Singular states of the 2D Euler fluid

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Outline

Main objective: understand emergence of coherent structures in turbulence. [fluids and plasmas]

Some success: the asymptotic, stationary, highly organized flows are identified in $2D$ (where there is inverse cascade). In addition we find some reasons for the *non-existence* of coherent stationary, organized flows in $3D$.

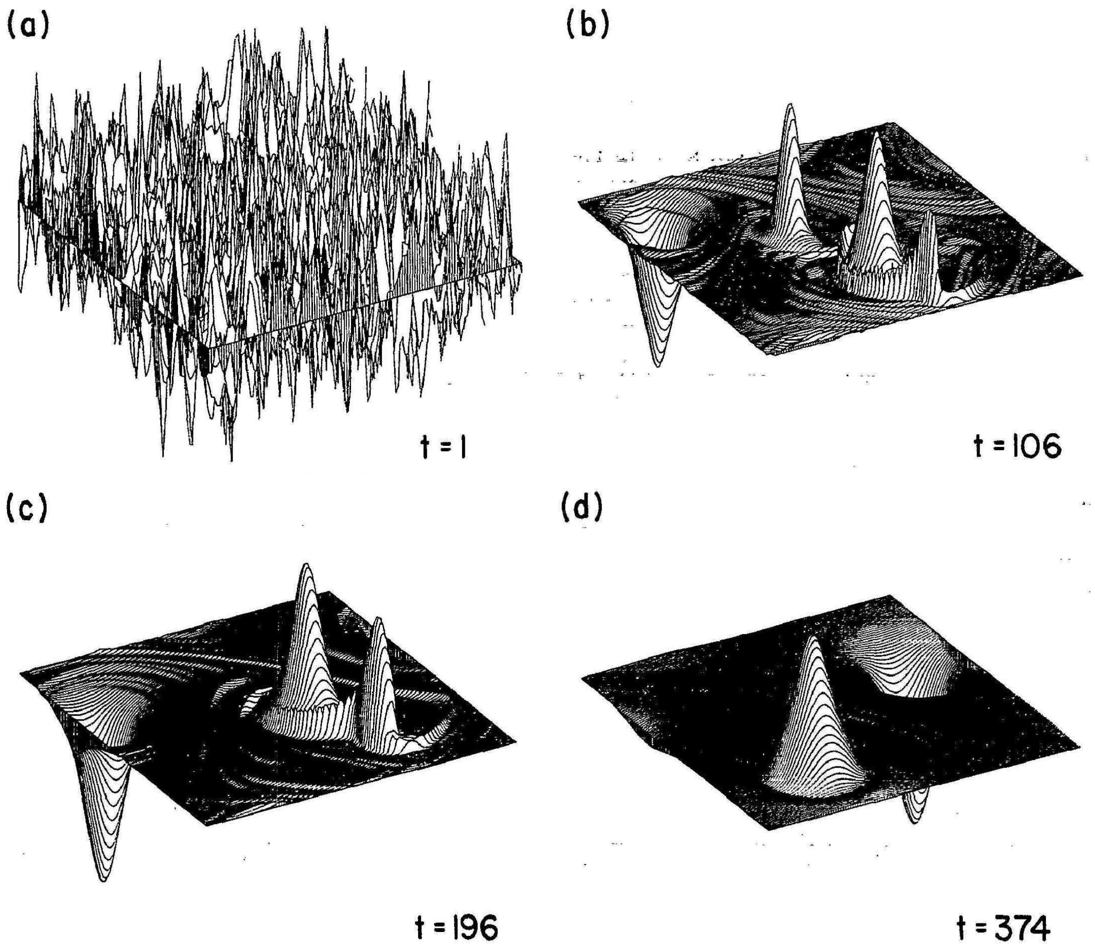
We will show that a field theoretical formulation for $2D$ fluid is able to provide such description

What happens in the evolving states, before the stationary organized flows? The emergence of structures consists of separation and clusterization of vorticity. We would like to know if there is random encounter and coalescence of vortices or if there is interaction: attraction between line-sign, - and repulsion between opposite sign vortices.

We will show that the field theoretical formulation provides an answer. A vortex acts upon the turbulent sea around by attracting elements of vorticity of like sign and repelling elements of vorticity of opposite sign

Two interesting consequences: possibility of singularity of vorticity (extreme events); spontaneous amplification of fluctuations.

Coherent structures in fluids and plasmas (numerical)



Numerical simulations of the Euler equation.

D. Montgomery, W.H. Matthaeus, D. Martinez, S. Oughton, Phys. Fluids A4 (1992) 3.

The equation for the $2D$ Euler fluid

$$\frac{d\omega}{dt} = \frac{\partial}{\partial t}\Delta\psi + \left[\left(-\nabla\psi \times \widehat{\mathbf{e}}_z\right) \cdot \nabla\right]\Delta\psi = 0. \tag{1}$$

An equivalent discrete model for the Euler equation

$$\frac{dr_k^i}{dt} = \varepsilon^{ij}\frac{\partial}{\partial r_k^j}\sum_{n=1,n\neq k}^{N}\omega_n G\left(\mathbf{r}_k - \mathbf{r}_n\right)\ ,\ i,j = 1,2\ ,\ k = 1,N \tag{2}$$

the Green function of the Laplacian

$$G\left(\mathbf{r},\mathbf{r}'\right) \approx -\frac{1}{2\pi}\ln\left(\frac{|\mathbf{r}-\mathbf{r}'|}{L}\right) \tag{3}$$

The statistical physics of the system of point-like vortices interacting in plane by a long range (Coulombian) potential has particular aspects: the phase space is finite, the temperature is negative. The discrete nature of the objects allows a combinatorial calculation of the entropy. Its extremum leads to

$$\Delta\psi + |\lambda| sinh(\psi) = 0 \tag{4}$$

The Lagrangian

of the $2D$ *Euler fluid (e.g water): Non-Abelian* $SU(2)$, *Chern-Simons,* 4^{th} *order*

$$\begin{aligned} \mathcal{L} &= -\varepsilon^{\mu\nu\rho} Tr\left(\partial_\mu A_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho\right) + \\ & iTr\left(\phi^\dagger D_0 \phi\right) - \frac{1}{2} Tr\left(\left(D_i \phi\right)^\dagger D_i \phi\right) + \frac{1}{4} Tr\left(\left[\phi^\dagger, \phi\right]\right)^2 \end{aligned} \quad (5)$$

where

$$D_\mu \phi = \partial_\mu \phi + \left[A_\mu, \phi\right]$$

The equations of motion are

$$iD_0 \phi = -\frac{1}{2} \mathbf{D}^2 \phi - \frac{1}{2}\left[\left[\phi, \phi^\dagger\right], \phi\right] \quad (6)$$

$$F_{\mu\nu} = -\frac{i}{2} \varepsilon_{\mu\nu\rho} J^\rho \quad (7)$$

The Hamiltonian density is

$$\mathcal{H} = \frac{1}{2} Tr \left((D_i \phi)^\dagger (D_i \phi) \right) - \frac{1}{4} Tr \left(\left[\phi^\dagger , \phi \right]^2 \right) \tag{8}$$

Using the notation $D_\pm \equiv D_1 \pm i D_2$

$$\begin{aligned} Tr \left((D_i \phi)^\dagger (D_i \phi) \right) &= Tr \left((D_- \phi)^\dagger (D_- \phi) \right) + \\ & \frac{1}{2} Tr \left(\phi^\dagger \left[\left[\phi , \phi^\dagger \right] , \phi \right] \right) \end{aligned}$$

Then the energy density is

$$\mathcal{H} = \frac{1}{2} Tr \left((D_- \phi)^\dagger (D_- \phi) \right) \geq 0 \tag{9}$$

and the Bogomol'nyi inequality is saturated at *self-duality*

$$D_- \phi = 0 \tag{10}$$

$$\partial_+ A_- - \partial_- A_+ + [A_+ , A_-] = \left[\phi , \phi^\dagger \right] \tag{11}$$

The *static* solutions of the *self-duality* equations

The algebraic *ansatz*:

$$\begin{aligned} [E_+, E_-] &= H \\ [H, E_\pm] &= \pm 2E_\pm \\ \mathrm{tr}\,(E_+E_-) &= 1 \\ \mathrm{tr}\left(H^2\right) &= 2 \end{aligned} \tag{12}$$

taking

$$\phi = \phi_1 E_+ + \phi_2 E_- \tag{13}$$

and

$$\begin{aligned} A_x &= \frac{1}{2}\left(-a^* + a\right) H \\ A_y &= \frac{1}{2i}\left(-a^* - a\right) H \end{aligned} \tag{14}$$

The gauge field tensor

$$F_{+-} = \left(\partial_+ a + \partial_- a^*\right) H$$

and from the first self-duality equation

$$\frac{\partial \phi_1}{\partial x} - i\frac{\partial \phi_1}{\partial y} + 2\phi_1 a = 0 \tag{15}$$

$$\frac{\partial \phi_2}{\partial x} - i\frac{\partial \phi_2}{\partial y} - 2\phi_2 a = 0 \tag{16}$$

and their complex conjugate from $(D_- \phi)^\dagger = 0$.

Notation : $\rho_1 \equiv |\phi_1|^2$, $\rho_2 \equiv |\phi_2|^2$

$$\Delta \ln(\rho_1 \rho_2) = 0 \tag{17}$$

$$\Delta \ln \rho_1 + 2(\rho_1 - \rho_1^{-1}) = 0 \tag{18}$$

We then have

$$\Delta\psi + \gamma \sinh(\beta\psi) = 0. \tag{19}$$

The miracle of *self-duality*: exclusive condition for coherent structures

Not a mapping. Suggested meaning

The **positive vortices**: (1) rotate anti-clockwise in plane: $\omega\widehat{\mathbf{e}}_z \sim \sigma$ spin is up; (2) move along the positive z axis: $\mathbf{p} = \widehat{\mathbf{e}}_z p_0$; (3) have positive chirality: $\chi = \frac{\sigma\cdot\mathbf{p}}{|\mathbf{p}|}$. The positive vortices can be represented as a point that runs along a positive helix, upward. In projection from the above the plane toward the plane we see a circle on which the point moves anti-clockwise.

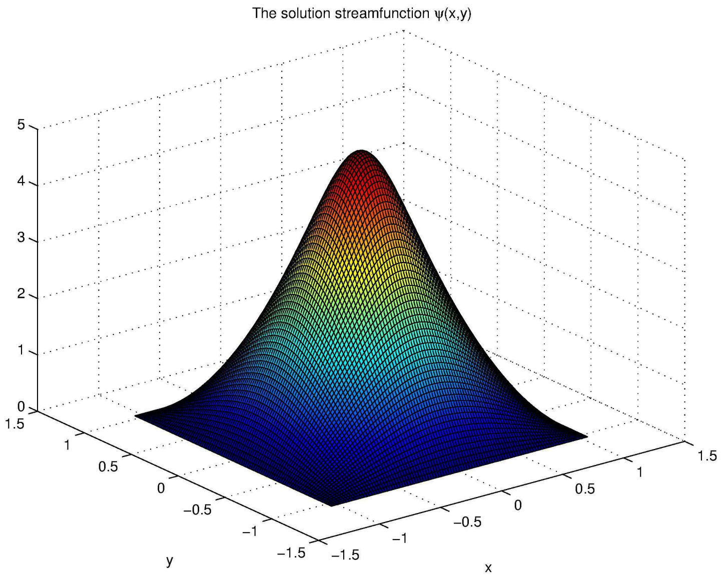
The **negative vortices**: (1) rotate clockwise in plane: $(-\omega)\,\widehat{\mathbf{e}}_z \sim -\sigma$ spin is down; (2) move along the negative z axis: $-\mathbf{p} = \widehat{\mathbf{e}}_z\,(-p_0)$, along $-z$; (3) have positive chirality: $\chi = \frac{\sigma\cdot\mathbf{p}}{|\mathbf{p}|}$. The negative vortices can be represented as a point that runs along a positive helix, the same as above, but runs downward. In projection from the above the plane toward the plane we see a circle on which the point moves clockwise.

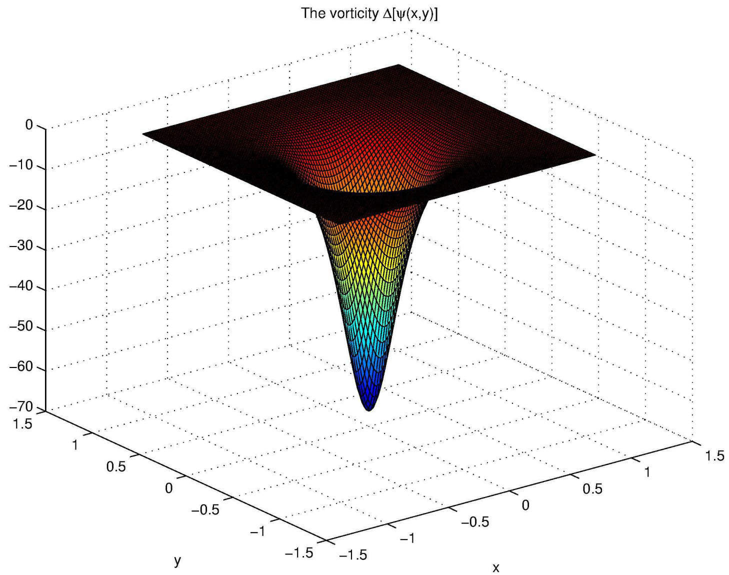
the negative vortex can be obtained from a positive vortex by reversing the direction of time

The two types of vortices behave as particles and anti-particles, having positive and respectively negative energy. They are together packed in the **mixed spinor**, $\phi \sim x^{\alpha\dot{\beta}}$.

The magnetic field is the vorticity, since

$$2\kappa F_{12} = iJ^0 = i\left[\phi,\phi^{\dagger}\right] \sim \omega \tag{20}$$

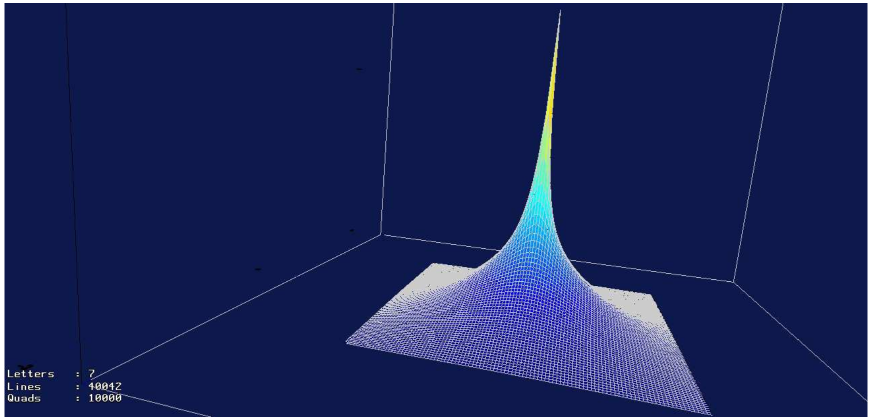




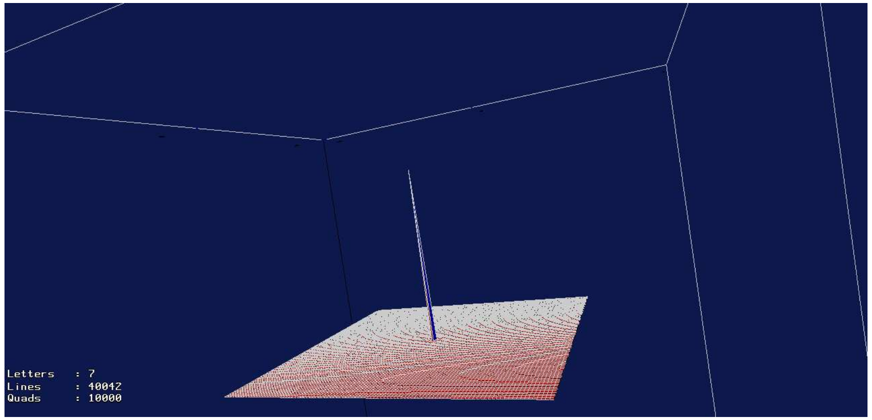
Solution streamfunction

Solution vorticity.

Solving *sinh*-Poisson eq.: strange solutions, extreme concentration of vorticity

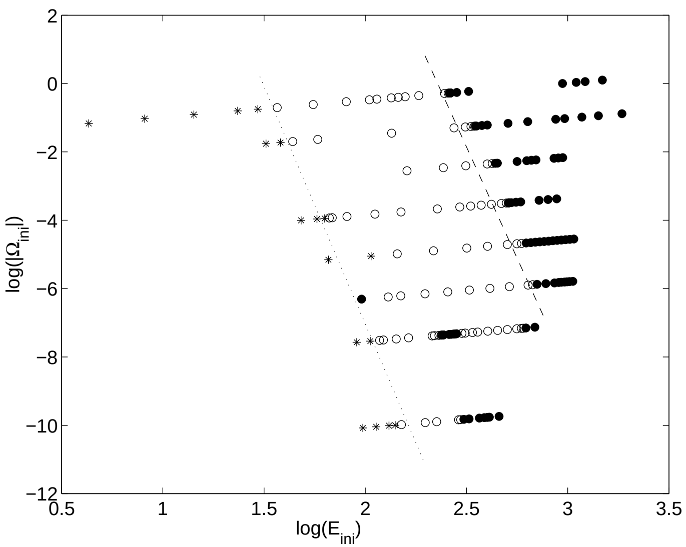


Solution streamfunction

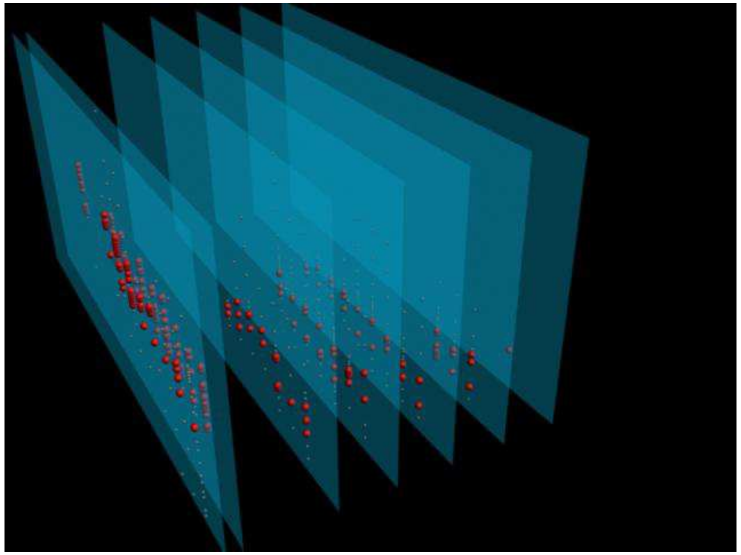


Solution vorticity.

Extreme concentration of vorticity is systematically found in numerical solutions



Regions in the space (E_{tot}, Ω_{tot}) for solutions ω : zero, regular and extreme



Red dots: regular solutions. Upper white dots: extreme ω.

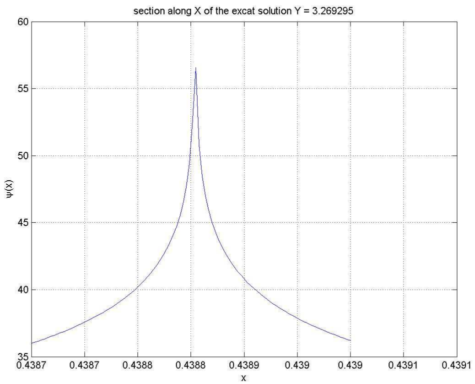
Exact analytic solution of the *sinh*-Poisson with singular vorticity

$$\psi = 4\tanh^{-1} \times \left\{ \frac{A\left[sn\left(rx,k_0\right)dn\left(ry,k_0\right)+sn\left(ry,k_0\right)dn\left(rx,k_0\right)\right]}{1+\sqrt{1-2A^2}cn\left(rx,k_0\right)cn\left(ry,k_0\right)} \right\}$$

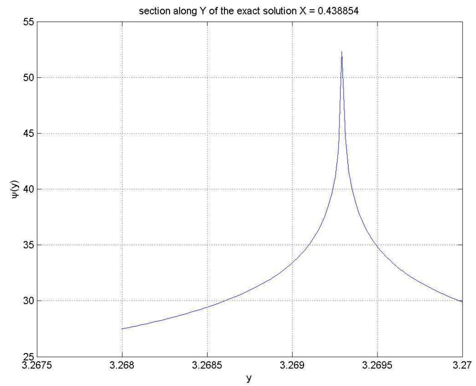
$$k_0 = \frac{1}{\sqrt{2}}\ ,\ 0 < A < \frac{1}{\sqrt{2}}$$

probably a CUSP

(Chow Gurarie, Phys. Fluids 16, 3296 (2004))



Solution streamfunction along X



Solution streamfunction along Y.

$$x = 0.438854 \quad y = 3.269395$$

Coherent structures in fluids and plasmas

Figure 1: Tornado vortex.

A comment by Chorin (in the book Vorticity and turbulence, page 80)

It is discussed the sinh-Poisson equation obtained from the combinatorial approach to the probability of distribution of point-like vortices. It is shown that for

$$\beta < -8\pi N$$

the Joyce Montgomery equation

$$\omega = -\Delta\psi = d\left[\exp\left(-\beta\psi\right) - \exp\left(\beta\psi\right)\right]$$

has no solutions with $\omega > 0$. Here β must be negative, $\beta = -\left|\beta\right|$ and

$$2d = \frac{N}{\int d^2x \exp\left(\beta\psi\right)} > 0$$

The reason is that the energy of interaction of two vortices, one fixed in origin and the other $N = 1$ is

$$E = -\frac{1}{4\pi}\ln\left(r\right)$$

the Gibbs factor is

$$\exp\left(-\frac{E}{T}\right) = \exp\left(+\beta\frac{1}{4\pi}\ln r\right) = r^{\frac{\beta}{4\pi}}$$

and the *canonical* partition function

$$Z = \int r dr d\theta \, r^{\frac{\beta}{4\pi}} = \text{const} \times r^{2+\frac{\beta}{4\pi}}$$

and for the *inverse temperature* $\beta = 1/T$

$$2 + \frac{\beta}{4\pi} < 0$$

$$\beta < -8\pi$$

we have

$$Z \to \infty \;\; \text{for} \;\; r \to 0$$

and this means that the partition function is dominated by states where $r \to 0$ *i.e.* positions of vortices very close of $r = 0$. All vortices are grouped there, which means $\omega \to \infty$. Chorin considers that this state is absurd, non-physical, since all vorticity collapses in a single point.

The problem of generation of large scale flows and coherent structures is reformulated as the problem of separation and clusterization of vorticities of opposite signs.

In particular, the late phase of organization of the flow (late phase in the emergency of coherent structures) is characterized by encounters and coalescences of like-sign (individualized) vortices.

The natural question: like-sign vortices: are-they attracting each other ? opposite-sign vortices: are-they repelling each other ?

Then we formulate the following problem: assume there is a strong, localized vortex, in a sea of low-amplitude turbulent field of vorticity. We want to find what is the effect of the vortex on the field of vorticity.

We dispose of a very powerful field theoretical framework. We found that it is able to give a response to this problem:

indeed a large, fixed vortex (1) attracts elements of vorticity of the same sign and (2) repels the elements of vorticity of opposite sign, - from the turbulent vorticity field around it.

To show this we have to place more emphasis on the *fermion content* of our field theoretical model.

We then start by showing that the general FT formulation contains a system of fermions interacting with an external magnetic field.

The FT matter field is a mixed spinor

$$\phi \sim x^{\alpha\dot{\beta}} = \phi_1 E_+ + \phi_2 E_- = \begin{pmatrix} 0 & \phi_1 \\ \phi_2 & 0 \end{pmatrix}$$

The equation of Self-Duality derived in Field Theory is $D_-\phi = 0$, or

$$\frac{\partial \phi_1}{\partial z} + a\phi_1 = 0 \text{ and } \frac{\partial \phi_2}{\partial z} - a\phi_2 = 0$$

The algebraic ansatz for the potential in FT $A_- = aH$.

The equation for the fermion fields, in external gauge potential $\mathbf{A}^f(\mathbf{x})$, at zero energy, is

$$\alpha \cdot \left(\mathbf{p} - e\mathbf{A}^f\right)\phi^f_{E=0} = 0$$

The matrices are $\alpha^1 = -\sigma^2$ and $\alpha^2 = \sigma^1$ and the column matrix $\phi^f = \begin{pmatrix} u \\ v \end{pmatrix}$. On components

$$\frac{\partial u}{\partial z} - ieA^f_- u = 0 \text{ and } -\frac{\partial v}{\partial \overline{z}} + ieA^f_+ v = 0$$

where $A_{\pm}^{f}$ are scalar function.

The two sets of equations become identical if

$$\begin{aligned} u &\rightarrow \phi_1 \\ v &\rightarrow \phi_2^{*} \\ A_{-}^{f} &\rightarrow -\frac{1}{ie}a \\ A_{+}^{f} &\rightarrow \frac{1}{ie}a^{*} \end{aligned}$$

Check:

We can map the two sets equations of fermions and the equations of FT.

The first Jackiw equation becomes after replacements

$$\frac{\partial u}{\partial z} - ieA_{-}^{f}u = 0 \rightarrow \frac{\partial \phi_1}{\partial z} + a\phi_1 = 0$$

The second Jackiw equation becomes

$$-\frac{\partial v}{\partial \overline{z}} + ieA_{+}^{f} v = 0 \rightarrow -\frac{\partial \phi_2^*}{\partial \overline{z}} + a^* \phi_2^* = 0$$

and after taking the conjugate

$$\frac{\partial \phi_2}{\partial z} - a\phi_2 = 0$$

This shows that we have the correct mapping.

Moreover, starting from the fermion model we can show that the gauge field is governed by a the Chern-Simons (Redlich) which can be seen as another confirmation of the two description.

Finally, in the system of fermions, the axial current exhibits *axial anomaly* and it leads to the result that the charge of the fermion (the axial anomaly) is equal with the local value of the vorticity.

The suggested conclusion is that the Field Theoretical model (Schrodinger, CS, 4-degree nonlinear self-interaction) is locally equivalent with a system of massless fermions in interaction with an external gauge field.

The two theories lead to the same results for local problem: *i.e.* localized ”external” magnetic field , which means a local $2D$ hump of vorticity, equivalently, a section of a string of vorticity perpendicular on plane.

Then we can use as much as possible the second, fermionic, model.

New properties become visible.

The effect of a strong, localized vortex on the vorticity field around

(equivalently: the effect of a tube of magnetic field on the fermion current)

We start from the gauged Dirac equation

$$\left(i\partial\!\!\!/ - eA\!\!\!/ - m\right)\psi = 0$$

and multiply

$$\left(i\partial\!\!\!/ - eA\!\!\!/ + m\right)\left(i\partial\!\!\!/ - eA\!\!\!/ - m\right)\psi = 0$$

with the result

$$\left[\left(i\partial_\mu - eA_\mu\right)^2 - \frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu} - m^2\right]\psi = 0$$

We note the occurrence of

$$\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]$$

which for the *magnetic field*

$$F_{12}$$

becomes

$$\sigma^{12} = \frac{i}{2}\left(\gamma^1\gamma^2 - \gamma^2\gamma^1\right) = \sigma_3$$

Then the term with the magnetic field is

$$\sigma_3 B$$

We know that this term will multiply

$$\phi = \left(\begin{array}{c} \Phi(x) \\ 0 \end{array} \right) \text{ for } E > 0$$

and

$$\phi = \left(\begin{array}{c} 0 \\ \chi(x) \end{array} \right) \text{ for } E < 0$$

and will give

$$\sigma_3 B \left(\begin{array}{c} \Phi(x) \\ 0 \end{array} \right) = \left(\begin{array}{c} B\Phi(x) \\ 0 \end{array} \right)$$

and respectively

$$\sigma_3 B \left(\begin{array}{c} 0 \\ \chi(x) \end{array} \right) = \left(\begin{array}{c} 0 \\ -B\chi(x) \end{array} \right)$$

Therefore the sign of this term is opposite for $E > 0$ and respectively for $E < 0$.

We conclude that

1. *the states with* $E < 0$ *will be REPELLED from the magnetic field*
2. *the states with* $E > 0$ *will be ATTRACTED by the tube of magnetic flux*

Since these are negative vorticity and respectively positive vorticity, we have that a tube of vorticity (a vortex) induces separation of the vorticities and, by attraction and repulsion, clusterization of like sign vorticity.

Conclusions

To our kowledge this is the direct explanation of the separation and clusterization of the like-sign vorticity elements in turbulence, a basic process in the formation of large scale coherent structures.

The possible consequences can be important: a fluctuation of the vorticity field, realized like a localized, individualized, vortex, will start to collect the like-sign vorticity and grow. This means that flow configurations that analytically are assumed to be stable may actually be spontaneously unstable, possibly algebraically and at a low rate.

Extreme concentration of vorticity appears to have a strong support.

System of interacting particles in plane

A system of particles in the plane interacting through a potential. The Hamiltonian is

$$H = \sum_{s=1}^{N} \frac{1}{2} m_s \mathbf{v}_s^2$$

where

$$m_s \mathbf{v}_s = \mathbf{p}_s - e_s \mathbf{A} \left(\mathbf{r}_s | \mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N \right)$$

the potential at the point $\mathbf{r}_s$

$$\mathbf{A} \left(\mathbf{r}_s | \mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N \right) \equiv \left(a_s^i \left(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N \right) \right)_{i=1,2}$$

$$a_s^i \left(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N \right) = \frac{1}{2\pi\kappa} \varepsilon^{ij} \sum_{q \neq s}^{N} e_q \frac{r_s^j - r_q^j}{\left| \mathbf{r}_s - \mathbf{r}_q \right|^2}$$

The vector potential $\mathbf{A}_s$ is the $curl$ of the Green function of the Laplacian

$$\frac{1}{2\pi} \varepsilon^{ij} \frac{r^j}{r^2} = \varepsilon^{ij} \partial_j \frac{1}{2\pi} \ln r \qquad \nabla^2 \frac{1}{2\pi} \ln r = \delta^2 (r)$$

The continuum limit is a classical field theory

- separate the matter degrees of freedom
- Consider the interaction potential as a *free* field $=$ new degree of freedom of the system, and find the Lagrangian which can give this potential.
- Couple the matter and the field by an interaction term in the Lagrangian

According to Jackiw and Pi the field theory Lagrangian

$$L = L_{matter} + L_{CS} + L_{interaction}$$

with

$$L_{matter} = \sum_{s=1}^{N} \frac{1}{2} m_s \mathbf{v}_s^2$$

The Chern-Simons part of the Lagrangian

$$\begin{aligned} L_{CS} &= \frac{\kappa}{2} \int d^2r \; \varepsilon^{\alpha\beta\gamma} \partial_\alpha A_\beta A_\gamma \\ &= \frac{\kappa}{2} \int d^2r \; \frac{\partial \mathbf{A}}{\partial t} \times \mathbf{A} - \int d^2r \; A^0 B \end{aligned}$$

where

$$x^{\mu} = (ct, \mathbf{r})$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = -\nabla A^0 - \frac{\partial \mathbf{A}}{\partial t}$$

The interaction Lagrangian is

$$L_{int} = \sum_{s=1}^{N} e_s \mathbf{v}_s \cdot \mathbf{A}(t, \mathbf{r}_s) - \sum_{s=1}^{N} e_s A^0 (t, \mathbf{r}_s)$$

Define the current

$$v^{\mu} = (c, \mathbf{v}_s)$$

$$j^{\mu}(t, \mathbf{r}) = \sum_{s=1}^{N} e_s v_s^{\mu} \delta(\mathbf{r} - \mathbf{r}_s)$$

the interaction Lagrangian can be written

$$\begin{aligned} L_{int} &= -\int d^2r\ A_\mu j^\mu \\ &= \int d^2r\ \mathbf{A}\cdot\mathbf{j} - \int d^2r\ A^0\rho \end{aligned}$$

The current at the continuum limit

$$j^\mu = (\rho, \mathbf{j})$$

with

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{j} = 0$$

Two steps to get the Hamiltonian form

1. *Eliminate the gauge-field variables in favor of the matter variables, by using the gauge-field equations of motion*.

The **equations of motion of the gauge field** are

$$\frac{\kappa}{2}\varepsilon^{\alpha\beta\gamma}F_{\alpha\beta} = j^{\mu} \tag{21}$$

$$B = -\frac{1}{\kappa}\rho$$

$$E^{i} = \frac{1}{\kappa}\varepsilon^{ij}j^{j}$$

2. *Define the canonical momenta.*

But not yet.

It is time to find the field that will represent the continuum limit of the density of discrete points

The right choice : a complex scalar field Φ.

Remember now that the momentum is the generator of the space translations which means that it has the form : $\partial/\partial x$.

(No subversive quantum activities)

Define the momenta as **covariant derivatives**

$$\begin{aligned}\mathbf{\Pi}\left(\mathbf{r}\right) &\equiv \left[\nabla - ie\mathbf{A}\left(\mathbf{r}\right)\right]\Psi\left(\mathbf{r}\right)\\ &= \mathbf{D}\Psi\left(\mathbf{r}\right)\end{aligned}$$

and the conjugate

$$\mathbf{\Pi}^{\dagger} \equiv \left(\mathbf{D}\Psi\right)^{\dagger}$$

The number density operator is

$$\rho = \Psi^{\dagger}\Psi$$

The **potential** $\mathbf{A}\left(\mathbf{r}\right)$ is constructed such as to solve the Chern-Simons relation between the field

$\mathbf{B} = \nabla \times \mathbf{A}$ and the charge density $e\rho$:

$$B = -\frac{e}{\kappa}\rho$$

The **potential** is then

$$\mathbf{A}(\mathbf{r}) = \nabla \times \frac{e}{\kappa} \int d^2 r' \; \mathbf{G}(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}')$$

where $\mathbf{G}(\mathbf{r} - \mathbf{r}')$ is the Green function of the Laplaceian in plane. The $curl$ of the Green function is

$$\nabla \times \mathbf{G}(\mathbf{r} - \mathbf{r}') = -\frac{1}{2\pi} \nabla \theta (\mathbf{r} - \mathbf{r}')$$

where

$$\tan \theta (\mathbf{r} - \mathbf{r}') = \frac{y - y'}{x - x'}$$

and θ is multivalued.

The Hamiltonian

$$H = \int d^2 r \; \mathcal{H}$$

is

$$\mathcal{H} = \frac{1}{2m} \left(\mathbf{D}\Psi\right)^* \left(\mathbf{D}\Psi\right) - \frac{g}{2} \left(\Psi^* \Psi\right)^2$$

with the equation of motion

$$i \frac{\partial \Psi\left(\mathbf{r},t\right)}{\partial t} = -\frac{1}{2m} \mathbf{D}^2 \Psi\left(\mathbf{r},t\right) + eA^0\left(\mathbf{r},t\right) - g\rho\left(\mathbf{r},t\right)\Psi\left(\mathbf{r},t\right) \tag{22}$$

The potential is related to the density ρ and to the current $\mathbf{j}$:

$$\mathbf{A}\left(\mathbf{r},t\right) = \nabla \times \frac{e}{\kappa} \int d^2 r \; \mathbf{G}\left(\mathbf{r}-\mathbf{r}'\right) \rho\left(\mathbf{r}',t\right) + \text{ gauge term}$$

$$A^0\left(\mathbf{r},t\right) = -\nabla \times \frac{e}{\kappa} \int d^2 r \; \mathbf{G}\left(\mathbf{r}-\mathbf{r}'\right) \mathbf{j}\left(\mathbf{r}',t\right) + \text{ gauge term}$$

Write Ψ as amplitude and phase $\Psi = \rho^{1/2} \exp\left(ie\chi\right)$ and inserting this expression into the equation of motion derived from the Hamiltonian the imaginary part gives the **equation of continuity**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

and the real part gives:

$$\begin{aligned} \nabla^2 \ln \rho &= 4m \left(eA^0 - g\rho \right) \\ &+2 \left(e\mathbf{A} - \frac{1}{2} \nabla \times \ln \rho \right) \left(e\mathbf{A} + \frac{1}{2} \nabla \times \ln \rho \right) \end{aligned}$$

The static self-dual solutions

All starts from the identity (Bogomolnyi)

$$\left|\mathbf{D}\Psi\right|^2 = \left|\left(D_1 \pm iD_2\right)\Psi\right|^2 \pm m\nabla\times\mathbf{j} \pm eB\rho$$

Then the *energy density* is

$$H = \frac{1}{2m}\left|\left(D_1 \pm iD_2\right)\Psi\right|^2 \pm \frac{1}{2}\nabla\times\mathbf{j} - \left(\frac{g}{2} \pm \frac{e^2}{2m\kappa}\right)\rho^2$$

Taking the particular relation

$$g = \mp\frac{e^2}{m\kappa}$$

and considering that the space integral of $\nabla\times\mathbf{j}$ vanishes,

$$H = \frac{1}{2m}\int d^2r\ \left|\left(D_1 \pm iD_2\right)\Psi\right|^2$$

This is non-negative and attains its minimum, zero, when Ψ satisfies

$$D_1\Psi \pm iD_2\Psi = 0$$

or

$$\mathbf{D}\Psi = i\mathbf{D}\times\Psi$$

which is the self-duality condition.

Then decomposing again Ψ in the phase and amplitude parts,

$$\mathbf{A} = \nabla\chi \pm \frac{1}{2e}\nabla\times \ln\rho$$

Introducing in the relation derived from Chern-Simons

$$B = \nabla \times \mathbf{A} = -\frac{e}{\kappa}\rho$$

we have

$$\nabla^2 \ln\rho = \pm 2\frac{e^2}{\kappa}\rho$$

which is the Liouville equation.

The Lagrangian of $2D$ *plasma in strong magnetic field: Non-Abelian* $SU(2)$*, Chern-Simons,* 6^{th} *order*

- gauge field, with "potential" A^{μ}, ($\mu = 0, 1, 2$ for (t, x, y)) described by the Chern-Simons Lagrangean;
- matter ("Higgs" or "scalar") field ϕ described by the covariant kinematic Lagrangean (*i.e.* covariant derivatives, implementing the minimal coupling of the gauge and matter fields)
- matter-field self-interaction given by a potential $V\left(\phi, \phi^{\dagger}\right)$ with 6^{th} power of ϕ;
- the matter and gauge fields belong to the *adjoint* representation of the algebra $SU(2)$

$$\begin{aligned}\mathcal{L} &= -\kappa\varepsilon^{\mu\nu\rho}\mathrm{tr}\left(\partial_\mu A_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right) \\ &\quad -\mathrm{tr}\left[\left(D^\mu\phi\right)^\dagger\left(D_\mu\phi\right)\right] \\ &\quad -V\left(\phi,\phi^\dagger\right)\end{aligned} \tag{23}$$

Sixth order potential

$$V\left(\phi,\phi^\dagger\right) = \frac{1}{4\kappa^2}\mathrm{tr}\left[\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)^\dagger\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)\right]. \tag{24}$$

The Euler Lagrange equations are

$$D_\mu D^\mu\phi = \frac{\partial V}{\partial\phi^\dagger} \tag{25}$$

$$-\kappa\varepsilon^{\nu\mu\rho}F_{\mu\rho} = iJ^\nu \tag{26}$$

The energy can be written as a sum of squares. The *self-duality* eqs.

$$\begin{aligned} D_{-}\phi &= 0 \\ F_{+-} &= \pm\frac{1}{\kappa^2}\left[v^2\phi - \left[\left[\phi,\phi^\dagger\right],\phi\right],\phi^\dagger\right] \end{aligned} \quad (27)$$

The algebraic *ansatz* : in the Chevalley basis

$$\begin{aligned} \left[E_+,E_-\right] &= H \\ \left[H,E_\pm\right] &= \pm 2E_\pm \\ \mathrm{tr}\left(E_+E_-\right) &= 1 \\ \mathrm{tr}\left(H^2\right) &= 2 \end{aligned} \quad (28)$$

The fields

$$\phi = \phi_1 E_+ + \phi_2 E_-$$

$$A_+ = aH, A_- = -a^* H$$

Equations for the components of the density of vorticity (here for $'+'$)

$$-\frac{1}{2}\Delta \ln \rho_1 = -\frac{1}{\kappa^2}\left(\rho_1 - \rho_2\right)\left[2\left(\rho_1 + \rho_2\right) - v^2\right] \tag{29}$$

$$-\frac{1}{2}\Delta \ln \rho_2 = \frac{1}{\kappa^2}\left(\rho_1 - \rho_2\right)\left[2\left(\rho_1 + \rho_2\right) - v^2\right] \tag{30}$$

$$\Delta \ln\left(\rho_1 \rho_2\right) = 0$$

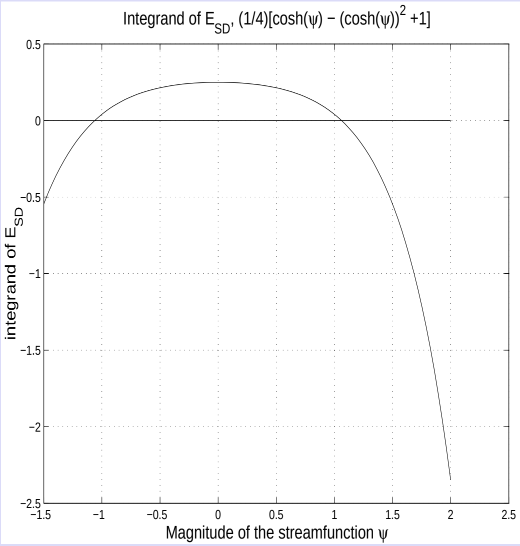
introduce a single variable

$$\rho \equiv \frac{\rho_1}{v^2/4} = \frac{v^2/4}{\rho_2} \tag{31}$$

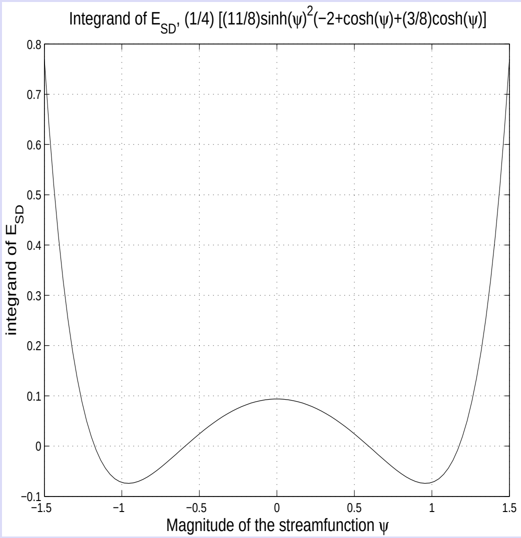
and obtain

$$-\frac{1}{2}\Delta \ln \rho = -\frac{1}{4}\left(\frac{v^2}{\kappa}\right)^2\left(\rho - \frac{1}{\rho}\right)\left[\frac{1}{2}\left(\rho + \frac{1}{\rho}\right) - 1\right] \tag{32}$$

The energy at Self-Duality for two choices of the Bogomolnyi form for the action functional



$$\Delta\psi - \sinh\psi\,(\cosh\psi - 1) = 0$$



$$\Delta\psi + \frac{1}{2}\sinh\psi\,(\cosh\psi - 1) = 0$$

This simplest form of the equation governing the stationary states of the CHM eq.

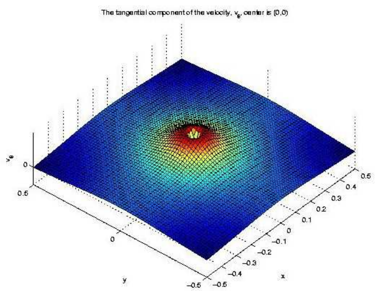
$$\Delta\psi + \frac{1}{2}\sinh\psi\,(\cosh\psi - 1) = 0$$

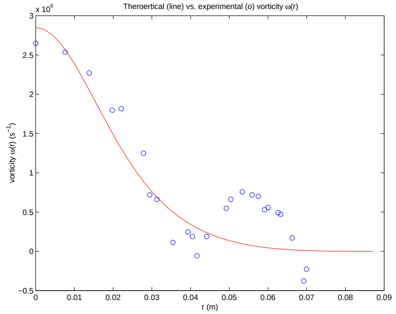
The 'mass of the photon' is

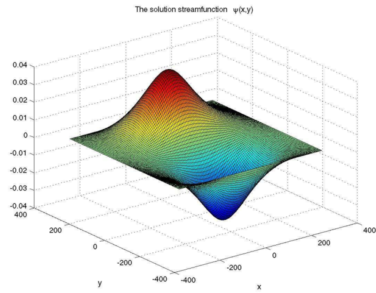
$$m = \frac{v^2}{\kappa} = \frac{1}{\rho_s}$$

$$\kappa \equiv c_s$$
$$v^2 \equiv \Omega_{ci}$$

Various applications







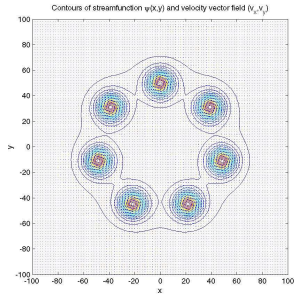
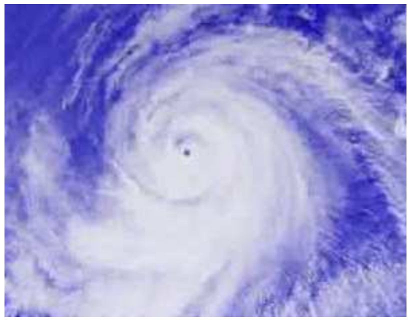
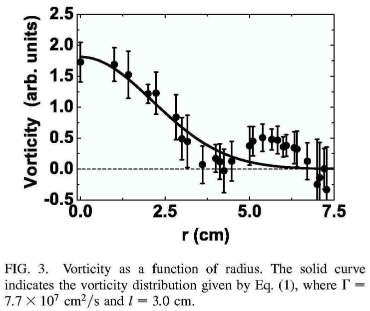
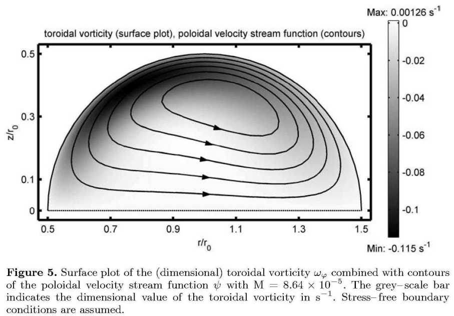
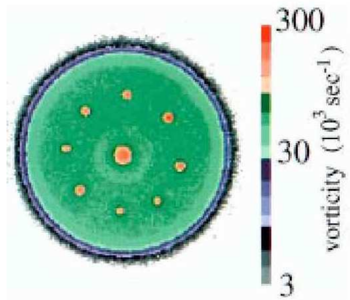


Figure 2: The atmospheric vortex, the plasma vortex, the flows in tokamak,the crystal of vortices in non-neutral plasma.







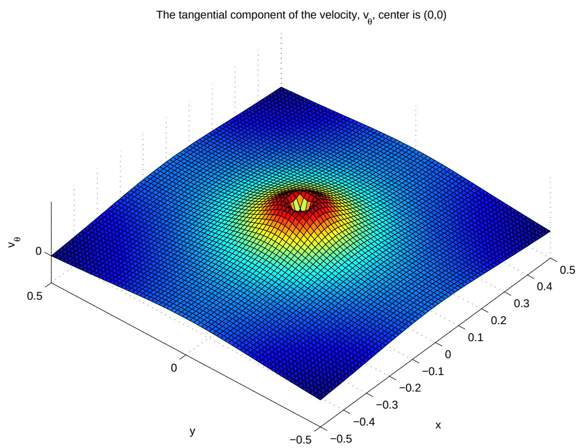


The tropical cyclone

Figure 3: The tangential component of the velocity, $v_\theta(x, y)$

This is an atmospheric vortex.

The tropical cyclone , comparisons



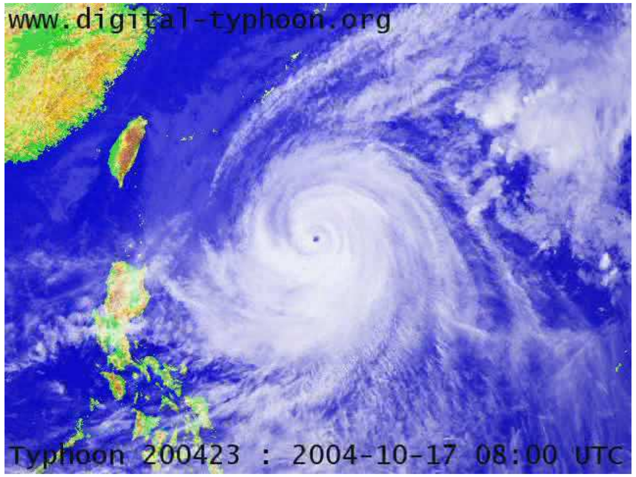
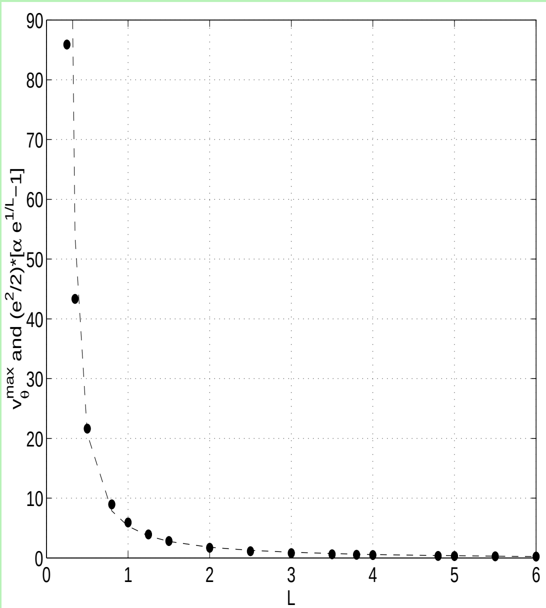


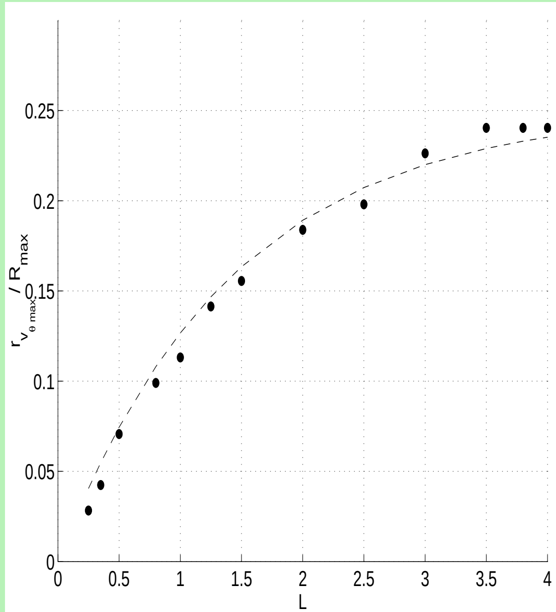
Figure 4: The solution and the image from a satelite.

The solution reproduces the *eye* radius, the radial extension and the vorticity magnitude.

Scaling relationships between main parameters of the tropical cyclone eye-wall radius, maximum tangential wind, maximum radial extension

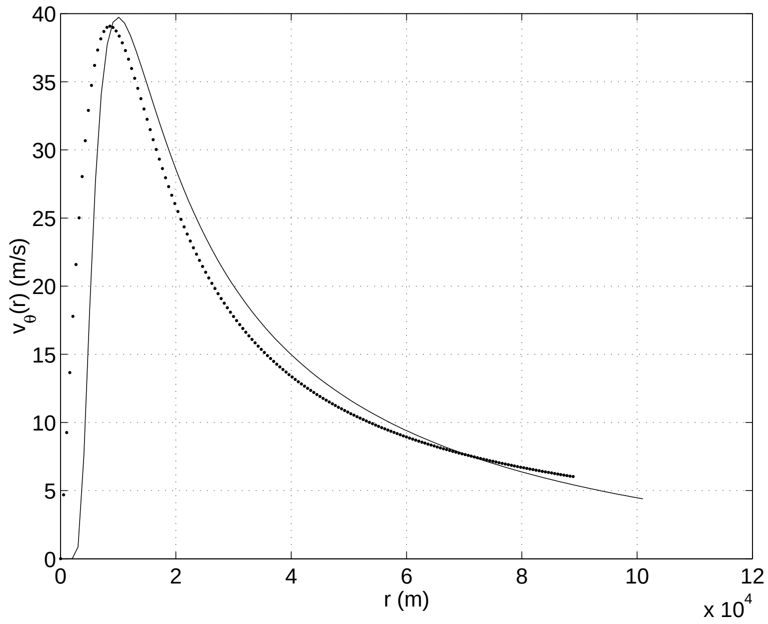


$$v_\theta^{\max}(L) \simeq \frac{e^2}{2}\left[\alpha \exp\left(\frac{\sqrt{2}}{R_{\max}}\right) - 1\right]$$



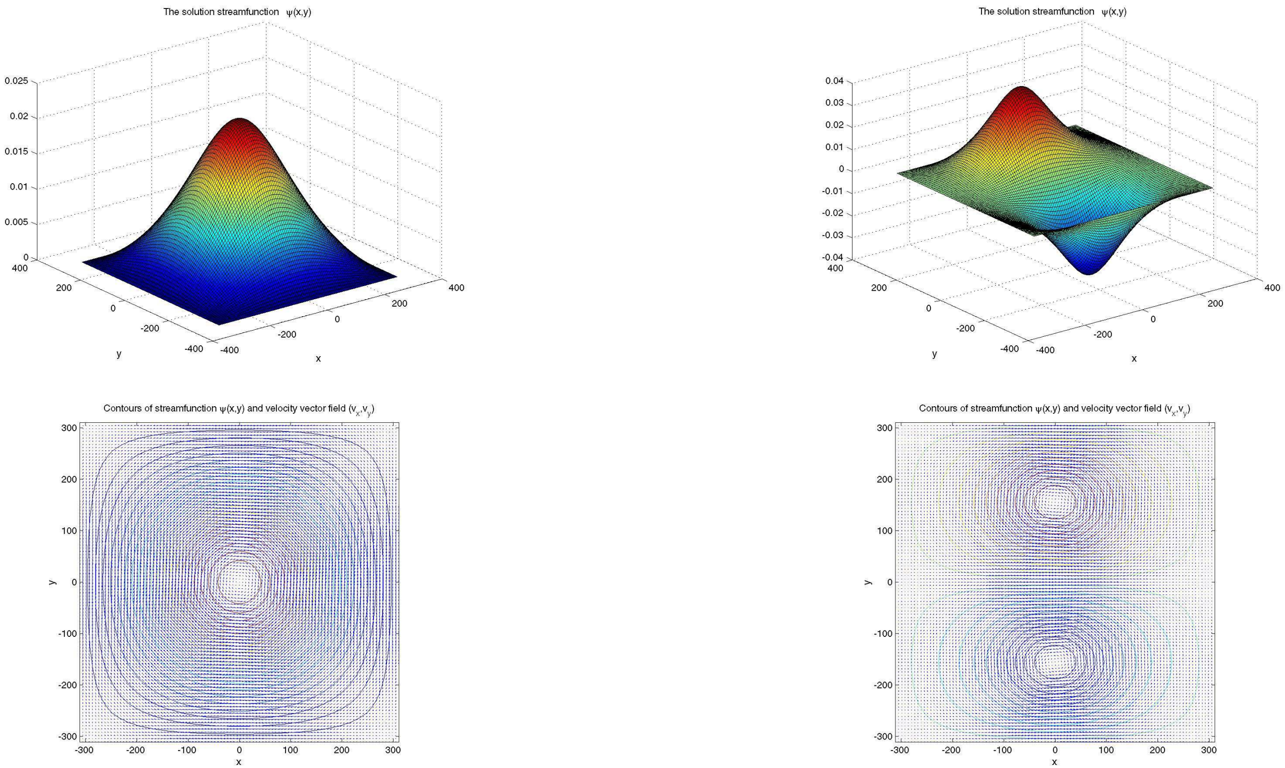
$$\frac{r_{v_\theta max}}{R_{max}} = \frac{1}{4}\left[1 - \exp\left(-\frac{R_{max}}{2}\right)\right]$$

Profile of the azimuthal wind velocity $v_\theta\left(r\right)$



Comparison between the Holland's empirical model for v_θ (continuous line) and our result (dotted line).

Tokamak plasma. Solution for $L = 307$: mono- and multipolar vortex



The plasma vortex : comparison of our results with the experiment

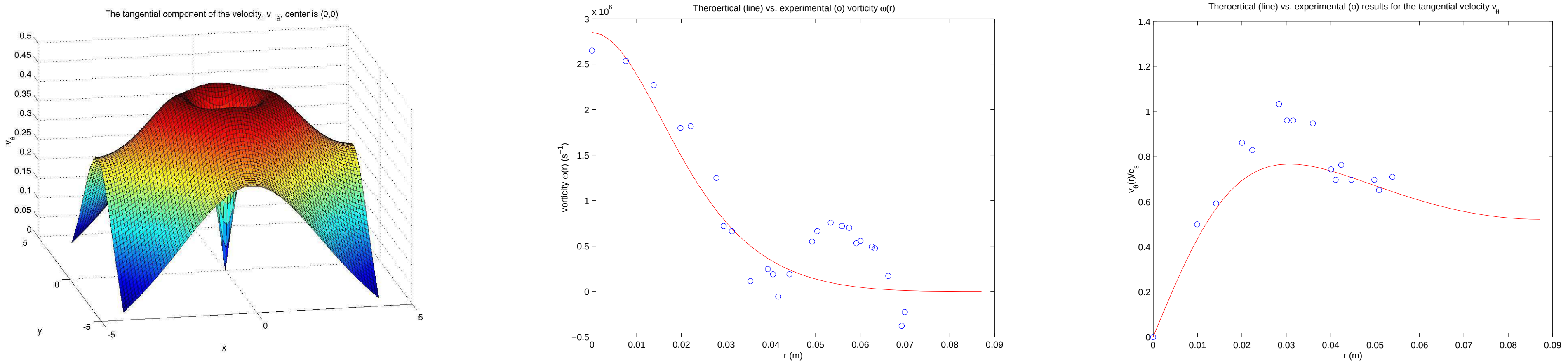


Figure 5: The calculated vortex and comparison with experiment.

Comparison of our vortex solution with experiment.

The crystals of plasma vortices

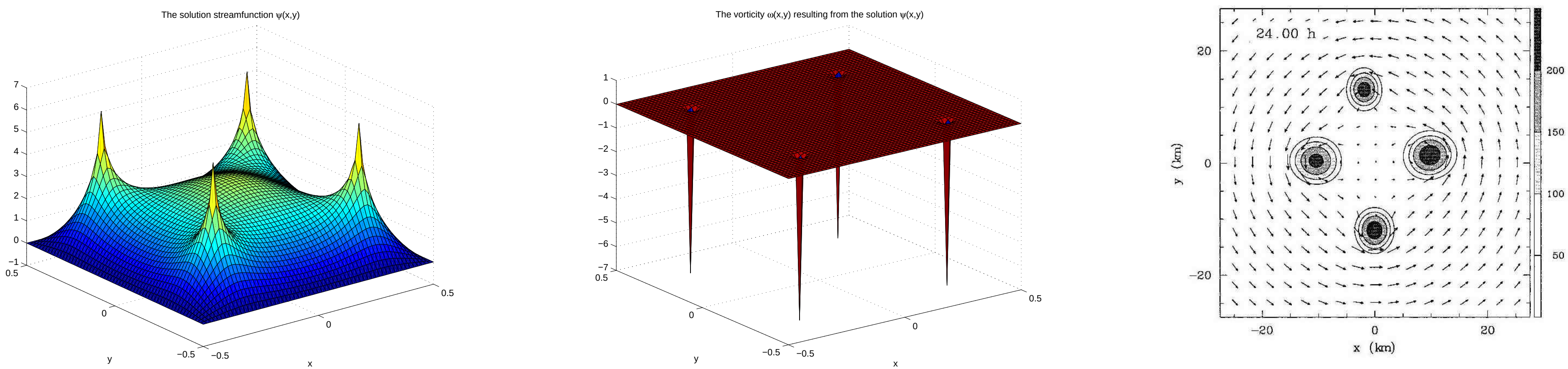


Figure 6: The crystals of plasma vortices.

Comparisons of crystal-type solutions with experiment.

Vortex crystals in non-neutral plasma

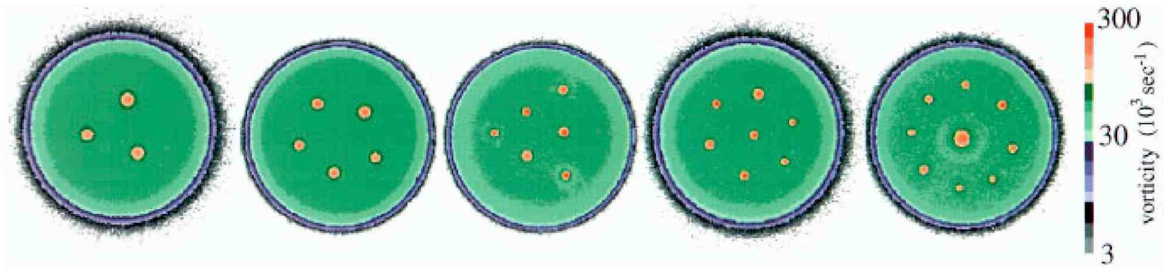
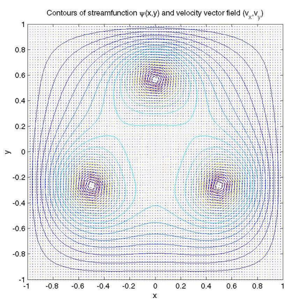
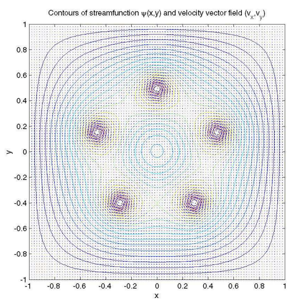
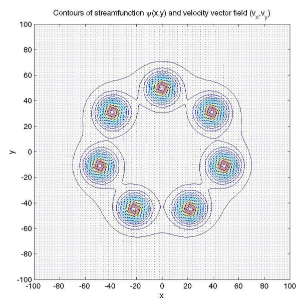


FIG. 1. Vortex crystals observed in magnetized electron columns (Ref. 8). The color map is logarithmic. This figure shows vortex crystals with (from left to right) $M = 3$, 5, 6, 7, and 9 intense vortices immersed in lower vorticity backgrounds. In a vortex crystal equilibrium, the entire vorticity distribution $\zeta(r,\theta)$ is stationary in a rotating frame; i.e., ζ is a function of the variable $-\psi + \frac{1}{2}\Omega r^2$, where ψ is the stream function and Ω is the frequency of the rotating frame.







Comparison of our vortex solution with experiment.