Lagrangian solutions to the Vlasov-Poisson system with a point charge

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in collaboration with
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Luminy, November 31, 2017

The Vlasov-Poisson system

Consider a plasma under the effect of a self-induced electric field E

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + (E \qquad) \cdot \nabla_v f = 0\,, \\ E(t,x) = (\frac{x}{|x|^3} * \rho)(t,x)\,, \\ \rho(t,x) = \int f(t,x,v)\, dv\,, \end{cases}$$

$f : [0, T] \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}_+$ density of particles in the plasma,

$\rho : [0, T] \times \mathbb{R}^3 \to \mathbb{R}_+$ spatial density,

$E : [0, T] \times \mathbb{R}^3 \to \mathbb{R}^3$ time-dependent self-induced electric field

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Well posedness:
Okabe & Ukai, Bardos & Degon, Pfaffelmoser, Lions & Perthame, . . .

The Vlasov-Poisson system with point-charge

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + (E + \gamma F) \cdot \nabla_v f = 0\,, \\ E(t,x) = (\frac{x}{|x|^3} * \rho)(t,x)\,, \\ \rho(t,x) = \int f(t,x,v)\, dv\,, \\ F(t,x) = \frac{x-\xi(t)}{|x-\xi(t)|^3} \end{cases}$$

$f(t,x,v)$ mass distribution of the plasma and $\xi(t)$ position of the point charge

$$\begin{cases} \dot{\xi}(t) = \eta(t)\,, & \xi(0) = \xi_0\,, \\ \dot{\eta}(t) = E(t,\xi(t))\,, & \eta(0) = \eta_0\,. \end{cases}$$

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$\gamma = \;\; 1$: the charge has the same sign of the plasma.

$\gamma = -1$: heavy charged particle which evolves in a sea of light particles of opposite sign.

V-P system with point-charge: repulsive case

Well posedness:

- *Caprino, Marchioro* (2d)
- *Marchioro, Miot, Pulvirenti* (3d): existence and uniqueness provided the initial datum $f_0 \in L^1 \cap L^\infty(\mathbb{R}^6)$ has compact support and satisfies

$$\min \{|x - \xi_0| \ : \ (x, v) \in \mathrm{supp}(f_0)\} \geq \delta_0$$

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- *Desvillettes, Miot, S.* (3d): existence of weak solutions for initial data $f_0 \in L^1 \cap L^\infty(\mathbb{R}^6)$ such that there exists $m_0 > 6$

$$\iint \left(|v - \eta_0|^2 + \frac{1}{|x - \xi_0|} \right)^{m/2} f_0(x, v) \, dx \, dv < \infty \qquad \forall \, m < m_0$$

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Price to pay:
the solution is no longer known to be *unique* and *Lagrangian*.

Lagrangian solutions

Goal: to recover the relation between the Eulerian and the Lagrangian picture

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System of characteristics

$$\begin{cases} \dot{X}(t,x,v) = V(t,x,v), & X(0,x,v) = x \\ \dot{V}(t,x,v) = E(t,X(t,x,v)) + F(t,X(t,x,v)), & V(0,x,v) = v \end{cases}$$

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We define *Lagrangian solution* a plasma density $f(t)$ and a trajectory $(\xi(t),\eta(t))$ of the point charge such that

$$\boxed{f(t,x,v) = f_0(X^{-1}(t,\cdot,\cdot)(x,v), V^{-1}(t,\cdot,\cdot)(x,v))}$$

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$$f(t,x,v) = f_0(X^{-1}(t,\cdot,\cdot)(x,v), V^{-1}(t,\cdot,\cdot)(x,v))$$

Difficulties:

- control on large velocities;
- the singular field F is a singular integral of measures.

Theorem 1 (Crippa, Ligabue, S. 2017)

Let $f_0 \in L^1 \cap L^\infty(\mathbb{R}^6)$*, such that the initial total charge*

$$\iint f_0(x,v)\,dxdv < 1$$

and the total energy

$$\iint \frac{|v|^2}{2} f_0(x,v)dxdv + \frac{|\eta_0|^2}{2} + \frac{1}{2}\iint \frac{\rho_0(x)\rho_0(y)}{|x-y|}dxdy + \iint \frac{\rho_0(x)}{|x-\xi_0|}dx$$

is finite. Assume that there exists $m_0 > 6$ *such that for all* $m < m_0$ *the energy moments*

$$\iint \left(|v-\eta_0|^2 + \frac{1}{|x-\xi_0|}\right)^{m/2} f_0(x,v)dxdv < \infty$$

Then there exists a global Lagrangian solution to the VP system with point charge.

Strategy of the proof

Generalised notion of flow: μ-Regular Lagrangian Flow

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Given an absolutely continuous measure μ with bounded density, a vector field $b(s,z) : [0,T] \times \mathbb{R}^6 \to \mathbb{R}^6$, and $t \in [0,T)$, a map

$$Z \in \mathcal{C}([0,T]; L^0_{\mathrm{loc}}(\mathbb{R}^6, d\mu)) \cap L^\infty([0,T]; \log\log L_{\mathrm{loc}}(\mathbb{R}^6, d\mu))$$

is a μ-regular Lagrangian flow in the renormalized sense starting at time t relative to b if the equation

$$\partial_s(\beta(Z(s,z))) = \beta^{'}(Z(s,z))b(s,Z(s,z))$$

holds in $\mathcal{D}^{'}((0,T))$ for μ-a.e. z, for every function $\beta \in \mathcal{C}^1(\mathbb{R}^6;\mathbb{R})$ that satisfies

$$|\beta(z)| \leq C(1+\log(1+\log(1+|z|^2))) \text{ and } |\beta^{'}(z)| \leq \frac{C\,|z|}{(1+|z|^2)(1+\log(1+|z|^2))}$$

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$\Longrightarrow$ Uniqueness and compactness for the flow

Strategy of the proof

Extension of the standard theory for transport equations. Main ingredients:

(H1) **Superlevels**: for all μ-regular Lagrangian flows relative to a vector field b, the λ-superlevels are controlled by a function $g(\lambda) \to 0$ as $\lambda \to \infty$;

(H2) **Structure of the vector field**:

$$b(t,x,v) = (b_1, b_2)(t,x,v) = (b_1(v), b_2(t,x))$$

with

$$b_1 \in \mathrm{Lip}(\mathbb{R}^3) \qquad \partial_{x_j} b_2 = \sum_{k=1}^{m} S_{jk} \mathrm{m}_{jk}$$

S_{jk} singular integrals on $\mathbb{R}^3$ and $\mathrm{m}_{jk} \in L^1((0,T); \mathcal{M}(\mathbb{R}^3))$;

(H3) **Local integrability of the vector field**:

$$b \in L^p_{\mathrm{loc}}([0,T] \times \mathbb{R}^6) \quad \text{for some } p > 1.$$

Strategy of the proof

In the context of the plasma-charge model:

$$b(t,x,v) = (v, E(t,x) + F(t,x))$$

(H1) **Superlevels**: the superlevels are bounded with respect to $\mu = f_0 \mathcal{L}^6$;

(H2) **Structure of the vector field**:

$$b(t,x,v) = (b_1(v), b_2(t,x)) = (v, E(t,x) + F(t,x))$$

with

$$b_1(t,v) = v \in \mathrm{Lip}(\mathbb{R}^3), \quad \partial_{x_j} b_2(t,x) = \frac{1}{|\cdot|^3} * (\rho + \delta_{\xi(t)})(t,x)$$

(H3) **Local integrability of the vector field**:

$$b(t,x,v) = (v, E(t,x) + F(t,x)) \in L^p_{\mathrm{loc}}([0,T] \times \mathbb{R}^6) \quad \text{for } p = \frac{3}{2}.$$

$$f_0^n(x,v) := f_0(x,v)\,\mathbf{1}_{\{(x,v)\in\mathbb{R}^6\,:\,n^{-1}<|x-\xi_0|<n,\ |v-\eta_0|<n\}}$$

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there exists a unique classical Lagrangian solution f^n, $(\xi^n(t),\eta^n(t))$

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Compactness + uniform estimates $\Longrightarrow$ existence of a Lagrangian solution

$$f(t, X(t,x,v), V(t,x,v)) = f_0(x,v)$$

Perspectives

- remove assumption $\int f_0(x, v)\, dx\, dv < 1$ (work in progress);
- uniqueness ?
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Thank you for your attention!