# Secular “Collisionless” Dynamics around Massive Central Bodies

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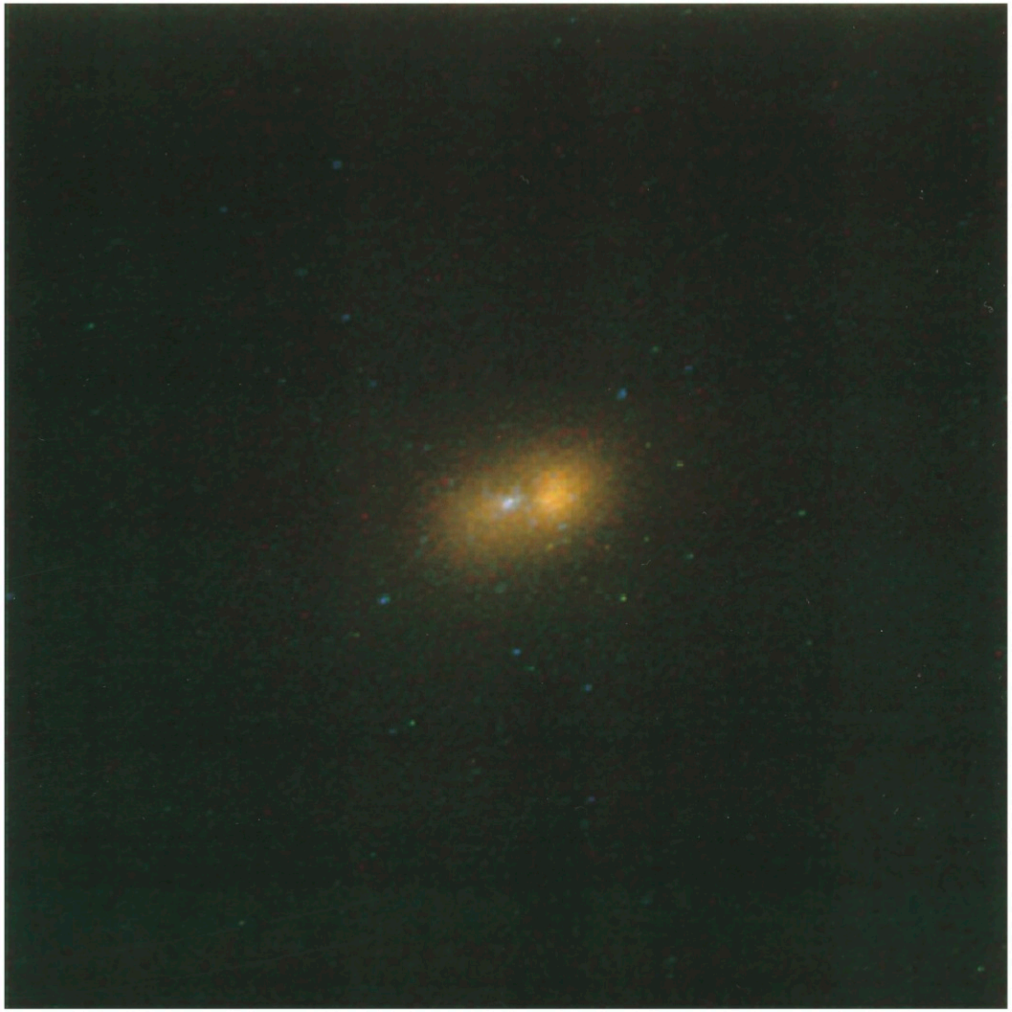
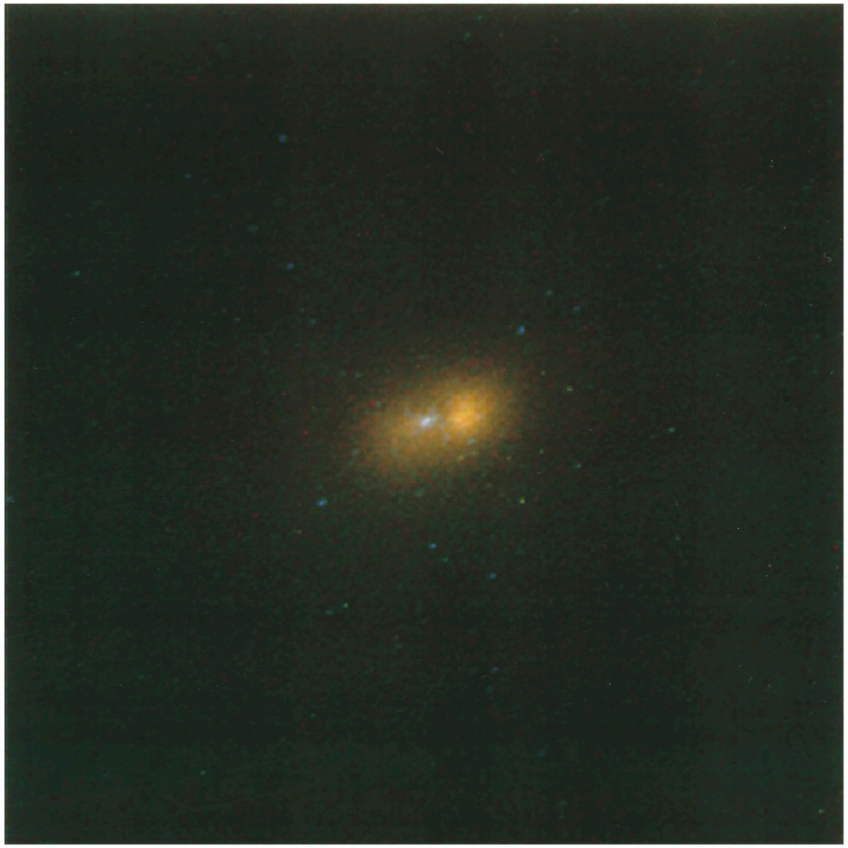
# M31's Double Nucleus: Under the Black Hole's Spell



Nasa/ESA and T. Lauer (NOAO)

# Planetary Antecedent: Uranus Epsilon Ring



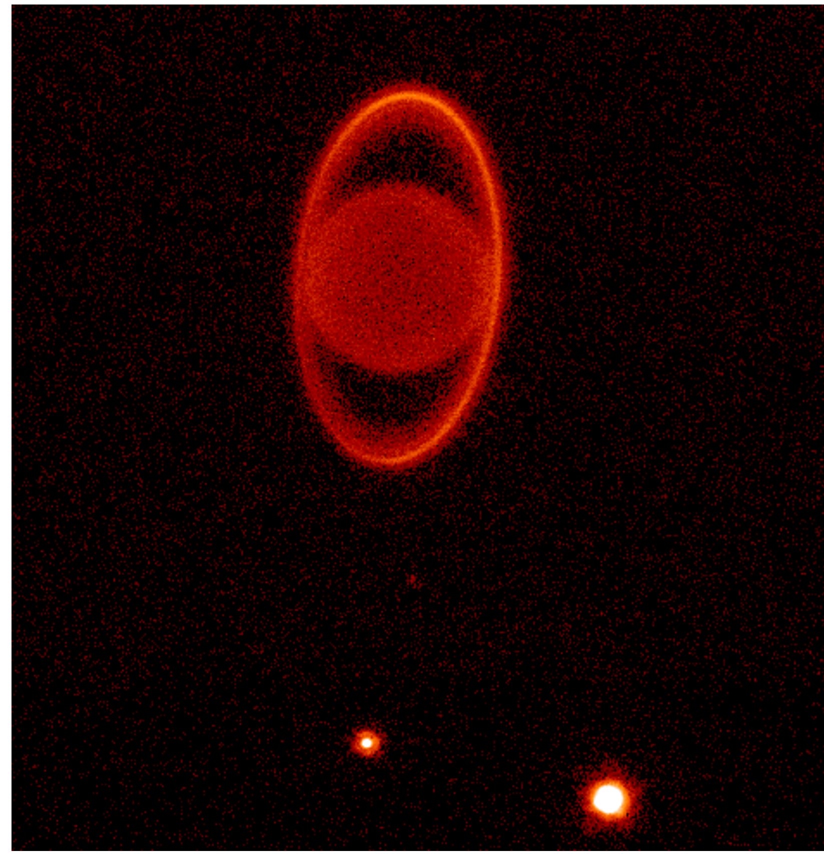
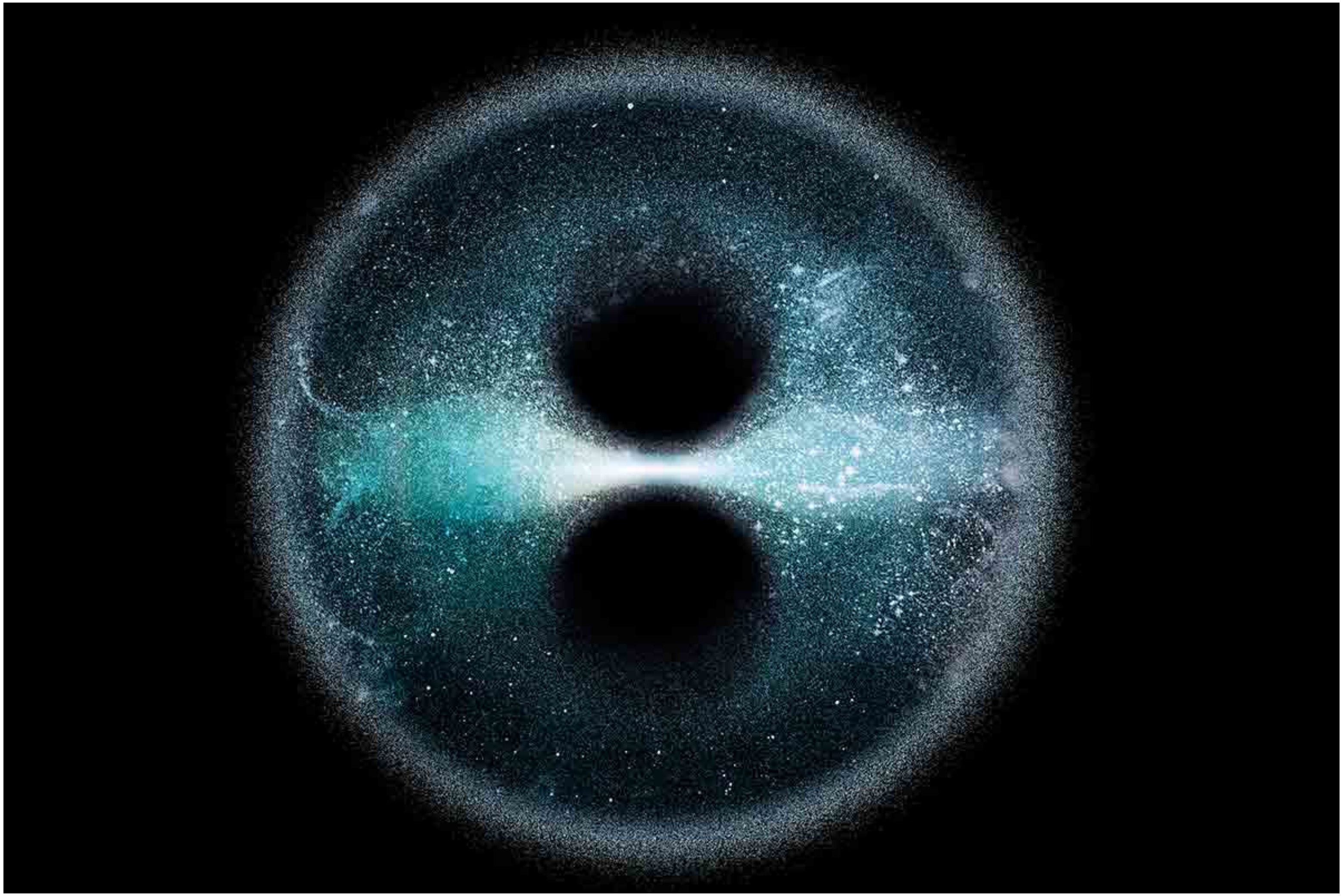


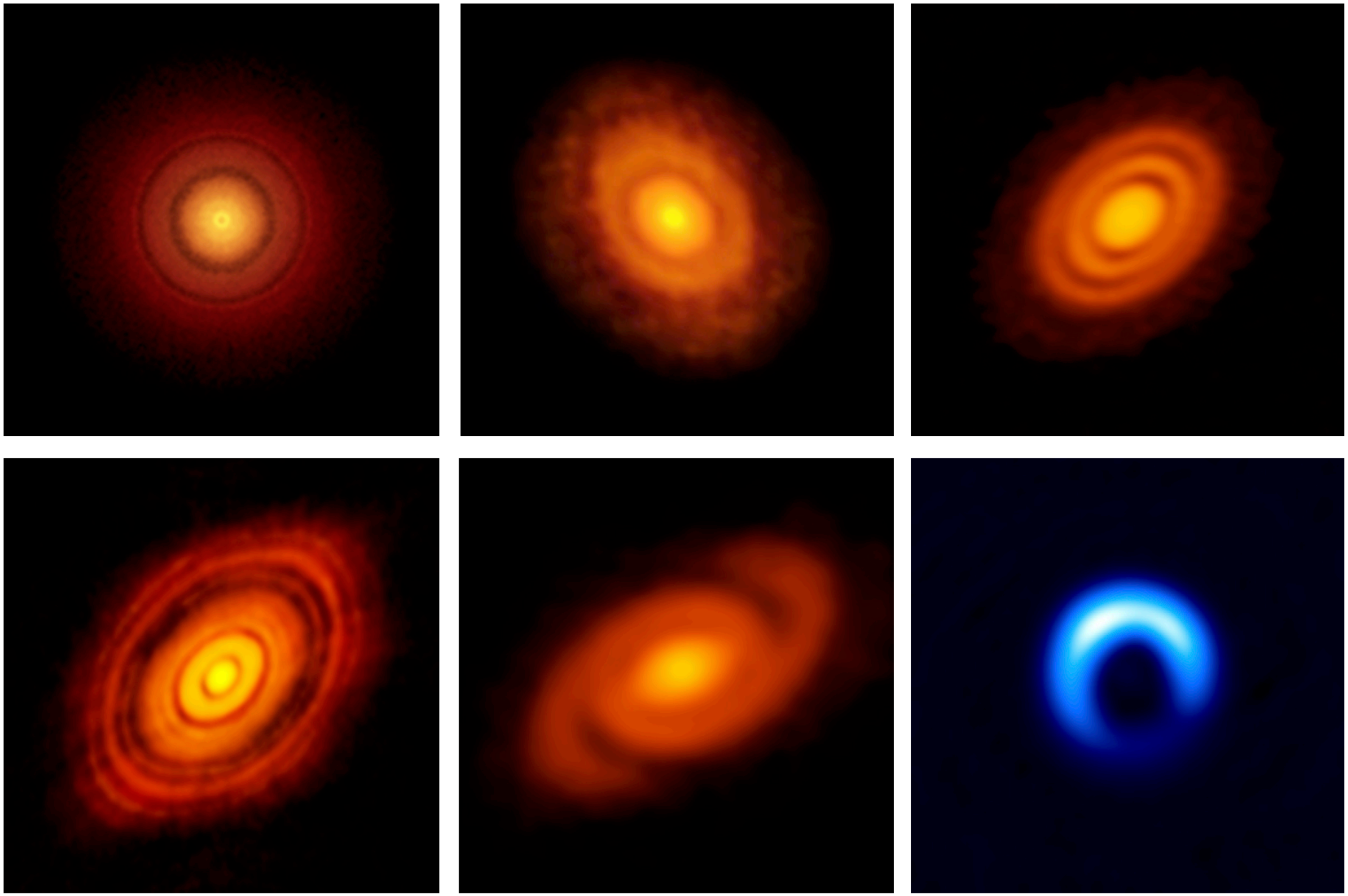
Image Courtesy of C. Dumas (Caltech/JPL) and J.-L Beuzit and F. Menard (CFHT)

# Planetary Context: The Oort Cloud



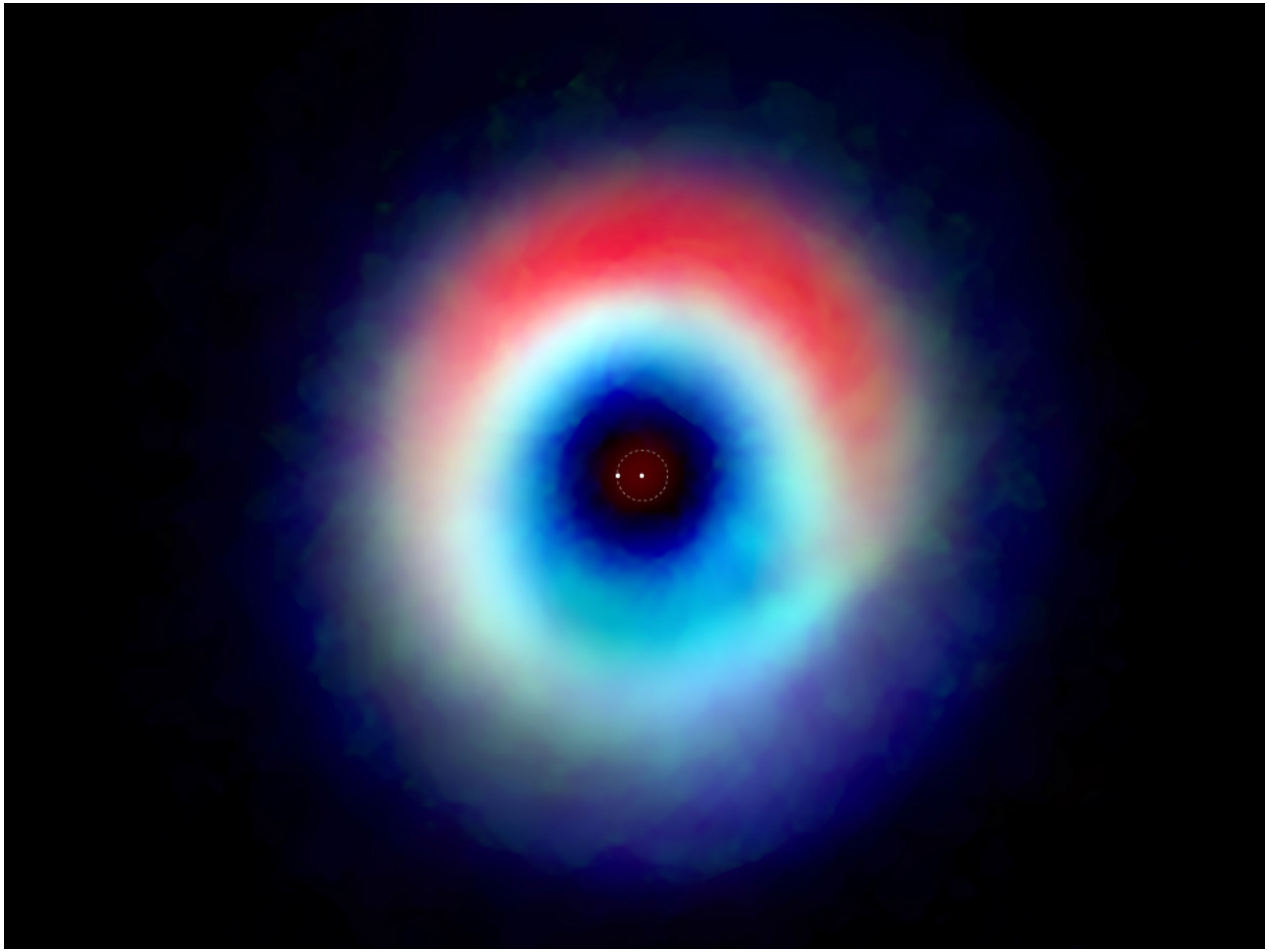
New Scientist, 2016

# Exo-Planetary Context: Proto-planetary Disks



ALMA (ESO/NAOJ/NRAO)

# Exo-Planetary Context: Circum-binary Disks



NRAO/AUI/NSF/Rice University/ESO/NAOJ

# Massive Central Bodies: Sphere of Influence

- Sphere of influence of MCB: $$r_{sphere} \sim \frac{GM_\bullet}{\sigma^2}$$
- Hierarchy of time scales: $$t_{orbit} \ll t_{sec} \ll t_{rr} \ll t_{relax}$$

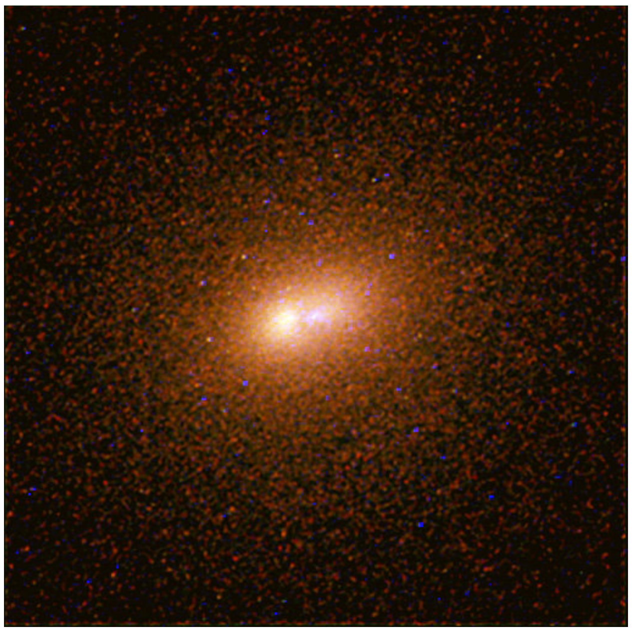
i. Keplerian orbital time: $$t_{orbit} \sim \left(\frac{r^3}{GM_\bullet}\right)^{1/2}$$

ii. Cluster self-gravity time: $$t_{sec} \sim \frac{M_\bullet}{M_c} t_{orbit}$$

iii. Resonant-Relaxation time: $$t_{rr} \sim \frac{M_\bullet}{m} t_{orbit} \sim N t_{sec}$$

iv. Two-body relaxation time: $$t_{relax} \sim \frac{{M_\bullet}^2}{Nm^2} t_{orbit} \sim \frac{M_\bullet}{M_c} t_{rr}$$

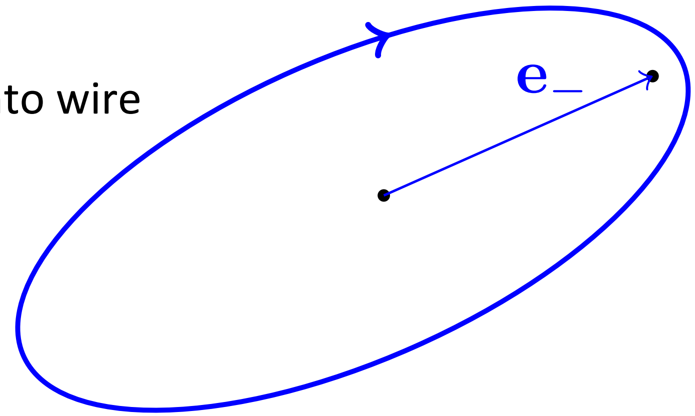
# Averaging Principle: Gauss Wire



T. Lauer (NOAO/AURA/NSF)

Star smeared into wire

$\longrightarrow$



# Averaging Principle: The Planetary Context

- Secular Invariance of Semi-Major Axes: Laplace(1773), Lagrange(1774, 1776)

- Secular "Stability" of the Solar System: Laplace-Lagrange theory

- Gauss(1818) Takes on Averaging: Wire with mass elements proportional to time spent on the orbit

# Averaging Principle: The Planetary Context

- Obstructions to Averaging: Resonances, Small Denominators (Poincare, KAM)
- Chaos and Mean Motion Resonances (Asteroids, Wisdom)
- Outer Solar System Chaos: Mean Motion Resonance Overlap (Murray and Holman)
- Secular Chaos: Mercury's escape (Laskar et al.)

# Towards the Large N Limit of Orbit-Averaged Celestial Mechanics!

# Averaging Principle: Stellar Dynamical Context

Over time scales longer than Keplerian orbital period, and shorter than two body relaxation, what is known:

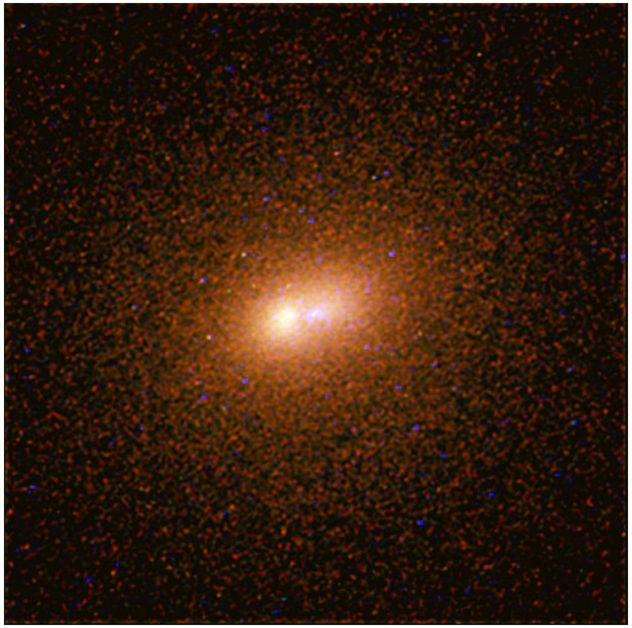
- Central Body Dominated, Nearly-Keplerian Motion: Orbit averaged into (Gaussian Wires) with Constant Keplerian Energy [S. Sridhar & J.T (1998))]

- Complete Integrability of the Planar Problem: Classification of orbits in generic non-axisymmetric, lopsided potential; Clarification of transition to chaos [S. Sridhar and JT 1998]

- Resonant Relaxation of Gaussian Wires Dominate Two-Body Relaxation [Rauch and Tremaine (1996)]

- Secular Instabilities of Disks and Spheres [JT (2002), Tremaine(2005), Polyachenko et. Al. (2007)]

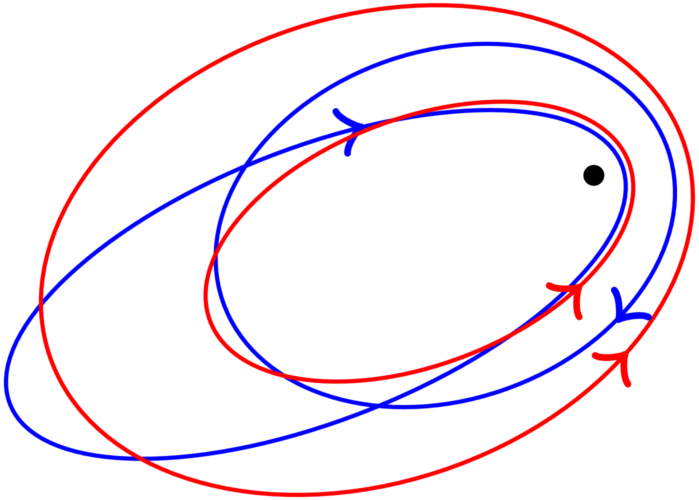
# Averaging Principle: Gauss Wires



T. Lauer (NOAO/AURA/NSF)

Disk of wires

$\longrightarrow$



# Secular Dynamics and (Non-Eq) Thermodynamics of Nearly-Keplerian Systems around MCB

- What determines observed kinematics, photometry, shape of Nearly Keplerian Systems?
- Are these systems subject to dynamical instabilities? If so can we characterize their saturated states, as the outcomes of a collisionless-relaxation process? Poisson-Vlasov in this limit?
- Do observed features reflect thermal equilibrium phases of a suitably defined self-gravitating gas? Phase Diagrams? Transitions? Type?
- What about the route to thermal equilibria? Can we construct properly grounded kinetic theory that is agile enough to handle the collisional processes at work in these systems?
- How does all this work itself out through the tumultuous conditions in which these systems are born, perturbed, and evolved? What does it say about origin, initial states, potential perturbers? About the feeding of super massive black holes? Tidal disruption events? Impact on rate of binary mergers in the sphere of influence of a super-massive black hole?

# Progress Report

- Counter-Rotating Nearly-Keplerian stellar disks are unstable: They evolve into lopsided uniformly precessing configurations [J.T. (MNRAS, 2002), Sridhar and Saini(MNRAS 2009), J.T. and Sridhar (MNRAS, 2012), Kazandjian and J.T. (MNRAS, 2013)]

- Nearly-Keplerian stellar disks (whether counter-rotating or not) are prone to violent secular instabilities: Collisionless orbit averaged Poisson-Vlasov for Waterbags [Kaur et. al (2017, Submitted), Sridhar and J.T. (MNRAS, 2016)]

- Microcanonical Thermal Equilibria of narrow, ring like, disks are, more often than not, lopsided [J.T and Tremaine (J. Phys. A, 2014)]

- First Principles Theory of "Resonant Relaxation" lays bare the kinetics of collisional relaxation onto thermal equilibria [Sridhar and J.T. (MNRAS, 2016-2017]

# Averaging Principle: Particle in External Potential

- Hamiltonian:

$$\mathcal{H} = \underbrace{\frac{v^2}{2} - \frac{GM}{r}}_{H_{Kepler}} + \mathbf{\Phi}(\mathbf{r})$$

- Action Angle Variables for Kepler: Delaunay:

$$(a, e, i, f, g, \Omega) \quad \rightarrow \quad (I, L, L_z, w, g, h)$$

- Orbit Averaging

$$\overline{\mathcal{H}} = \oint \frac{\mathrm{d}w}{2\pi} \mathcal{H} = \oint \frac{\mathrm{d}\eta}{2\pi} \left[1 - \sqrt{1 - (L/I)^2} \cos \eta\right] \mathcal{H}$$

$$= -\frac{1}{2} \left(\frac{GM}{I}\right)^2 + \overline{\Phi},$$

# Gauss Wire in 3D: Vectorial Formulation

- Lenz and "Angular Momentum" Vectors:

$$\mathbf{L} = \mathbf{r} \times \mathbf{v} = \sqrt{(GMa)}\,\mathbf{j}$$

$$\mathbf{e} = \frac{1}{GM}\mathbf{v} \times (\mathbf{r} \times \mathbf{v}) - \frac{\mathbf{r}}{r}$$

- In Terms of Orbit Normal/Periapse:

$$\mathbf{j} = (1 - e^2)^{1/2}\mathbf{n}, \quad \mathbf{e} = e\mathbf{u}$$

# Equations of Motion

$$\frac{d\mathbf{j}}{dt} = -\frac{1}{\sqrt{GMa}} \left[\mathbf{e} \times \nabla_{\mathbf{e}} H + \mathbf{j} \times \nabla_{\mathbf{j}} H\right]$$

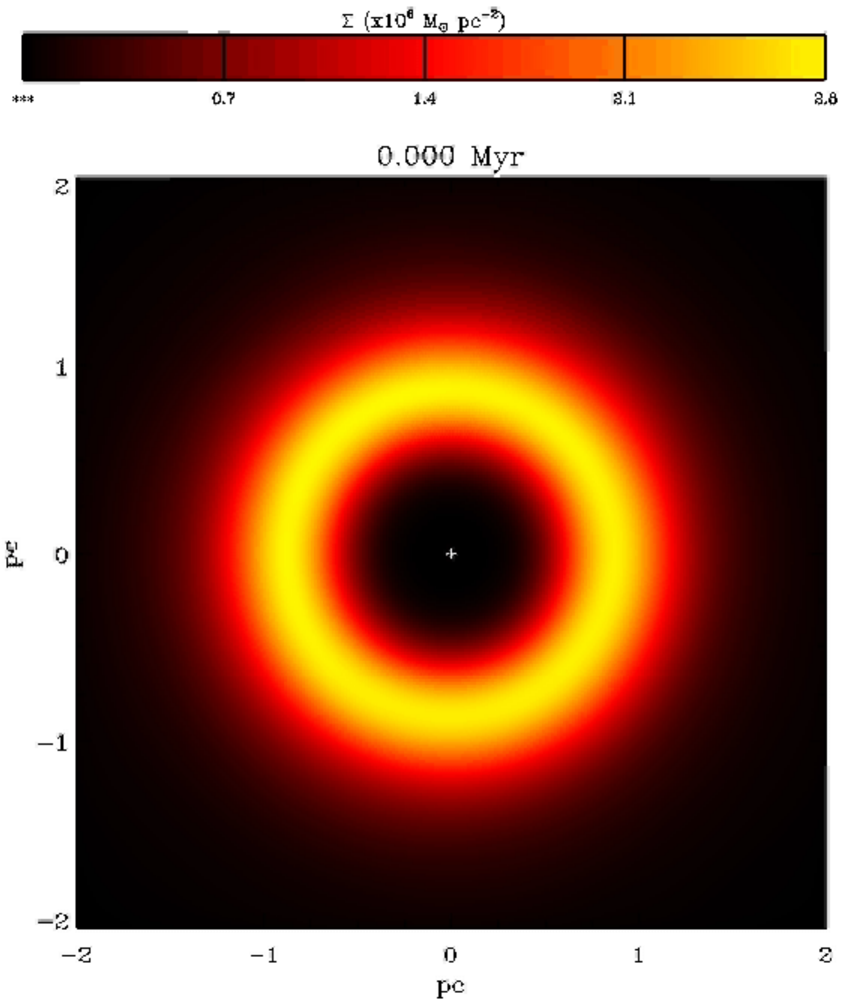
$$\frac{d\mathbf{e}}{dt} = -\frac{1}{\sqrt{GMa}} \left[\mathbf{j} \times \nabla_{\mathbf{e}} H + \mathbf{e} \times \nabla_{\mathbf{j}} H\right]$$

$$\mathbf{j} \cdot \mathbf{e} = 0, \ \mathbf{j}^2 + \mathbf{e}^2 = 1$$

# Gauss Wires at Work

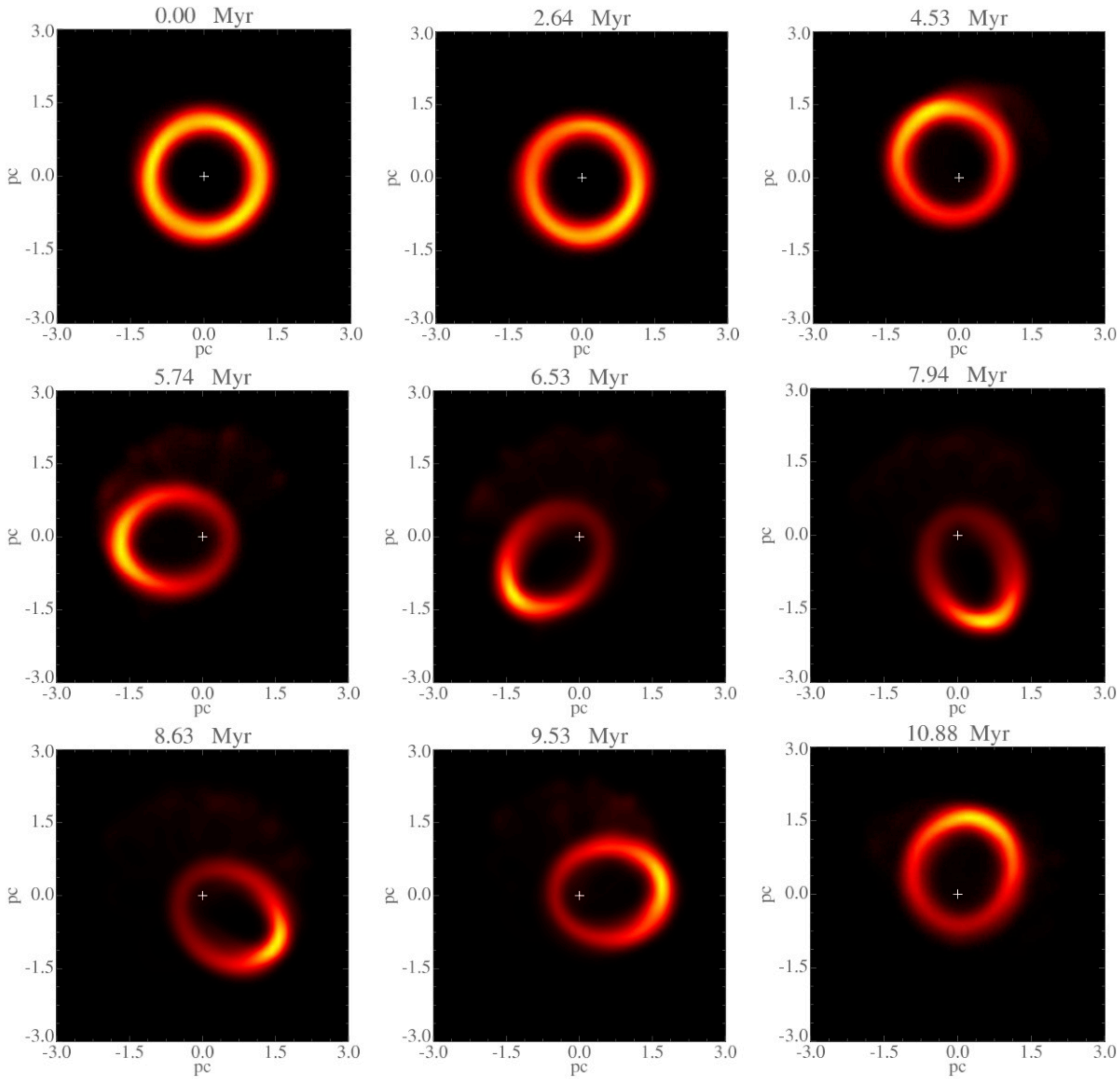
- Planetary theory to first order in the masses and all orders in eccentricity and inclination
- Secular evolution of hot nearly-keplerian systems
- Collisionless relaxation of secularly unstable clusters
- Collisional resonant relaxation of triaxial clusters around central massive bodies

# Counter-Rotating Disks: Instability onto Saturation




$$M_p = 10^{-3} M_\star, M_r = 0.2 M_p, a \simeq 1\mathrm{pc}, b = 0.1\mathrm{pc}$$

# Counter-Rotating Disks: Instability onto Saturation

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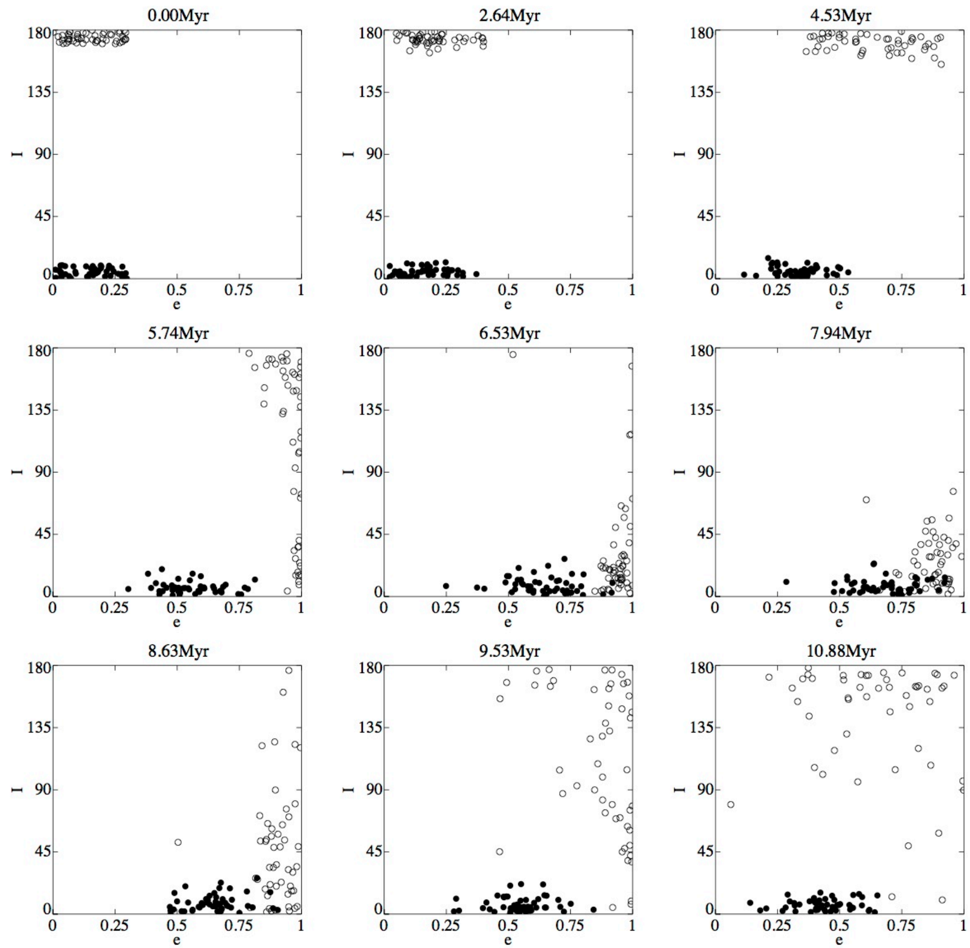
# Counter-Rotating Disks: Instability onto Saturation




$$M_p = 10^{-3} M_\star, M_r = 0.2 M_p, a \simeq 1\text{pc}, b = 0.1\text{pc}$$

# Collisionless Boltzmann-Poisson for Particles

- Single Particle DF in 6D:

$$\frac{\partial f}{\partial t} + [\, f \,,\, H \,]_{(6)} = 0 \,,$$

$$H(\mathbf{r},\mathbf{u},t) = \frac{u^2}{2} - \frac{GM_\bullet}{r} + \varepsilon\varphi(\mathbf{r},t) + \varepsilon\mathbf{r}\cdot\mathbf{A}_\bullet(t)$$

$$[\, f \,,\, H \,]_{(6)} = \frac{\partial f}{\partial \mathbf{r}}\cdot\frac{\partial H}{\partial \mathbf{u}} - \frac{\partial f}{\partial \mathbf{u}}\cdot\frac{\partial H}{\partial \mathbf{r}} \,.$$

- Explicitly:

$$\frac{\mathrm{d}f}{\mathrm{d}t} \equiv \frac{\partial f}{\partial t} + \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\cdot\frac{\partial f}{\partial \mathbf{r}} + \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}\cdot\frac{\partial f}{\partial \mathbf{u}}$$

$$= \frac{\partial f}{\partial t} + \mathbf{u}\cdot\frac{\partial f}{\partial \mathbf{r}} + \left(-\frac{GM_\bullet}{r^2}\hat{\mathbf{r}} - \varepsilon\frac{\partial \varphi}{\partial \mathbf{r}} - \varepsilon\mathbf{A}_\bullet\right)\cdot\frac{\partial f}{\partial \mathbf{u}} = 0 \,.$$

# From Kepler to Delaunay

- Non-canonical, Osculating, Keplerian Orbital Elements:

$$(a, e, i, f, g, \Omega)$$

- Canonical, Action-Angle like, Delaunay Variables:

$$\mathcal{D} \equiv \{I, L, L_z; w, g, h\},$$

$$\text{Actions} = \left[I = \sqrt{GM_\bullet a}\; L = I\sqrt{1-e^2}, L_z = L\cos i\right], \ \text{Angles} = [w\, g,\ h]$$

$$E_\mathrm{k}(I) = -1/2(GM/I)^2,\ \Omega_\mathrm{k}(I) = \frac{\mathrm{d}E_\mathrm{k}}{\mathrm{d}I} = \frac{(GM_\bullet)^2}{I^3}\,.$$

# Poisson-Vlasov in Delaunay

- Poisson Brackets in Delaunay Variables:

$$[\,\chi_1\,,\,\chi_2\,]_{(6)} \;=\; \left(\frac{\partial \chi_1}{\partial w}\frac{\partial \chi_2}{\partial I} - \frac{\partial \chi_1}{\partial I}\frac{\partial \chi_2}{\partial w}\right) + \left(\frac{\partial \chi_1}{\partial g}\frac{\partial \chi_2}{\partial L} - \frac{\partial \chi_1}{\partial L}\frac{\partial \chi_2}{\partial g}\right) + \left(\frac{\partial \chi_1}{\partial h}\frac{\partial \chi_2}{\partial L_z} - \frac{\partial \chi_1}{\partial L_z}\frac{\partial \chi_2}{\partial h}\right).$$

- Mean Field and Non-Inertial contribution in Delaunay Variables:

$$\varphi(\mathcal{D},t) \;=\; -\,GM_\bullet \int \frac{f(\mathcal{D}',t)}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}\mathcal{D}', \qquad \mathbf{A}_\bullet(t) \;=\; GM_\bullet \int f(\mathcal{D},t)\,\frac{\hat{\mathbf{r}}}{r^2}\,\mathrm{d}\mathcal{D}\,.$$

$$\downarrow$$

$$\frac{\partial f}{\partial t} \;+\; \Omega_{\mathrm{k}}\frac{\partial f}{\partial w} \;+\; \varepsilon\,[\,f\,,\,\varphi + \mathbf{r}\cdot\mathbf{A}_\bullet\,]_{(6)} \;=\; 0\,.$$

# Orbit Averaged Potential: Wire-Wire Kernel

The Orbit-Averaged Mean Field of Cluster:

$$\Phi(\mathcal{R}, \tau) \;=\; \int \mathrm{d}\mathcal{R}'\, F(\mathcal{R}', \tau)\, \Psi(\mathcal{R}, \mathcal{R}')\,,$$

where

$$\Psi(\mathcal{R}, \mathcal{R}') \;=\; -GM_{\bullet} \oint \oint \frac{\mathrm{d}w}{2\pi}\, \frac{\mathrm{d}w'}{2\pi}\, \frac{1}{|\mathbf{r} - \mathbf{r}'|}$$

# Reduced Phase Space of Wires

From Delaunay to Gauss (from 6D to [4+1]D):

$$\mathcal{D} \equiv \{I, L, L_z; w, g, h\} \to \mathcal{R} \equiv \{I, L, L_z; ., g, h\},$$

Conserved Action (semi-major axis), and resulting EOM:

$$I = \sqrt{GM_\bullet a} = \mathrm{constant}\,,$$

$$\frac{\mathrm{d}L}{\mathrm{d}\tau} = -\frac{\partial H}{\partial g}\,, \qquad \frac{\mathrm{d}g}{\mathrm{d}\tau} = \frac{\partial H}{\partial L}\,; \qquad \frac{\mathrm{d}L_z}{\mathrm{d}\tau} = -\frac{\partial H}{\partial h}\,, \qquad \frac{\mathrm{d}h}{\mathrm{d}\tau} = \frac{\partial H}{\partial L_z}\,.$$

# Orbit-Averaged Poisson-Vlasov

- Fourier in fast Keplerian Motion:

$$f(\mathcal{D},t,\tau) \;=\; \frac{1}{2\pi}\,F(\mathcal{R},\tau) \;+\; \frac{\varepsilon}{2\pi}\sum_{n=-\infty}^{\infty} f_n(\mathcal{R},t,\tau)\exp\left[\mathrm{i}nw\right] \;+\; O(\varepsilon^2)$$

- Vlasov for orbit averaged DF:

$$\frac{\mathrm{d}F}{\mathrm{d}\tau} \;=\; \frac{\partial F}{\partial \tau} \;-\; \frac{\partial H}{\partial g}\frac{\partial F}{\partial L} \;+\; \frac{\partial H}{\partial L}\frac{\partial F}{\partial g} \;-\; \frac{\partial H}{\partial h}\frac{\partial F}{\partial L_z} \;+\; \frac{\partial H}{\partial L_z}\frac{\partial F}{\partial h} \;=\; 0$$

With Hamiltonian:

$$H(\mathcal{R},\tau) \;=\; \Phi(\mathcal{R},\tau) \;+\; H^{\mathrm{rel}}(I,L,L_z) \;+\; \Phi^{\mathrm{tid}}(\mathcal{R},\tau)$$

# Collisionless Boltzmann for Wires: A Cheat Sheet

$$[\,\chi_1\,,\,\chi_2\,] \;=\; \left(\frac{\partial\chi_1}{\partial g}\frac{\partial\chi_2}{\partial L} - \frac{\partial\chi_1}{\partial L}\frac{\partial\chi_2}{\partial g}\right) + \left(\frac{\partial\chi_1}{\partial h}\frac{\partial\chi_2}{\partial L_z} - \frac{\partial\chi_1}{\partial L_z}\frac{\partial\chi_2}{\partial h}\right)$$

$$H(\mathcal{R},\tau) \;=\; \Phi(\mathcal{R},\tau) \;+\; H^{\mathrm{rel}}(I,L,L_z) \;+\; \Phi^{\mathrm{tid}}(\mathcal{R},\tau)$$

$$\frac{\mathrm{d}F}{\mathrm{d}\tau} \;=\; \frac{\partial F}{\partial\tau} \;-\; \frac{\partial H}{\partial g}\frac{\partial F}{\partial L} \;+\; \frac{\partial H}{\partial L}\frac{\partial F}{\partial g} \;-\; \frac{\partial H}{\partial h}\frac{\partial F}{\partial L_z} \;+\; \frac{\partial H}{\partial L_z}\frac{\partial F}{\partial h} \;=\; 0$$

$$P(I) \;=\; \int dL\,dL_z\,dg\,dh\,F(I,\,L,\,L_z,\,g,\,h,\,\tau)$$

# Collisionless Gauss-Wire Equilibria

- Secular Jeans Theorem: Any stationary solution is a function of time-independent integrals of the secular dynamics, and any such function is a stationary solution
- General stationary spherical DF must be of the form: F(I, L, Lz); general stationary axisymmetric DF must of the form F(I, L, H) (Jacobi Hamiltonian replaces Hamiltonian for rotating equilibria)
- Spherical Equilibria with F(I, L) monotonic in L and no loss cone are stable to l=1 modes, and a have a neutral l=1 mode when F(I, 0)=0 (Tremaine (2005); Spherical equilibria with negative nodal precession, and F(I, L) non-monotonic in L, can be unstable to l>= 3 modes (Polyachenko et al. (2007).
- Axisymmetric razor thin disks with F(I, L) monotonic in L are neutrally stable to all secular perturbations (Sridhar and JT, 2016).

# The Road to Thermal Equilibrium: Kinetic Theory of Resonant Relaxation

- Generalize Gilbert (1968) to the realm of Gaussian Wires [Sridhar and J.T (MNRAS, 2016-2017)]
- First principles theory of Resonant Relaxation: The first formally grounded approach to Rauch and Tremaine (1996)
- Interactions between (apse/node)-resonant populations feeding 2-wire wakes which then feed the orbit-averaged Boltzmann equation with collisional flux
- Recover a Fokker-Planck equation with explicit expressions for dissipative and fluctuating components
- The Fun Begins!

# Razor Thin Disks: 6 to (4+1) to (2+1) D

- Collapse onto a Disk:

$$\mathcal{R} = (I, L, g)$$

- Vlasov for orbit averaged DF:

$$I = \sqrt{GM_\bullet a} = \text{constant}\,, \qquad \frac{\mathrm{d}L}{\mathrm{d}\tau} = -\frac{\partial H}{\partial g}\,, \qquad \frac{\mathrm{d}g}{\mathrm{d}\tau} = \frac{\partial H}{\partial L}\,.$$

$$\frac{\mathrm{d}F}{\mathrm{d}\tau} = \frac{\partial F}{\partial \tau} - \frac{\partial H}{\partial g}\frac{\partial F}{\partial L} + \frac{\partial H}{\partial L}\frac{\partial F}{\partial g} = \frac{\partial F}{\partial \tau} + [\,F\,,\,H\,] = 0$$

with Hamiltonian:

$$H(\mathcal{R},\tau) = \Phi(\mathcal{R},\tau) + H^{\mathrm{rel}}(I,L) + \Phi^{\mathrm{tid}}(\mathcal{R},\tau)$$

# Razor Thin, Mono-energetic Disks: 6 to (4+1) to (2+1) to 2D !

- Thin and Mono-energetic [same semi-major axis !]:

$$\mathcal{R} = (I, L, g) \rightarrow \mathcal{R}_a = (\ell, g)$$

where

$$\ell = \sqrt{1-e^2} \quad L = \sqrt{GM_\bullet a}\,\ell$$

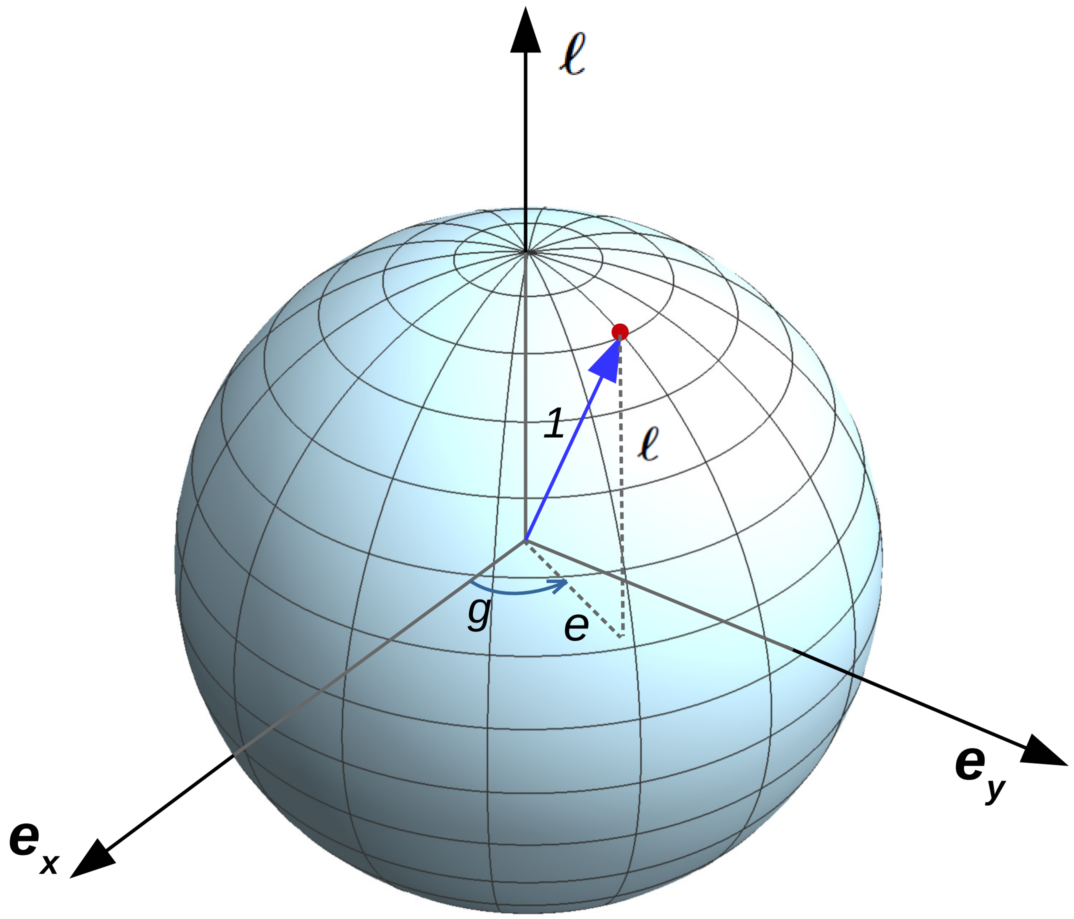
- Wire-Wire Kernel, Hamiltonian, EOM...etc:

$$\phi(\mathbf{e}_i, \mathbf{e}_j) = -\frac{4\log 2}{\pi} + \frac{1}{2\pi}\log(\mathbf{e}_i - \mathbf{e}_j)^2$$

$$\Gamma(\mathbf{e}_i) = \frac{1}{N}\sum_{j\neq i}\phi(\mathbf{e}_i, \mathbf{e}_j)$$

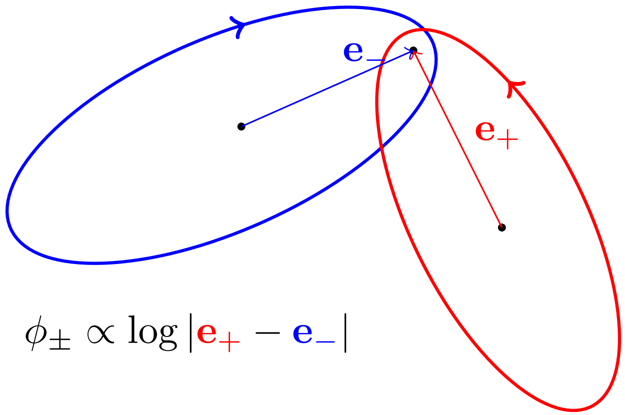
$$\frac{dk_i}{dt} = 2s_i\sqrt{1-e_i^2}\frac{\partial\Gamma}{\partial h_i}, \quad \frac{dh_i}{dt} = -2s_i\sqrt{1-e_i^2}\frac{\partial\Gamma}{\partial k_i}, \quad s_i = \pm 1$$

# Compact Phase Space: The Lenz-Vector Sphere




Kaur et. al. (Submitted, 2017)

# Compact Phase Space: Lenz-Vector Dynamics



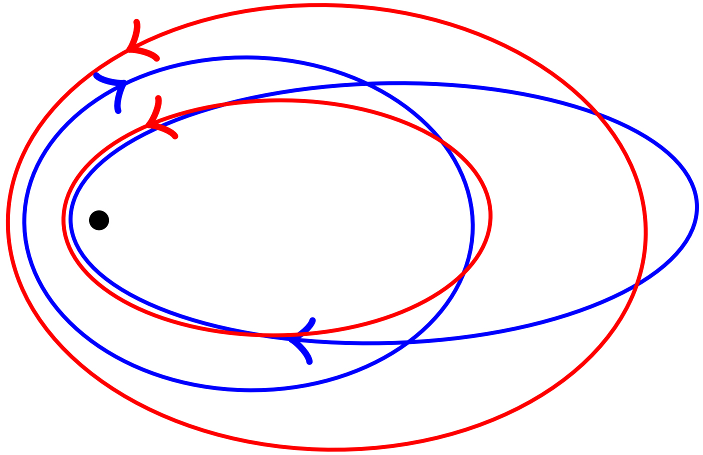

$$\phi_\pm \propto \log|\mathbf{e}_+ - \mathbf{e}_-|$$

$$\phi(\mathbf{e}_i, \mathbf{e}_j) = -\frac{4\log 2}{\pi} + \frac{1}{2\pi}\log(\mathbf{e}_i - \mathbf{e}_j)^2$$

$$\Gamma(\mathbf{e}_i) = \frac{1}{N}\sum_{j\neq i}\phi(\mathbf{e}_i, \mathbf{e}_j)$$

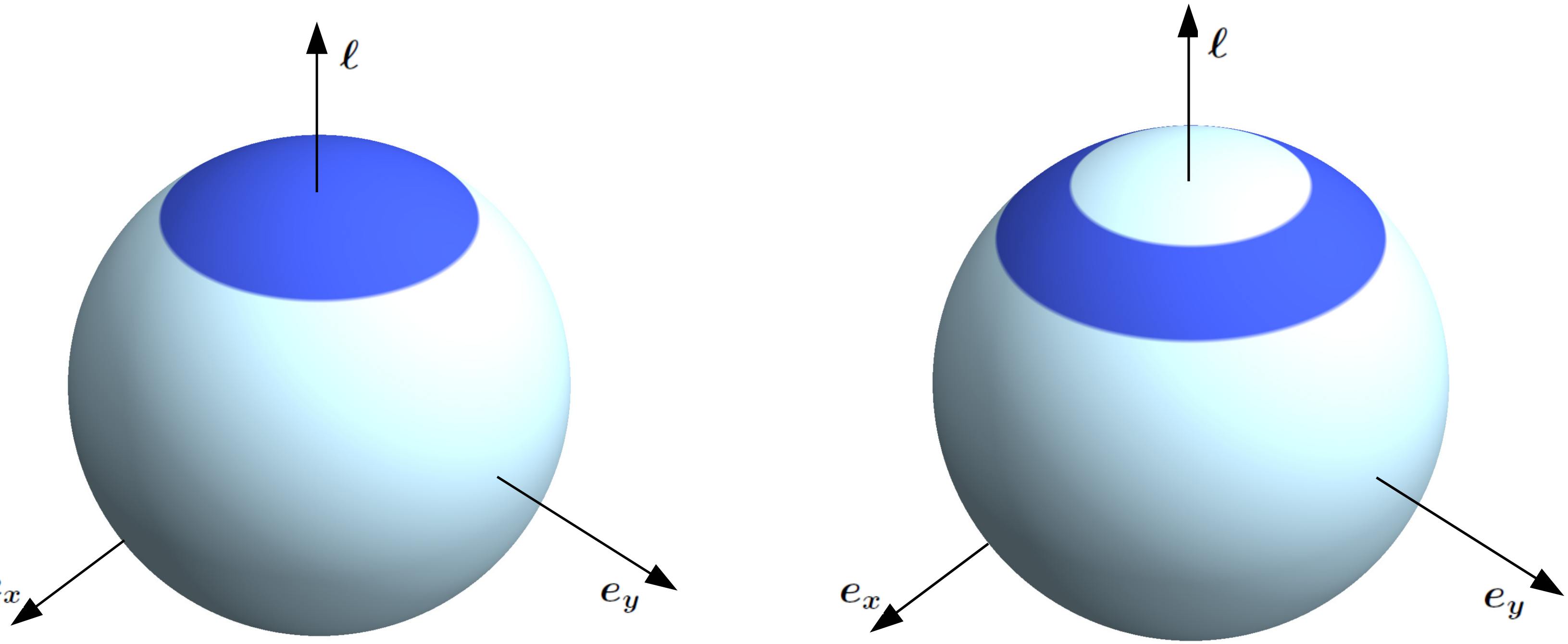
$$\frac{dk_i}{dt} = 2s_i\sqrt{1-e_i^2}\,\frac{\partial\Gamma}{\partial h_i},\;\; \frac{dh_i}{dt} = -2s_i\sqrt{1-e_i^2}\,\frac{\partial\Gamma}{\partial k_i},\;\; s_i = \pm 1$$

$\downarrow$ Collisionless Limit



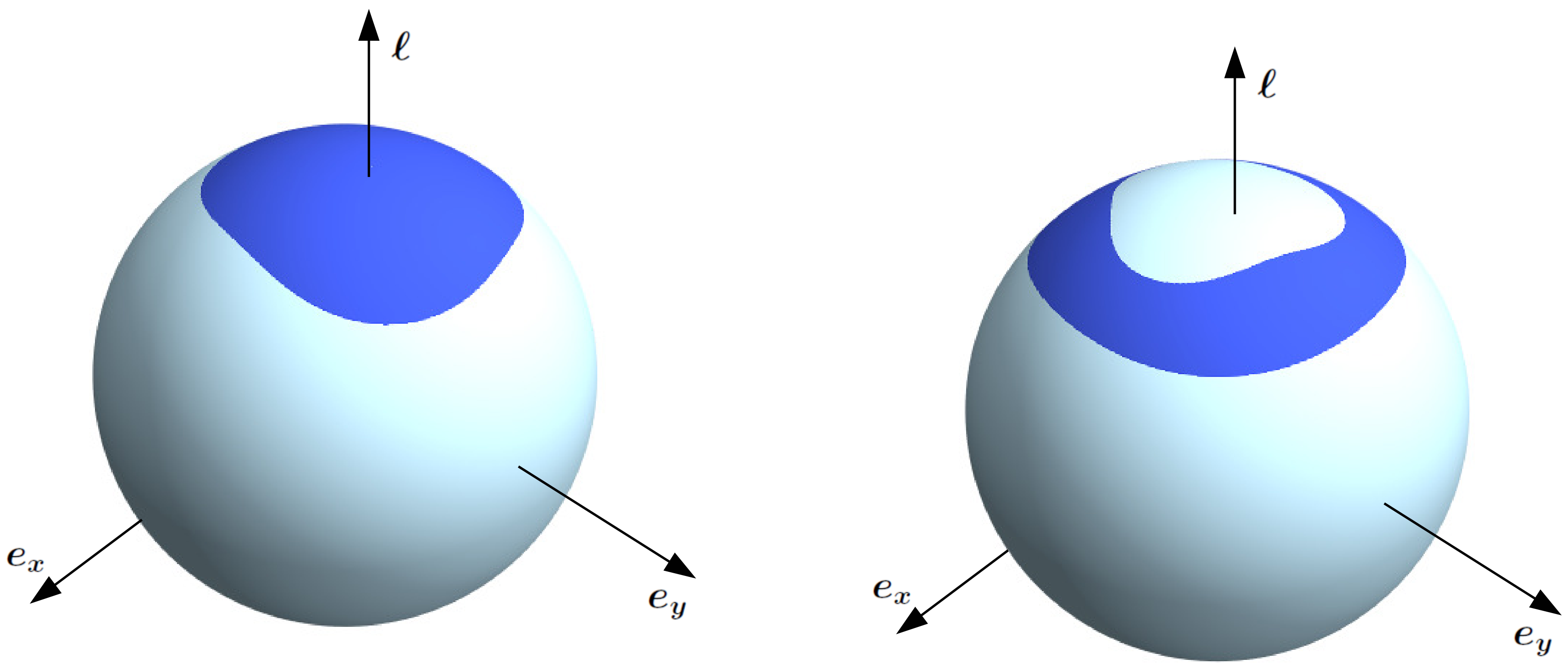
$$\frac{\partial f}{\partial t} + \frac{\sqrt{1-e^2}}{e}\left[\frac{\partial\Gamma}{\partial g}\frac{\partial f}{\partial e} - \frac{\partial\Gamma}{\partial e}\frac{\partial f}{\partial g}\right] = 0$$

# Waterbags on the Lenz-Vector Sphere: Prograde Equilibria

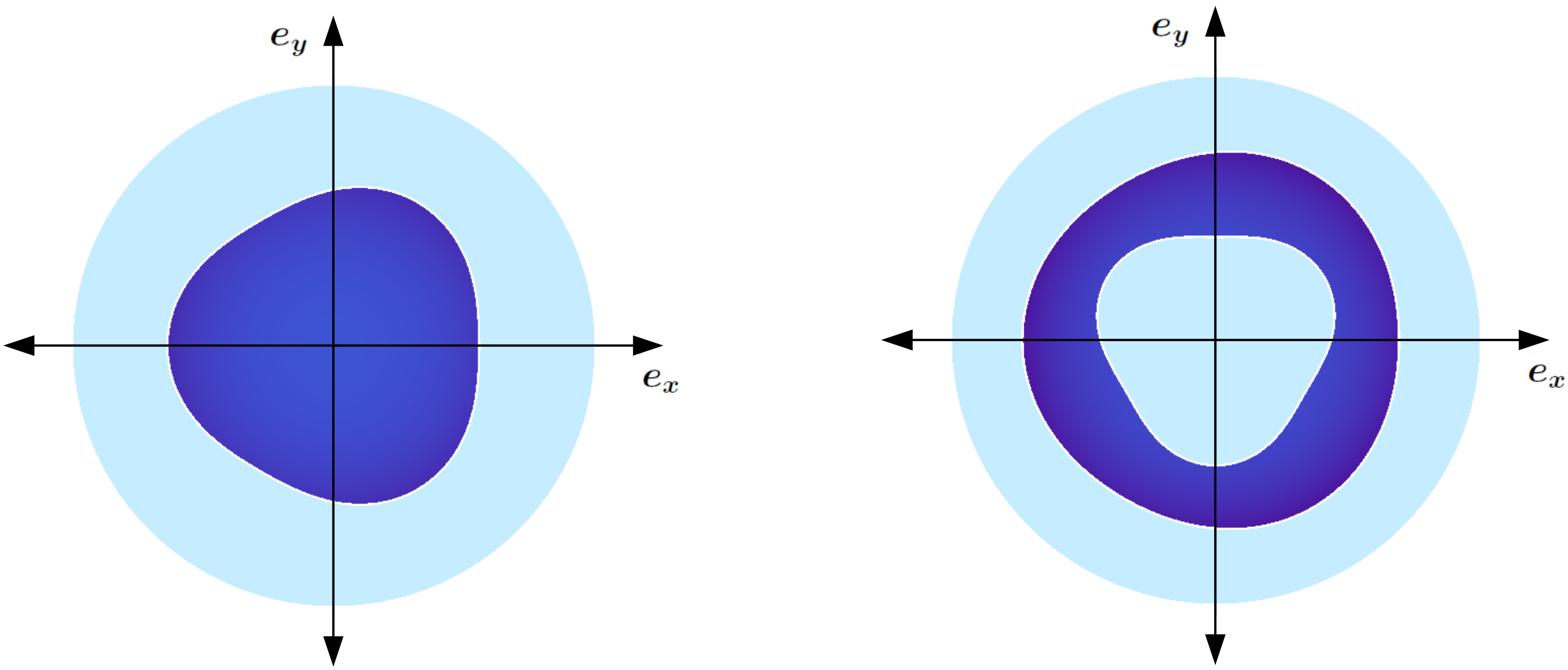



Kaur et. al. (Submitted, 2017)

# Waterbags on the Lenz-Vector Sphere: Perturbations

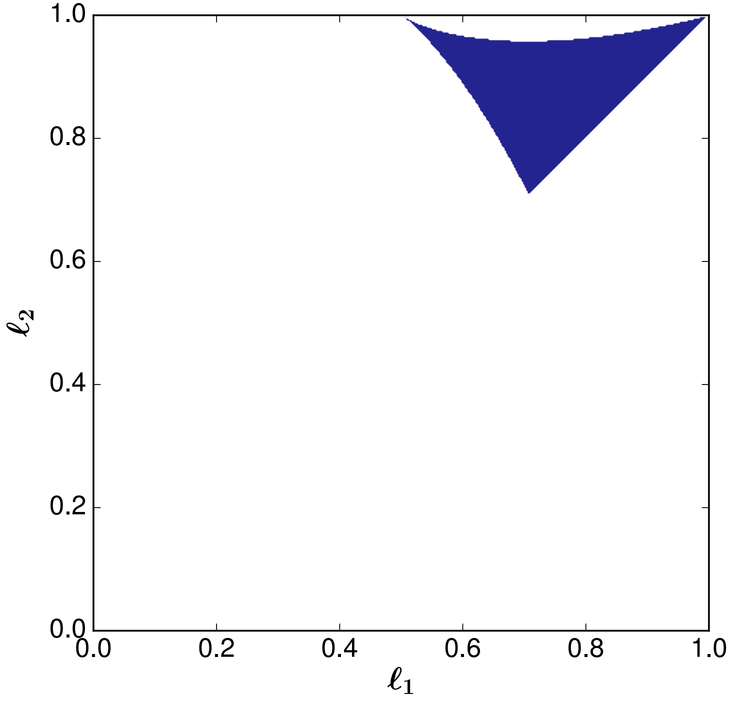


# Waterbags on the Disk: Edge Modes

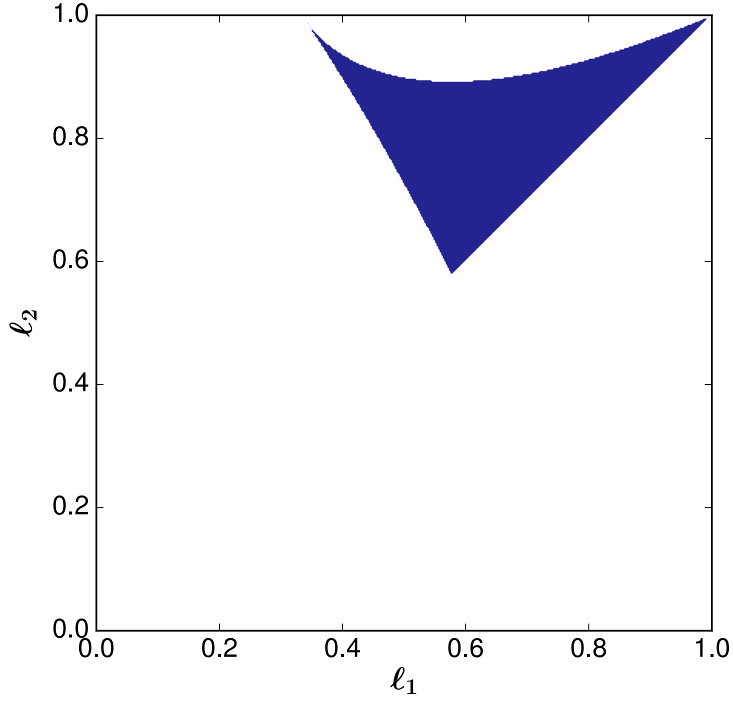


# Mono-Energetic Waterbags: Linear Stability of Edge Modes

m = 3 unstable

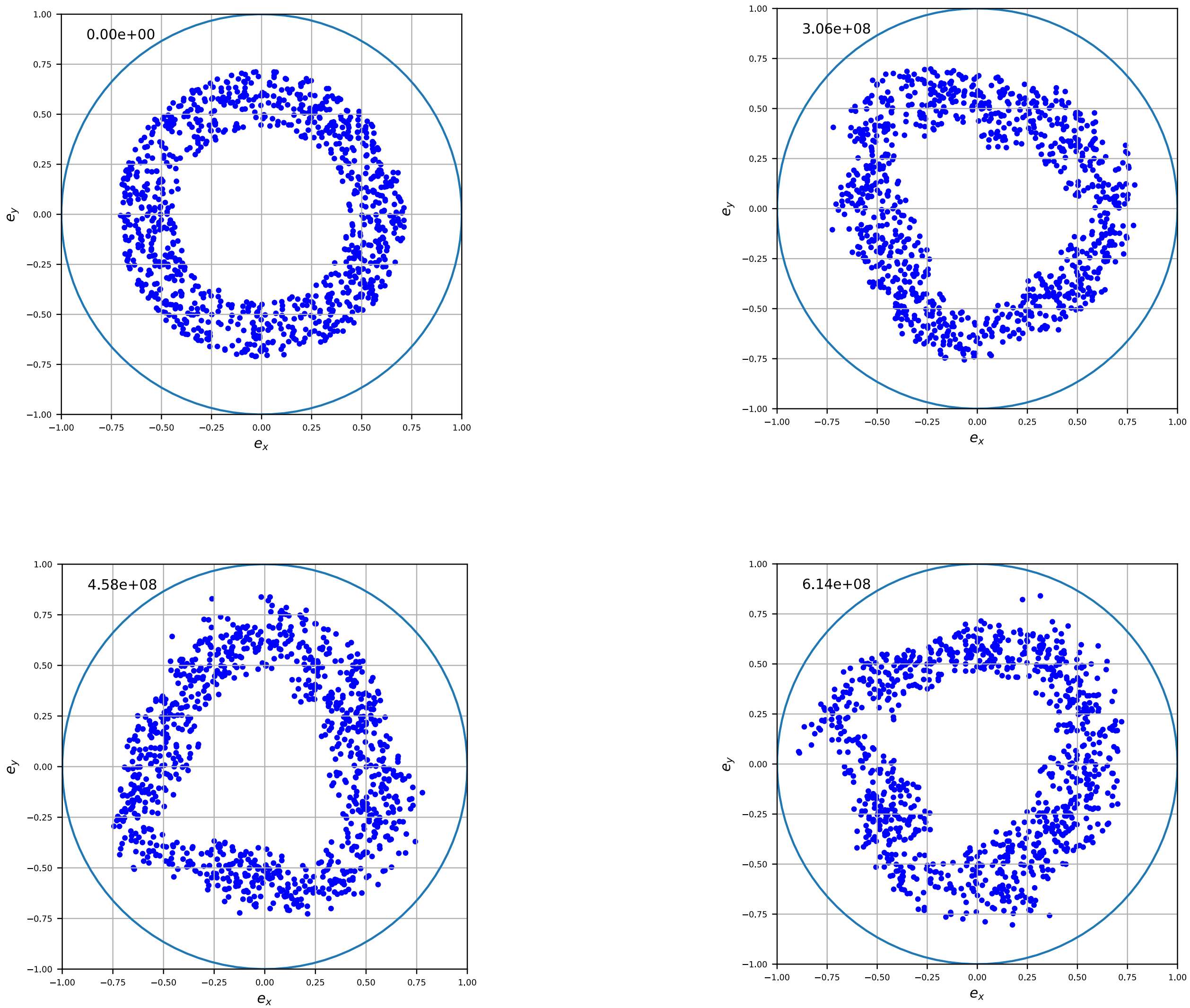



m=4 unstable



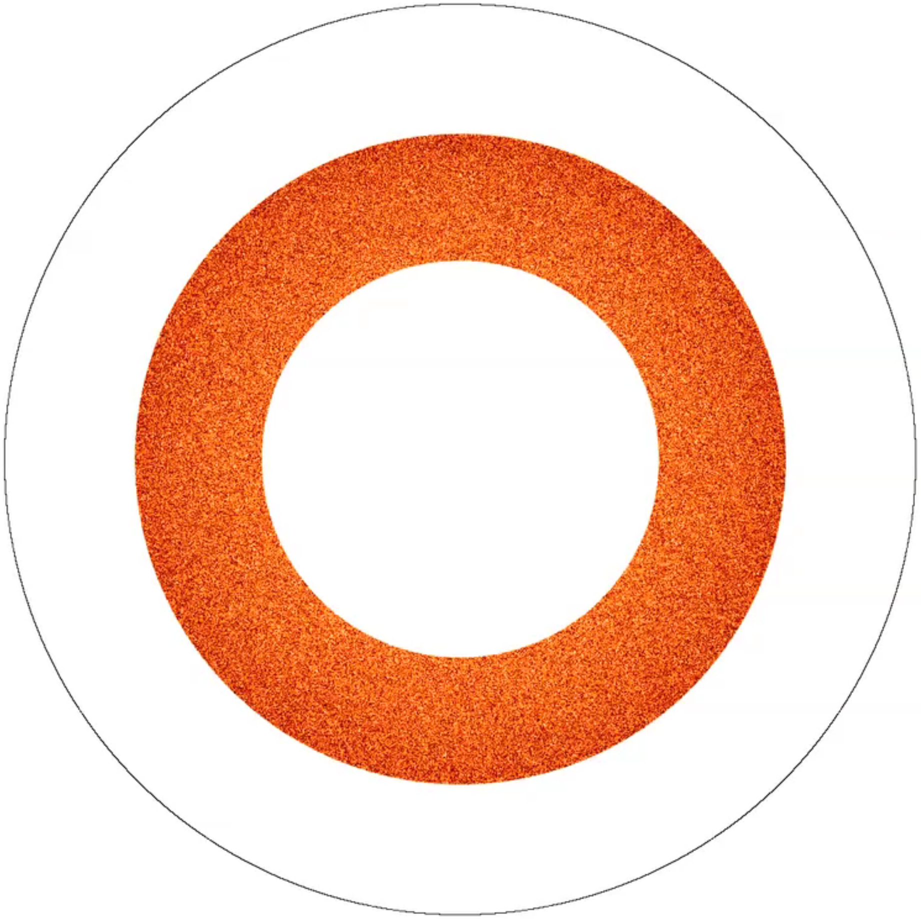

Kaur et. al. (2017, Submitted)

# Mono-Energetic Waterbags: N-Wire Simulations



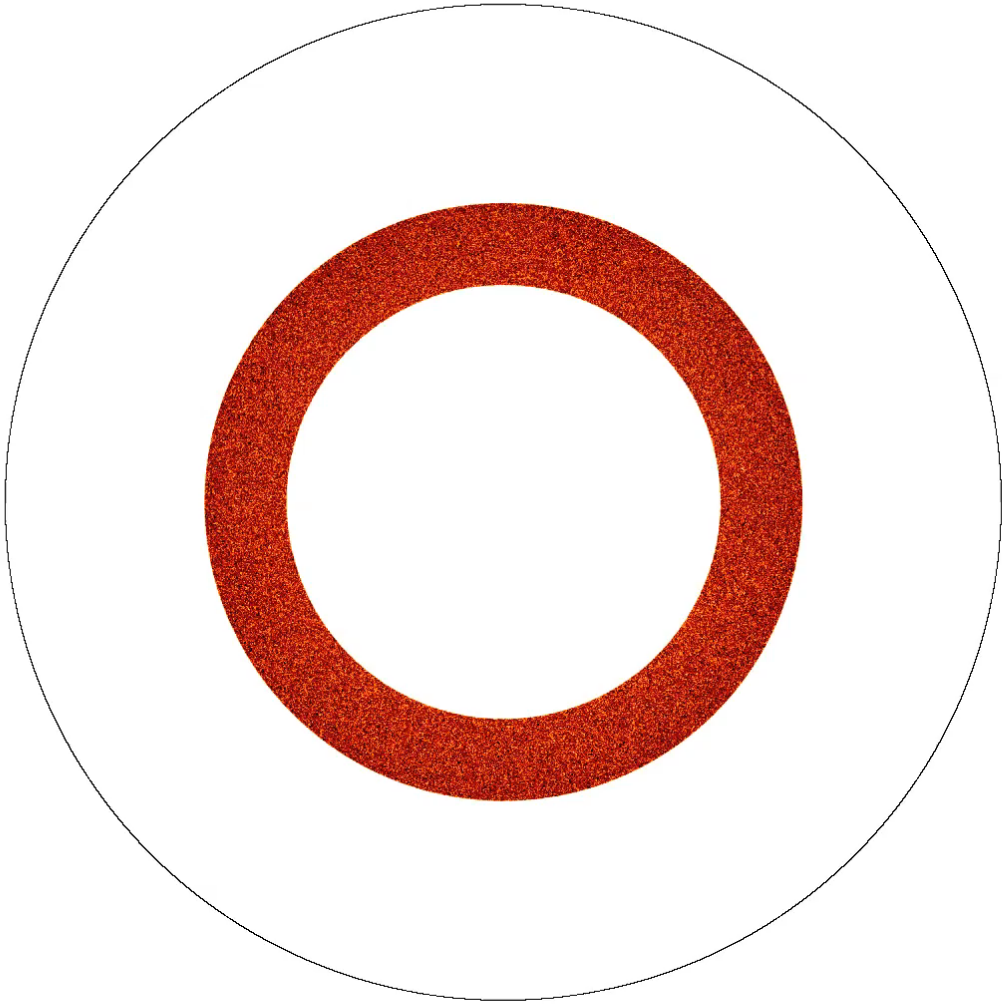

Kaur et. al (2017, Submitted)

# From N-Wire to PM: Lenz-Vecotr vortex (m=3 mode)!



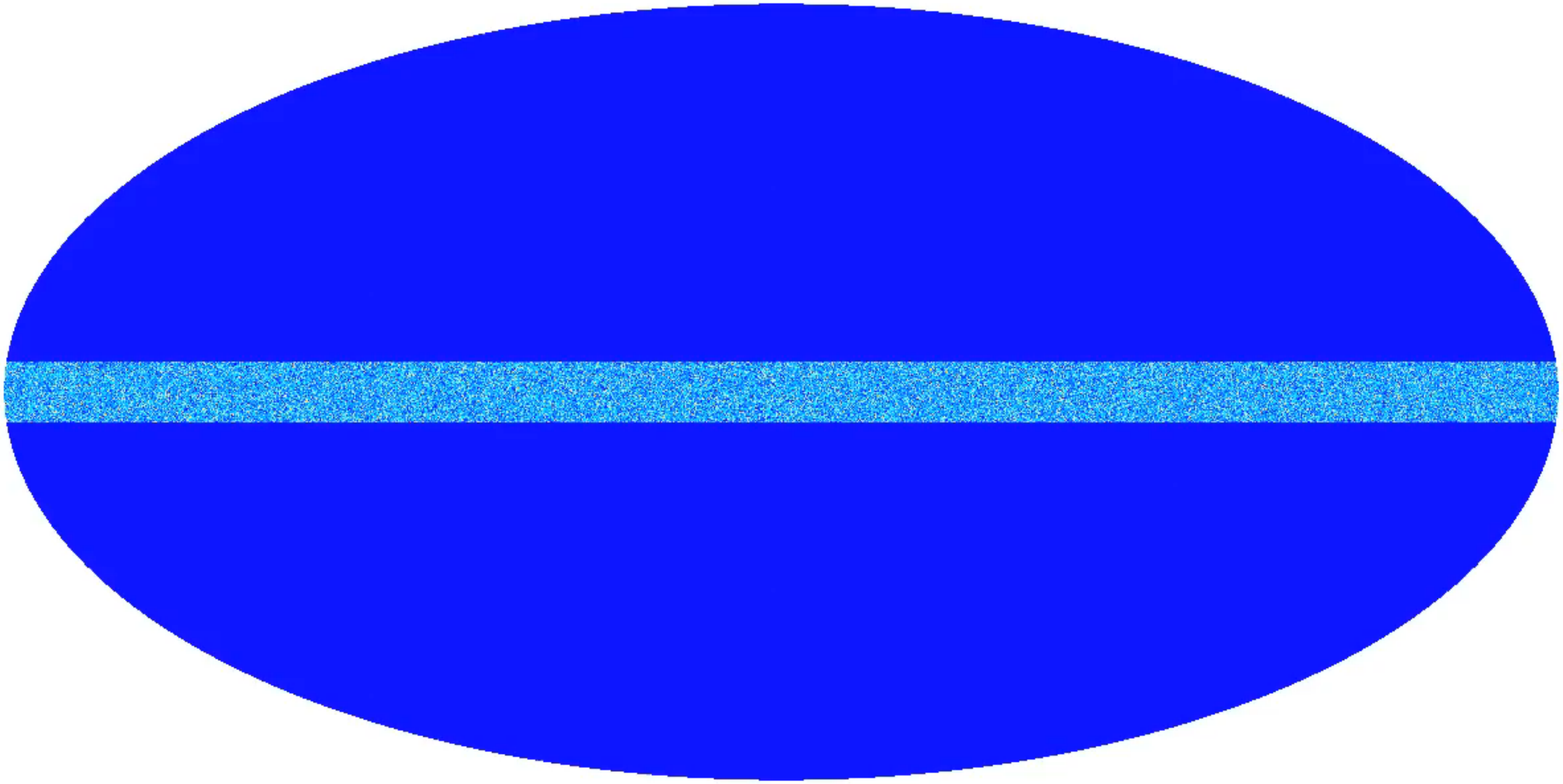
Movie: Colombi and Touma, In Preparation

# From N-Wire to Mean Field: m=4 Instability



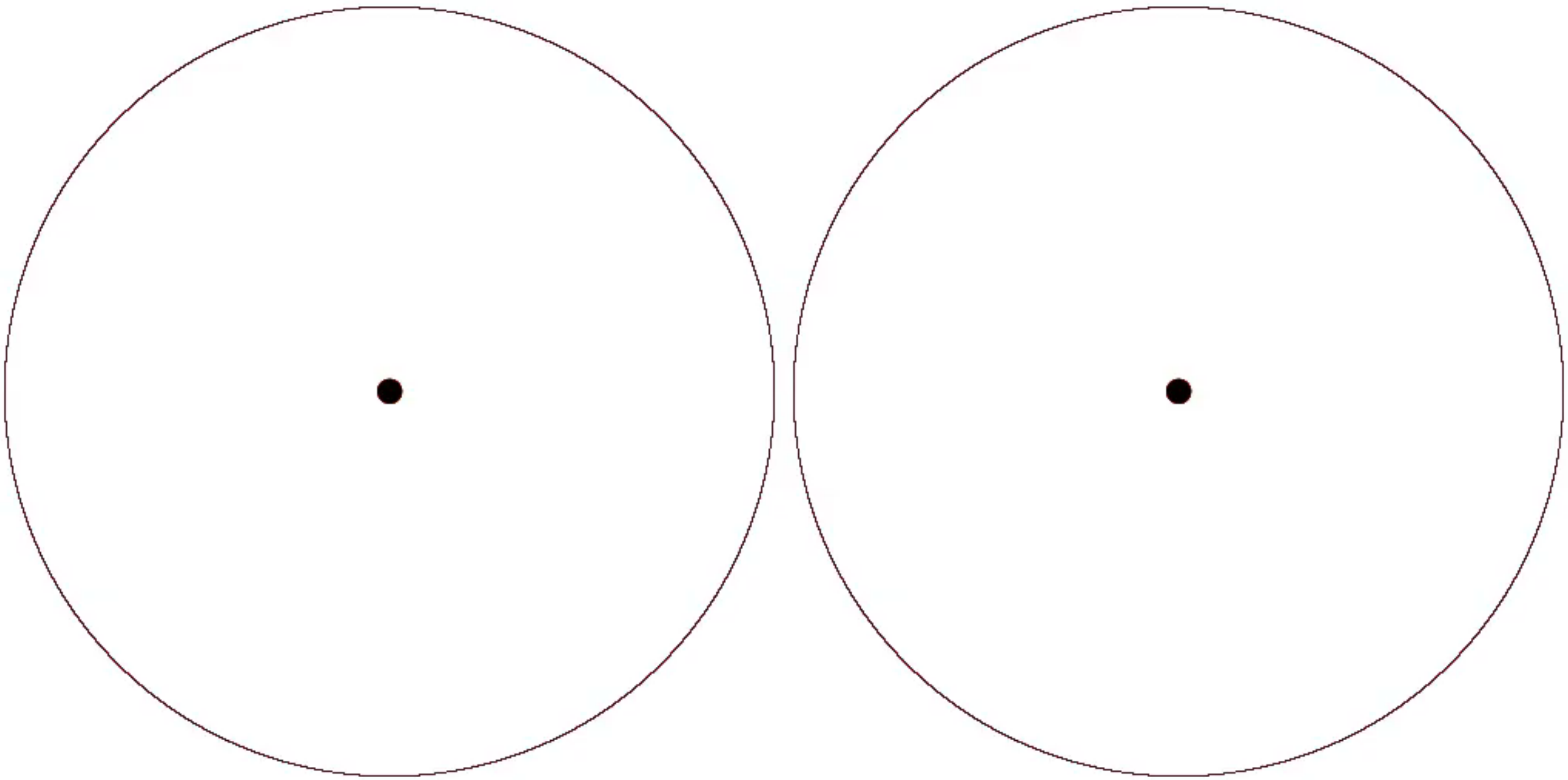
Movie: Colombi and Touma, In Preparation

# From N-Wire to Mean Field: Equatorial Band



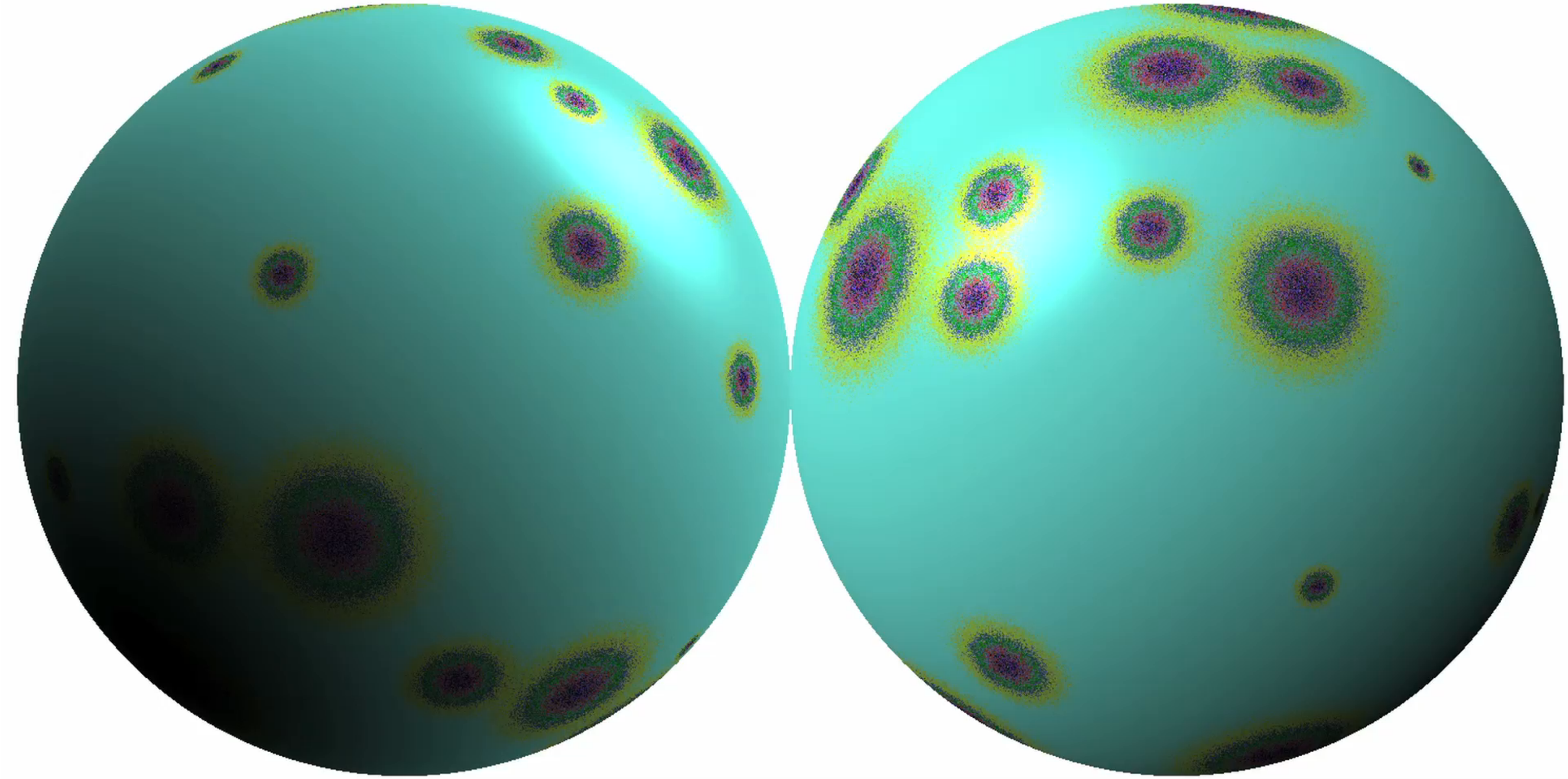
Movie: Colombi and Touma, In Preparation

# From N-Wire to Mean Field: Counter-Rotating Wires



Movie: Colombi and Touma, In Preparation

# Marbleized L-Sphere: Stephane's Magic



Movie: Colombi and Touma, In Preparation

# Conclusions (Once Again!)

- Counter-Rotating Nearly-Keplerian stellar disks are unstable: They evolve into lopsided uniformly precessing configurations [J.T. (MNRAS, 2002), Sridhar and Saini(MNRAS 2009), J.T. and Sridhar (MNRAS, 2012), Kazandjian and J.T. (MNRAS, 2013)]
- Nearly-Keplerian stellar disks (whether counter-rotating or not) are prone to violent secular instabilities: Collisionless orbit averaged Poisson-Vlasov for Waterbags [Kaur et. al (2017, Submitted), Sridhar and J.T. (MNRAS, 2016)]
- Microcanonical Thermal Equilibria of narrow, ring like, disks are, more often than not, lopsided [J.T and Tremaine (J. Phys. A, 2014)]
- First Principles Theory of "Resonant Relaxation" lays bare the kinetics of collisional relaxation onto thermal equilibria [Sridhar and J.T. (MNRAS, 2016-2017]