;:=\; \{H, V\} \;=\; \mathbf{d}H \wedge \mathbf{d}V\;\pi$$

E.g: $\dot{\mathbb{B}}_\alpha = \mathbf{d}H \wedge \mathbf{d}\mathbb{B}_\alpha\,\pi \quad (= \dot{B} \wedge \alpha)$ where $\mathbf{d}\,\mathbb{B}_\alpha = \alpha \wedge \mathbf{d}B$

$\mathbf{d}B \neq dB \;(= 0)$ $\qquad \mathbf{d}B\;\partial_B \;=\; \mathbf{1}$ $\qquad \mathbf{d}B\;\partial_E \;=\; 0$

$$\Longrightarrow \qquad \mathbf{d}\mathbb{B}_\alpha\,\pi \;=\; \alpha \wedge d\,\partial_E \;\in\; \chi^1\Phi$$

$$\Longrightarrow \qquad \dot{B} \;=\; d\,E^*$$

Likewise:

$$\dot{E} \;=\; d\,B^* \;+\; J \qquad \text{where} \qquad J \;:=\; e \int_{T_q^*\,\mathcal{Q}} \mathbf{i}_{W\pi_0}\, f$$

("the current") and:

$$W := p\; d\,\bar{p}/(m\,\gamma) \;\in\; \Omega^1\mathcal{T}$$

is the "velocity" 1-form. Hence:

$$\Longrightarrow \qquad W\pi_0 \;\in\; \chi^1\mathcal{T} \qquad \Longrightarrow \qquad \mathbf{i}_{W\pi_0}\, f \;\in\; \Omega^5\mathcal{T}$$

W is the fiber-derivative of the energy $\qquad m\,\gamma \;=\; \sqrt{m^2 + p\,\bar{p}}$

So we get a 2-form on $\mathcal{Q}$ after integration over the 3-d domain $T_q^*\,\mathcal{Q}$ (this is permitted here!)

The Vlasov equation is a Lie dragging:

$$\dot{f} \;=\; L_{Z\pi_0}\, f \qquad \text{where} \qquad Z := v.(1 \;+\; e\,\hat{B}) \;+\; e\,E^* \;\in\; \Omega^1\mathcal{T}$$

Indeed the derivative of the energy is a 1-form, the "velocity": $v := d(m\gamma) \;\in\; \Omega^1\mathcal{T}$

We recognize the advection term, associated to "1" (like $\vec{v}\,.\,\nabla$), and the Lorentz force: $e\;(\vec{v} \times \vec{B} \;+\; \vec{E})$

In a curved "configuration space" $\mathcal{Q}$ we recover the extra terms due to the curvature:

$$v = \frac{1}{m\,\gamma}\;[p\;d\,\bar{p} \;+\; \frac{1}{2}\,p\,(g'(q)\,dq)\,p]$$

Finally we also reget the last Maxwell equation as a Casimir invariant:

For all $\alpha \in \Omega^0 \mathcal{Q}$:

$$\mathbb{K}_\alpha = \mathbb{K}_\alpha(f, E, B) := \int_{\mathcal{Q}} \alpha \left[dE + e \int_{T_q^* \mathcal{Q}} f \right] \qquad \text{verifies} \quad \mathbf{d}\mathbb{K}_\alpha \, \pi = 0$$

In particular it is a constant of the motion generated by H. If we start with:

$$dE \; + \; e \int_{T_q^* \mathcal{Q}} f \; = \; 0$$

then this quantity will remain 0 for any t.

The same is true for $\int_{\mathcal{T}} f \; f^{*n}$ $\quad (\forall \, n \in \mathbb{N})$

where we use the symplectic Hodge dual $f^* \; := \; \mathbf{i}_{\pi_0 \wedge \pi_0 \wedge \pi_0} \, f$

Remarks

This model is gauge invariant: no potentials, only fields.

The Poincaré group is included in the Poisson-isomorphisms of this Maxwell-Vlasov algebra. So this model has relativistic covariance.

Viewing the fields as independent variables (rather than generated by the matter) removes the need of the “retarded fields”.

The constraint $dB = 0$ is actually more fondamental than a Casimir invariant: it is needed to insure the Jacobi identity.

Perturbation theory in the strong B region

(as in Tokamaks)

We split the Hamiltonian into:

$$H = H_0 + V \qquad H_0 := \frac{1}{2} B^* \wedge B \; + \; f\, m\gamma \qquad V := \frac{1}{2} E^* \wedge E$$

This splitting of H induces our perturbation method: finding some actions variables for the simple H_0 (the "guiding-center" gyro-momentum, and another constant of motion: the "bounce averaging") and then try to perturb them under the "retro-action" of the matter on the fields when adding V (the "gyro-center"), by a KAM-like method.

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