Brownian regime of finite-N corrections to particle motion in the XY hamiltonian mean field model

Bruno Vieira Ribeiro

IFB - Câmpus Estrutural

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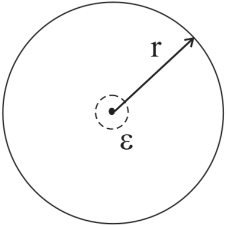
Summary

Long-range interactions

Interaction potentials that decay as $r^{-\alpha}$ with $\alpha < D$ (spatial dimension) for large r.



Potential energy

$$V \propto \int_{\epsilon}^{R} C(D) r^{D-1} \times \frac{1}{r^{\alpha}}$$

Examples

- Self-gravitating systems (galaxies, globular clusters);
- Charged plasmas;
- Spin models;
- Simplified models (Ring model, HMF, 2D-gravity, self-gravitating sheets...)

Properties of long-ranged systems

- Non-additive;
- Ensemble inequivalence;
- Non-gaussian stationary states (finite N);
- Anomalous diffusion and aging;
- Violent realxation (phase mixing).

Evolution

- Violent relaxation: rapid relaxation to a (quasi)stationary state;

$$\Downarrow$$

- The system stays on a (non-Gaussian) quasi-stationary state whose life-time diverges with the number of particles;

$$\Downarrow$$

- For finite N, the system finally relaxes to the true (Gaussian) thermodynamic equilibrium state.

The model

$$H_{\mathrm{HMF}} = \sum_{i=0}^{N} \frac{p_i^2}{2} + \frac{V}{2N} \sum_{i,j} [1 - \cos(\theta_i - \theta_j)], \tag{2.1}$$

$$\begin{cases} \dot{\theta}_i = p_i \\ \dot{p}_i = -\frac{V}{N} \sum_{j=1}^{N} \sin(\theta_i - \theta_j), \end{cases} \tag{2.2}$$

- system of N coupled pendula.

$$\mathbf{M} = \frac{1}{N} \sum_{i=1}^{N} \mathrm{e}^{\mathrm{i}\theta_i} \equiv M \mathrm{e}^{\mathrm{i}\varphi} \equiv (M_x, M_y),$$

- We can rewrite:

$$\dot{p}_i = -VM \sin(\theta_i - \varphi)$$

Pendular motion

Eq. of motion:

$$\dot{p}_i = -VM\sin(\theta_i - \varphi) \;\rightarrow \text{Pendulum with amplitude } M.$$

Energy per particle given by

$$e_i = \frac{p_i^2}{2} + V[1 - M\cos(\theta_i - \varphi)], \tag{2.3}$$

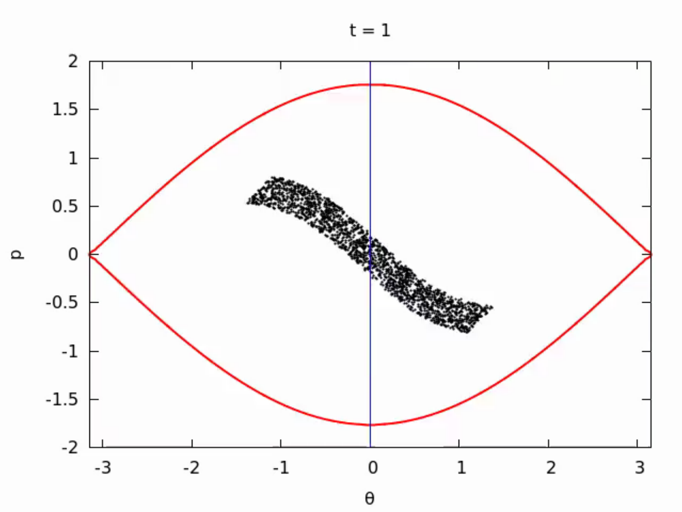
- Associated separatrix: width $2\sqrt{VM}$ and energy $E_s = V(1 + M)$.

Particels

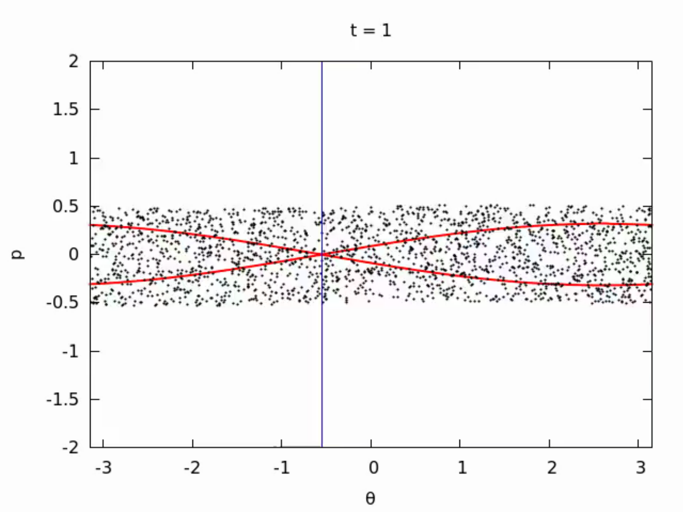
- High Energy Particles (HEP) - $e_i > E_s$: visit the whole ring;
- Low Energy Particles (LEP) - $e_i < E_s$: confined to a periodic motion.

▶ Supercritical case ($U > U_c$): width of separatrix goes to zero, all particles are HEP.

Attractive case: Confined



Translational



Subregimes: Subcritical Bounded

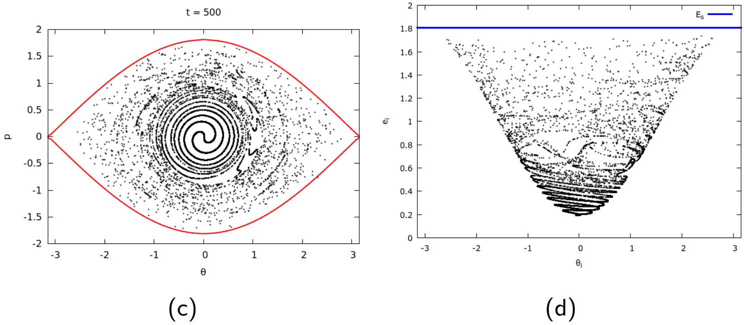
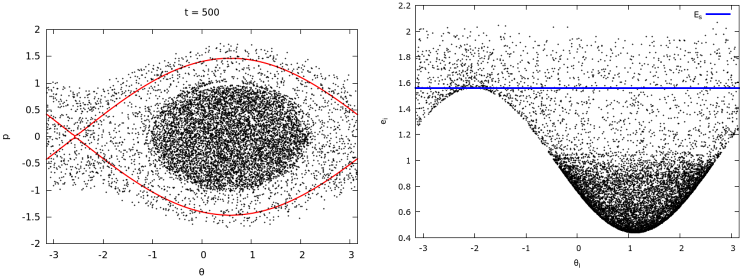


Figure: (c): Phase portrait of the system in the QSS. The line represents the instanteneous separatrix. (d): Particle energy as a fucntion of its position on the ring. $N = 10^4$ and $U \approx 0,3$.

Subcritical Translational



(a) (b)

Figure: (a): (c): Phase portrait of the system in the QSS. The line represents the instanteneous separatrix. (b): Particle energy as a fucntion of its position on the ring. $N = 10^4$ e $U \approx 0,5$.

Repulsive Case

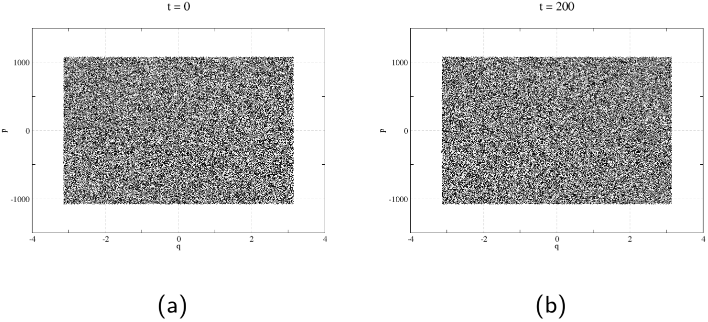


Figure: Phase portraits: (a) initial state and (b) state after simulation time of $t = 200$. Note that there is no phase transition in this case, i.e. , system remains in an homogeneous state. Graphics are made with 10^5 particles.

Vlasov limit

Rigorously

$$\frac{\partial f}{\partial t} + p\frac{\partial f}{\partial q} - \frac{\mathrm{d}V}{\mathrm{d}q}\frac{\partial f}{\partial p} = 0$$

$$\begin{aligned} V(q)[f] &= 1 - M_x[f]\cos(q) - M_y[f]\sin(q) \\ M_x[f] &= \int_{-\pi}^{\pi}\int_{-\infty}^{\infty} f(q,p,t)\cos(q)\mathrm{d}q\mathrm{d}p \\ M_y[f] &= \int_{-\pi}^{\pi}\int_{-\infty}^{\infty} f(q,p,t)\sin(q)\mathrm{d}q\mathrm{d}p \end{aligned}$$

- QSS $\rightarrow$ stable steady state of the Vlasov equation.

Equilibrium from maximum entropy

- Repulsive ($V = -1$): $M = 0$ is the only solution $\rightarrow$ system always reaches a state of equilibrium;

- Attractive ($V = 1$): Admits, also, $M \neq 0$ solution. Some T_c exists separating an homogeneous equilibrium from a clustered one.

Repulsive case

- Starting from homogeneous configuration;
- MD simulations $\rightarrow M(t)$ fluctuates;
- Empirically assumed to have diffusive nature.

Analytical results

Assuming

$$q_j(t) = q_{j0} + v_{j0}t + Q_j, \quad Q_j \ll q_j \tag{3.1}$$

So

$$M\mathrm{e}^{\mathrm{i}\varphi} \simeq M^{(0)}\mathrm{e}^{\mathrm{i}\varphi^{(0)}} := \frac{1}{N}\sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}(q_{j0}+v_{j0}t)}$$

and

$$\ddot{q}_j \simeq M^{(0)} \sin(q_j - \varphi^{(0)}).$$

Rescaling

- Small noise $\rightarrow$ effect on large times.
- $t' = t/N''$ and $N = N' \cdot N''$.

$$\mathbf{M}\,\mathrm{d}t = \frac{1}{N}\sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}(q_{j0}+v_{j0}t)}\,\mathrm{d}t = N'^{-\alpha}\sqrt{2\pi}\,\mathrm{d}W_{t'}^{N}, \tag{3.2}$$

where we introduce the complex-valued process

$$\mathrm{d}W_{t'}^{N} = \frac{1}{\sqrt{2\pi}N'^{1-\alpha}}\sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}q_{j0}}\mathrm{e}^{\mathrm{i}N''v_{j0}t'}\,\mathrm{d}t' \tag{3.3}$$

Eqs. of motion

- Setting $p_i' = N'' p_i$ and $N'' = N'^{\alpha}$,

$$
\begin{aligned}
\mathrm{d}q_i &= p_i' \,\mathrm{d}t', \\
\mathrm{d}p_i' &= \sqrt{2\pi}\,[\sin(q_i)\,\mathrm{d}\Re(W_{t'}^N) - \cos(q_i)\,\mathrm{d}\Im(W_{t'}^N)].
\end{aligned}
$$

Initial Conditions

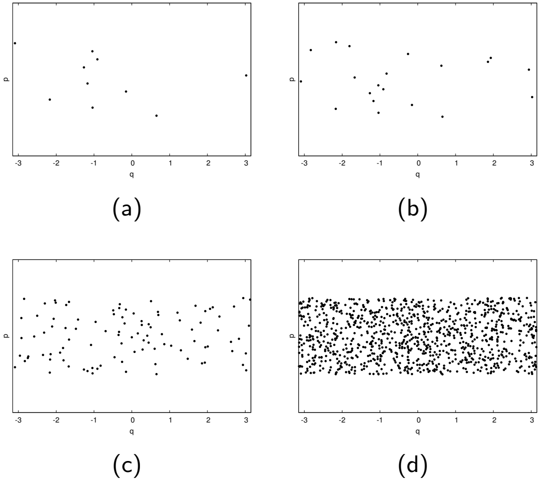


Figure: (a): 10 particles, (b) 20 particles, (c) 100 particles e (d) 1000 particles. Distribution approaches a "waterbag".

$N \to \infty$ limit

- Process $W(t')$ approaches a Wiener process $W_{t'}$, thus

$$
\begin{aligned}
\mathrm{d}q_i &= p_i' \mathrm{d}t', \\
\mathrm{d}p_i' &= \sqrt{2\pi} \frac{N''^{1/2}}{N'^{1-\alpha}} \left[\sin(q_i) \circ \mathrm{d}\Re(W_{t'}) - \cos(q_i) \circ \mathrm{d}\Im(W_{t'})\right].
\end{aligned}
$$

- Selecting $N''^{1/2} = N'^{1-\alpha}$ along with $N'' = N'^{\alpha} = N/N'$, yields $\alpha = 2/3$, $N' = N^{3/5}$ and $N'' = N^{2/5}$.
- By Proposition 4.2 of Elskens (2012),

$$
\begin{aligned}
q_i(t') &= q_{i0} + p_{i0}' t' + \sqrt{2\pi} \int_0^{t'} B_i(s) \mathrm{d}s, \\
p_i'(t') &= p_{i0}' + \sqrt{2\pi}\, B_i(t').
\end{aligned}
$$

Variations

$$\begin{aligned}\delta q_i(t') &\sim t'\delta p_i' \sim \mathrm{O}\left(t'^{3/2}\right),\\ \delta p_i'(t') &\sim \sqrt{2\pi}\,B(t') \sim \mathrm{O}\left(t'^{1/2}\right).\end{aligned}$$

So, as a correction to ballistic motion, we get

$$q_j(t) = q_{j0} + v_{j0}t + \delta q_i(t') \sim q_{j0} + v_{j0}t + \mathrm{O}\left(\frac{t}{N''}\right)^{3/2}.$$

- Validity

$$\frac{t}{N''} \ll 1,$$

For finite N, the approach breaks down when $t \sim N'' \sim N^{2/5}$.

Numerical findings

Molecular dynamics: follow particles trajectories in velocity space.

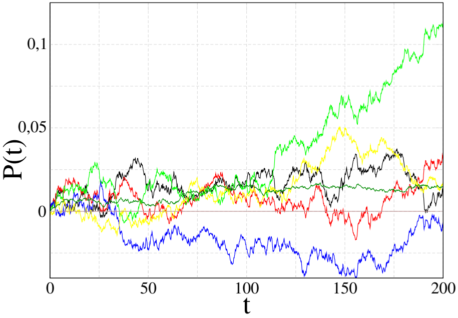
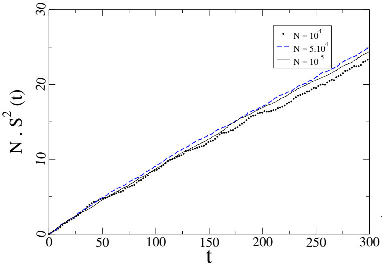


Figure: Trajectories of 6 different particles in velocity space. Number of simulated particles: 10^4.

Dispersion



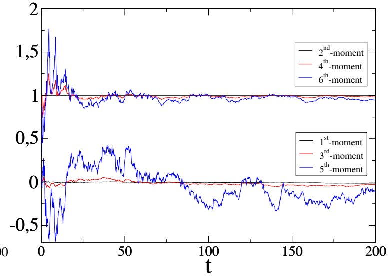


Figure: (a) Evolution of momentum deviation average square $S^2(t)$. System sizes $N = 10^4$, $N = 5 \cdot 10^4$ and $N = 10^5$, sample size $N_s = 0.1N$, total simulation time $t = 300$, time step $\delta t = 0.1$. (b)Moments of $P_j(t) = p_j(t) - p_{j0}$. System size $N = 10^6$, sample size $N_s = 10^5$, total simulation time $t = 200$, time step $\delta t = 0.1$.

Conclusions

- Choosing proper initial conditions (monokinetic beams), for the repulsive case, we show the existence of a regime of Brownian corrections in velocity space implied by the Vlasov limit;
- These corrections propagate initial independence (molecular chaos);
- Mean field forces converge, rigorously, to Wiener processes;
- Lower time estimate for our approximations;
- Numerical results confirm our findings.

Thank You