Vlasov Equation and Violent Relaxation – The Self-Gravitating Ring Model

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Long-range interactions

Potentials $\propto r^{-\alpha}$ with $\alpha < D$ for large r (D the spatial dimension).

- Violent relaxation $\rightarrow$ Quasi-stationary state $\rightarrow$ Thermodynamic equilibrium (finite N).
- Dynamics in the $N \rightarrow \infty$ limit:

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + \frac{1}{2N} \sum_{i,j=1}^{N} v(\mathbf{r}_i - \mathbf{r}_j).$$

Vlasov equation:

$$\frac{d}{dt} f(\mathbf{p}, \mathbf{r}, t) = \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} + \mathbf{F} \cdot \frac{\partial}{\partial \mathbf{p}} \right) f(\mathbf{p}, \mathbf{r}, t) = 0.$$

$$\mathbf{F}(\mathbf{r}, t) = -\frac{\partial}{\partial r} U(\mathbf{r}, t), \qquad U(\mathbf{r}) = \int f(\mathbf{p}, \mathbf{r}, t) v(\mathbf{r} - \mathbf{r}') d\mathbf{p}' d\mathbf{r}'.$$

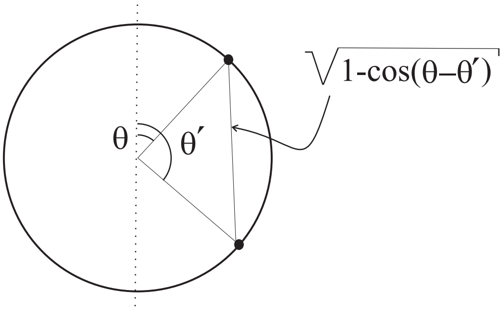
- The violent relaxation is described by the Vlasov equation for N sufficiently large.

One-Dimensional Models

- Simpler but yet retaining many of the important physical aspects.
- The Vlasov equation can be solved numerically, e. g. using a semi-Lagrangian method on a GPU.
- A direct comparison of the numeric solution of the Vlasov Equation and Molecular Dynamics simulations is thus possible.

The Ring model

Sota (2001): self-gravitating particles of unit mass on a ring of radius R



$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} - \frac{1}{N} \sum_{i<j=1}^{N} \frac{1}{\sqrt{2}\sqrt{1-\cos(\theta_i-\theta_j)+\epsilon}}.$$

$(t \to \sqrt{N}t)$.

The Hamiltonian Mean Field model $\to \epsilon \gg 1$

Antony and Ruffo (1995)

- Obtained in the limit with ϵ large:

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} - \frac{1}{N} \sum_{i<j=1}^{N} \left[1 - \cos(\theta_i - \theta_j)\right].$$

- Exactly solvable:

$$\mathbf{M} = (M_x, M_y) = \frac{1}{N} \sum_{i=1}^{N} (\cos\theta_i, \sin\theta_i).$$

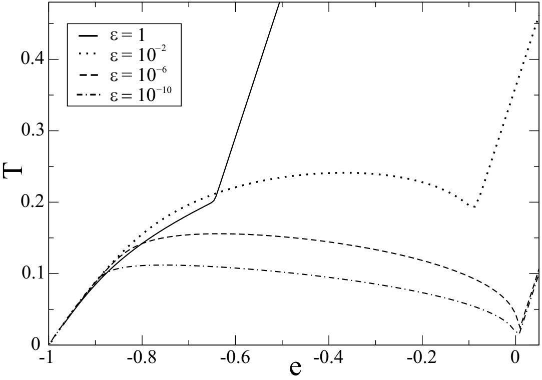
$$f_{\rm eq}(p,\theta) = \frac{\sqrt{\beta}}{(2\pi)^{3/2}\, \mathrm{I}_0(\beta)} e^{-\beta\left(p^2/2 - M\cos(\theta)\right)}, \qquad M = \frac{\mathrm{I}_1(\beta M)}{\mathrm{I}_0(\beta M)}.$$

- Simulations scale with N, and not with N^2 as in self-gravitating systems:

$$F_i = -\sin(\theta_i) M_x + \cos(\theta_i) M_y.$$

Caloric curve for the ring model

Variational method (Tatekawa et al. 2005):



Convergence to the Vlasov equation

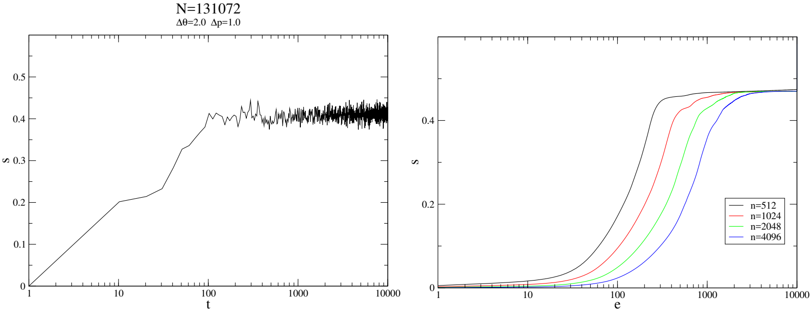
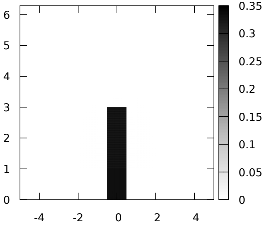


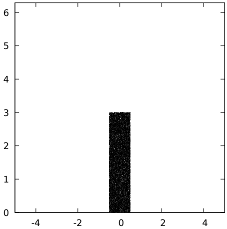
Figure: Left Panel: Entropy for the HMF model from MD, $N = 131\,072$, waterbag initial condition $\Delta p = 1.0$ and $\Delta\theta = 2.0$. Right Panel: Entropy from the Vlasov equation with grid resolutions $n \times n$.

Numerical solution of the Vlasov Eq. and MD

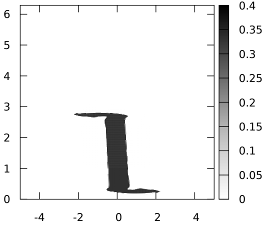
Ring model: $N = 20\,480$, $\Delta\theta = 3.0$, $\Delta p = 1.0$

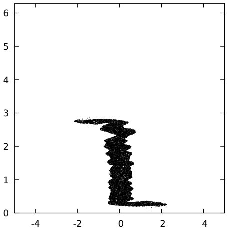
$$t = 0$$



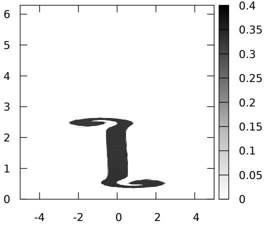


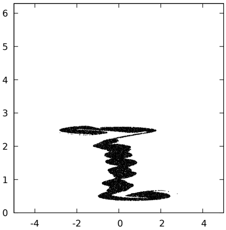
$t = 0.3$



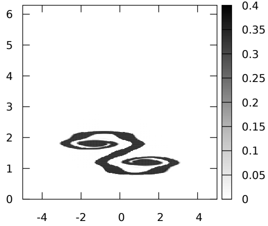


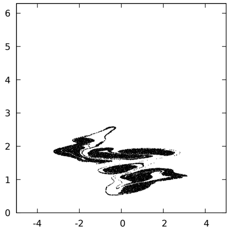
$t = 0.5$



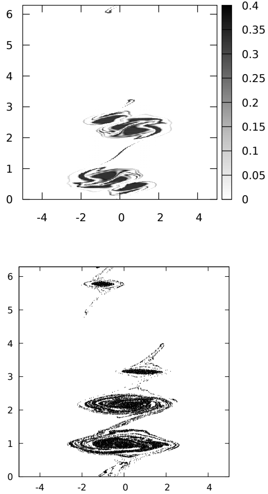


$t = 1.0$

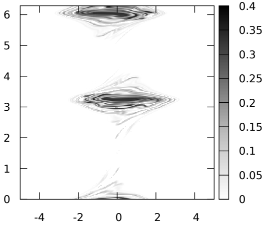


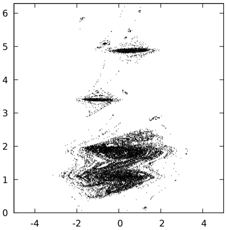


$t = 2.5$

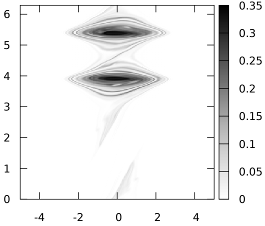


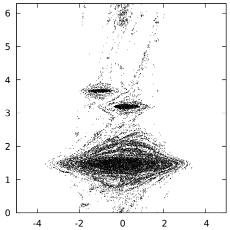
$t = 5.0$



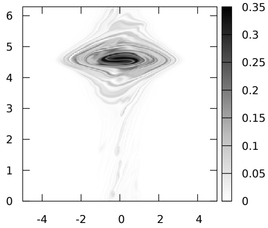


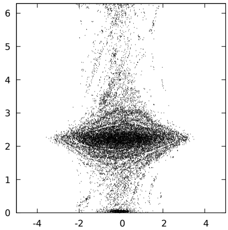
$t = 10.0$



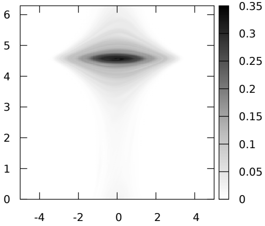


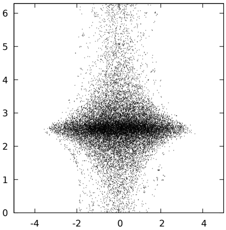
$t = 20.0$





$t = 50.0$





Convergence to the Vlasov equation

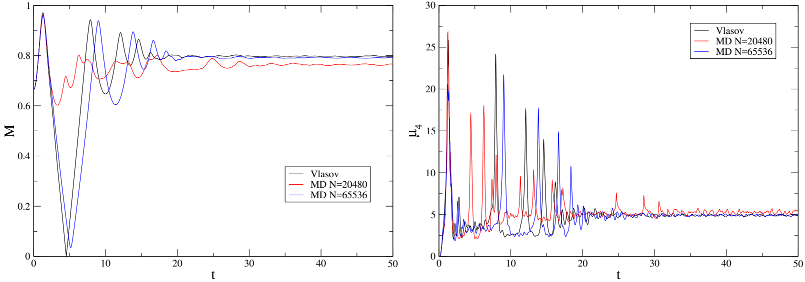


Figure: Solution of Vlasov Eq. and MD for the Ring Model $\epsilon = 10^{-3}$. $\mu_k = \langle p^k \rangle$.

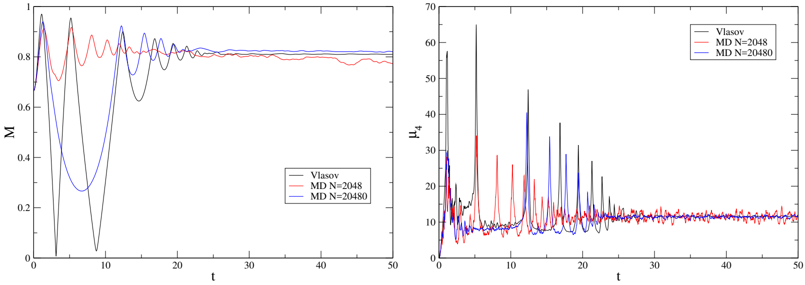


Figure: Comparison of Vlasov solution and MD for the Ring Model $\epsilon = 10^{-4}$.

- Although the intermediate step can deviate significantly from Vlasov equation solution, the state from Vlasov dynamics after the violent relaxation coincides with good accuracy to Molecular Dynamics (at least for N not too small).
- This can be explained from the Core-Halo approach (Levin et al. 2008) with a maximum (coarse grained) entropy principle (Rocha Filho 2016):

$$f_{\mathrm{QS}}(q,p) = \eta\Theta\left(\epsilon_F - e(p,\theta)\right) + \chi\Theta\left(e(p,\theta) - \epsilon_F\right)\Theta\left(\epsilon_H - e(p,\theta)\right),$$

$$e(p,\theta) = \frac{p^2}{2} + U(\theta).$$

- Four parameters: η (core phase value), χ (Halo phase), ϵ_F (core maximum energy) and ϵ_H (halo energy).
- Determined from: Value of the phase of initial distribution, total energy, normalization and maximization of entropy.

Finite N corrections – Homogeneous states

- Liouville Equation:

$$\frac{\partial f_N}{\partial t} + \{f_N, H\} = 0.$$

- BBGKY hierarchy:

$$\frac{\partial}{\partial t} f_s = \sum_{j=1}^{s} \hat{L}_j^0 f_s + \frac{1}{N} \sum_{j<k=1}^{s} \hat{L}'_{jk} f_s + \sum_{j=1}^{s} \int \mathrm{d}(\mathbf{s}+\mathbf{1})\, \hat{L}'_{j,s+1} f_{s+1}.$$

$$\hat{L}_i^0 \equiv -\vec{v}_i \cdot \frac{\partial}{\partial \vec{r}_i},$$

$$\hat{L}'_{ij} \equiv -\frac{1}{m} \vec{F}(\vec{r}_i - \vec{r}_j) \cdot \left(\frac{\partial}{\partial \vec{v}_i} - \frac{\partial}{\partial \vec{v}_j}\right),$$

- Two-particle correlation:

$$f_2(\mathbf{1},\mathbf{2};t) = f(\mathbf{1};t)f(\mathbf{2};t) + g_2(\mathbf{1},\mathbf{2};t).$$

- Kinetic equation

$$\partial_t f(\mathbf{1};t) = L_1^0 f(\mathbf{1};t) + \int \mathrm{d}\mathbf{2}\, L'_{12}\, g_2(\mathbf{1},\mathbf{2};t),$$

with (up to dominant terms in N^{-1}):

$$\frac{\partial}{\partial t} g_2(\mathbf{1},\mathbf{2};t) = \left[\hat{L}_1^0 + \hat{L}_2^0\right] g_2(\mathbf{1},\mathbf{2};t)$$
$$+\hat{L}'_{12}\left[g_2(\mathbf{1},\mathbf{2};t) + f(\mathbf{1};t)\, f(\mathbf{2};t)\right]$$
$$+\int \mathrm{d}\mathbf{3}\left\{\hat{L}'_{13} f(\mathbf{1};t) g_2(\mathbf{2},\mathbf{3};t) + \hat{L}'_{23} f(\mathbf{2};t) g_2(\mathbf{1},\mathbf{3};t)\right\}.$$

- Splitting the integrals in the previous equation in a small region where the force is more important (divergence), and the rest of the space, we have

$$\int \mathrm{d}\mathbf{2}\, L'_{12}\, g_2(\mathbf{1},\mathbf{2};t) = I_1 + I_2,$$

  where I_1 is of order $1/N$ correction (see previous talk), and I_2 is the usual Balescu-Lenard collisional integral, which vanishes exactly for a 1D homogeneous states. The next term in the expansion is of order N^{-2} (Rocha Filho 2014).
- There are two well separated time scales: the $1/N$ contribution of the divergence of the potential from I_1, and the $1/N^2$ contribution from the collisional integral I_2.

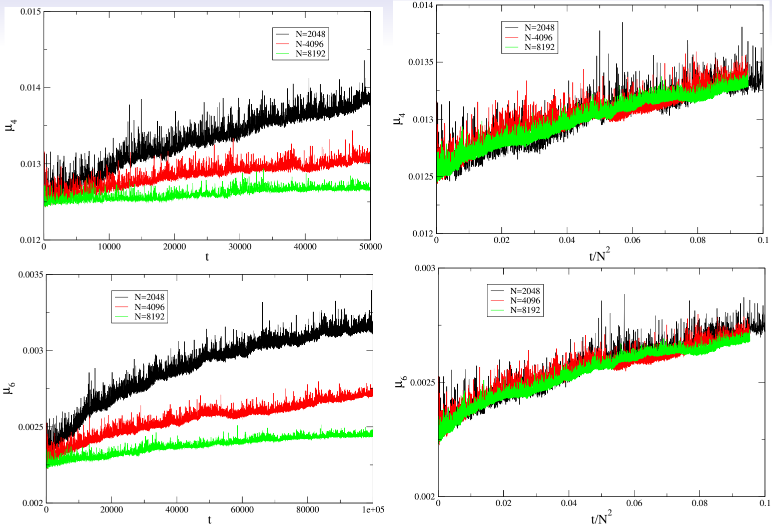


Figure: a) μ_4 with $\epsilon = 1.0$ for a homogeneous waterbag initial condition, $\Delta p = 1.0$ and $\Delta t = 1.0$. (b) Same as (a) but with $t \to t/N^2$. (c) and (d) same as (a) and (b), but for μ_6.

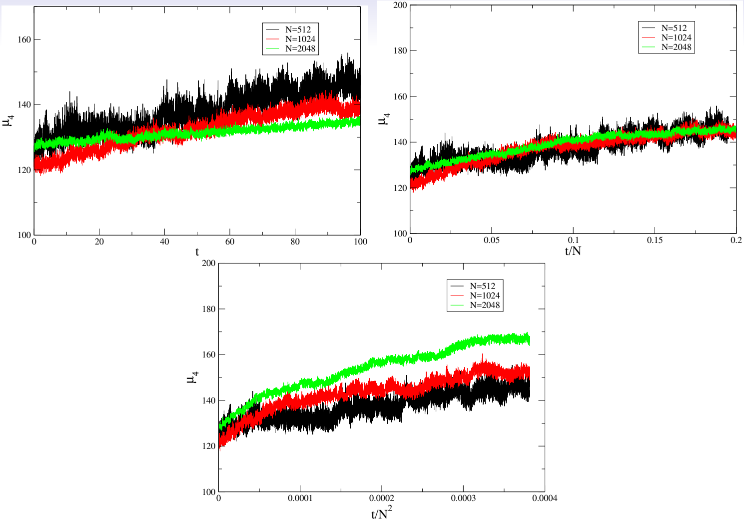


Figure: a) μ_4 with $\epsilon = 10^{-6}$ for a few particle numbers for a homogeneous waterbag initial condition with $\Delta p = 10.0$ and time step $\Delta t = 10^{-5}$. (b) Same as (a) but with the time rescaled as $t \to t/N$. (c) Same as (a) but with the time rescaled as $t \to t/N^2$.

Conclusions

- Convergence of particle dynamics to Vlasov equation is much slower in the presence of a diverging potential.
- Although intermediate states during violent relaxation, the final states seem to coincide with the final state from Vlasov equation which seems to be related to a maximum (coarse grained) entropy principle.
- Finite N corrections to the Vlasov equation depends strongly on the behavior of the potential at short distances.
- Although some progress have been made since Lynden-Bell theory (1967), a complete and consistent theory for violent relaxation is still lacking.

Thank You for Your Attention!